



Fuzzy Reliability Estimation for Benktander Distribution

Naser Odat^{1,*}

¹Department of Mathematics, Jadara University, P.O. Box (733), postal code 21111, Irbid, Jordan

Email: nodat@jadara.edu.jo

Abstract

The fuzzy reliability estimate for the Benktander distribution, a model appropriate for heavy-tailed data, is investigated in this work. By adding membership functions and α -cuts, we extend the Benktander distribution to a fuzzy framework and compute its probability density function and reliability function. The fuzzy reliability is estimated using two methods: maximum likelihood and Bayesian approaches. The Bayesian method uses special loss functions, gamma priors, and squared error. The effectiveness of these estimators is examined in a simulated study using varying sample sizes and parameter values. The findings show that, especially for smaller samples, Bayesian techniques—in particular, the cautious Bayes estimator—perform better in terms of accuracy and stability than maximum likelihood estimation. The results emphasize how crucial it is to choose suitable prior distributions and loss functions while doing reliability analysis.

Keywords: Benktander Distribution; Fuzzy Reliability; Maximum likelihood; Bayesian estimator

1. Introduction

In disciplines like engineering, finance, and environmental research, where knowing when systems or components fail is crucial, reliability analysis is crucial. While real-world situations always contain uncertainty and imprecision, traditional dependability, models generally presume accurate data. This is addressed by fuzzy dependability theory, which uses membership functions to symbolize uncertainty levels. This study is based on the Benktander distribution, which is well known for its adaptability in simulating heavy-tailed occurrences

A fuzzy measure has been used to introduce and formulate the fuzzy reliability notion. [1] introduced fuzzy sets and fuzzy logic, which helped to establish the theory of fuzzy sets. Fuzzy set theory and mathematics were later used in a wide range of study domains [2].

The efficacy of fuzzy reliability estimation for heavy-tailed distributions has been shown by recent developments. For example, [3] used ranked set sampling to study fuzzy exponential reliability, while [4] created a fuzzy framework for the Lomax distribution. [5] expanded on this study by confirming that fuzzy approaches are robust for Lomax distribution reliability in the presence of imprecise data. The Benktander distribution, a versatile model for extreme occurrences in actuarial science and risk analysis [6], is still not well understood in the context of fuzzy dependability, although these contributions.

This study examines both Bayesian and conventional (maximum likelihood) methods for estimating fuzzy reliability for the Benktander distribution. Gamma priors and asymmetric loss functions are added to the Bayesian framework to accommodate different levels of uncertainty. The performance of the estimators is assessed by a simulation exercise, and useful suggestions are offered for their use. By shedding light on the trade-offs between various estimating techniques, the findings add to the expanding corpus of research on fuzzy reliability.

In engineering, finance, and environmental research, reliability analysis is essential, yet conventional approaches frequently fall short in capturing uncertainty in the real world. By putting forth a fuzzy reliability framework for the Benktander distribution—a heavy-tailed model appropriate for severe events—this paper fills the gap. By combining Bayesian estimation and fuzzy set theory, our work expands on traditional methods and provides reliable tools for imprecise data.

This paper makes three major contributions:

1. A new fuzzy reliability framework that incorporates α -cut-based membership functions for the Benktander distribution.
2. A comparison of Bayesian and MLE estimators under asymmetric (precautionary) and symmetric (squared error) loss functions.
3. Useful recommendations for choosing estimators that are backed by simulations and based on sample size and uncertainty levels.

2. Benktander Distribution

[6] initially presented the Benktander distribution, a specific probability model intended for heavy-tailed data, in the context of actuarial risk analysis. It offers greater flexibility for applications in environmental science, economics, and reliability engineering by addressing the drawbacks of classical distributions (such as Weibull and exponential) in simulating extreme occurrences [7].

The random variable X is assumed to have a Benktander distribution with a shape parameter β , where $\beta > 0$. We use $X \sim \text{Benk}(\beta)$ to denote it. Its probability density function is as follows:

$$f(x|\beta) = \beta e^{\beta(1-x)}, \quad x > 1 \quad (2.1)$$

$$R(t) = p(x \geq t) = \int_t^{\infty} \beta e^{\beta(1-x)} dx = e^{\beta(1-t)}$$

3. Fuzzy Reliability

The idea of fuzzy reliability, which expands on conventional reliability theory to consider uncertainty, is introduced in this section. Using membership functions and α -cuts, which measure the level of conviction in system survival, we create the fuzzy reliability function. Important derivations include the piecewise membership function (3.2) and fuzzy probability formula (3.1), which are followed by reliability expressions based on the α -cut (3.4), (3.6). Reliability analysis with inaccurate or insufficient data is made possible by these tools.

Several researchers have introduced and expanded the theory of fuzzy reliability [8] and [9]. Time fuzzy sets have been the subject of investigation by several researchers, including the works cited in [10] and [11]. [12] studied the reliability for standard Kumaraswamy distribution. Let T be the continuous random variable that represents a system's "or unit or component's" failure time. The fuzzy reliability can then be derived using the fuzzy probability formula [13] by,

$$\tilde{R}(t) = p(T \geq t) = \int_t^{\infty} \mu(x) f(x) dx, \quad 1 < t \leq x < \infty \quad (3.1)$$

where the degree to which each element of a given universe belongs to the fuzzy set is represented by the membership function $\mu(x)$.

Now, assume that $\mu(x)$ is,

$$\mu(x) = \begin{cases} 0, & x < t_1 \\ \frac{x-t_1}{t_2-t_1}, & t_1 < x < t_2, \quad t_1 > 0 \\ 1, & x > t_2 \end{cases} \quad (3.2) \quad \text{According to [10], the lifetime}$$

$x(\alpha)$ for $\mu(x)$ can be found using the function of fuzzy numbers computational approach. This corresponds to a certain value of α -cut, $\alpha \in [0,1]$ as follows:

$$\begin{cases} x(\alpha) \leq t_1, & \alpha = 0 \\ t_1 + \alpha(t_2 - t_1), & 0 < \alpha < 1 \\ x(\alpha) \geq t_2, & \alpha = 1 \end{cases} \quad (3.3)$$

$$\tilde{R}(t)_{\alpha=0} = \int_{t_1}^{t_1} f(x) dx = 0 \quad (3.4)$$

$$\tilde{R}(t)_{0 < \alpha < 1} = \int_{t_1}^{x(\alpha)=t_1+\alpha(t_2-t_1)} \mu(x) f(x) dx \quad (3.5)$$

$$\tilde{R}(t)_{\alpha=1} = \int_{t_1}^{t_2} f(x) dx \quad (3.6)$$

4. Benktander Fuzzy Reliability

The notion of fuzzy reliability and the Bayesian method for estimating the fuzzy reliability of the Benktander distribution are examined here. Assuming that, like in (2.1), $f(x)$ is the Benktander distribution's pdf, the First Procedure is as follows: As stated in (3.1), the fuzzy reliability definition $\tilde{R}(t) = \int_t^\infty \mu(x) f(x) dx$

Where $\mu(x)$ as in (3.2) and $x(\alpha)$ as (3.3), then,

$$\tilde{R}(t)_{\alpha=0} = \alpha \int_{t_1}^{t_1} 0 e^{\beta(1-x)} dx = 0 \quad (4.1)$$

$$\tilde{R}(t)_{0 < \alpha < 1} = \int_{t_1}^{x(\alpha)=t_1+\alpha(t_2-t_1)} \frac{x-t_1}{t_2-t_1} \beta e^{\beta(1-x)} dx \quad (4.2)$$

$$= \frac{e^{\beta(1-t_1)}}{\beta(t_2-t_1)} - e^{\beta(1-x)} \left(\alpha + \frac{1}{\beta(t_2-t_1)} \right)$$

$$\tilde{R}(t)_{\alpha=1} = \int_{t_1}^{t_2} \beta e^{\beta(1-x)} dx = e^{\beta(1-t_1)} - e^{\beta(1-t_2)} \quad (4.3)$$

4.1. Benktander maximum likelihood:

Let β be the maximum likelihood estimator that can be derived from the likelihood function $L(\beta|x)$.

$$L(\beta|x) = \beta^n e^{\sum_{i=1}^n \beta(1-x_i)}$$

Taking the natural logarithm, $L(\beta|x)$ will be,

$$\ln(L) = n \ln(\beta) + \beta \sum_{i=1}^n (1-x_i)$$

Derive the natural log-likelihood function, $\ln(L)$, with respect to β and then equating to zero we get the maximum likelihood estimator of β by

$$\hat{\beta} = \frac{-n}{\sum_{i=1}^n (1-x_i)}$$

4.2. Fuzzy Standard Bayes estimation for Benktander

We have the likelihood function as

$$L(\beta|x) = \beta^n e^{\sum_{i=1}^n \beta(1-x_i)}$$

Assume that β has a gamma prior distribution, $g(\beta)$, with parameters a and b , in accordance with the Bayesian process.

$$g(\beta) = \frac{b^a}{\Gamma a} \beta^{a-1} e^{-\beta b}$$

The posterior is

$$\pi(\theta/x) = \frac{L(\beta|x)g(\beta)}{\int L(\beta|x)g(\beta)d\beta}$$

$$\pi(\beta/x) = \frac{\beta^n e^{\sum_{i=1}^n \beta(1-x_i)} \frac{b^a}{\Gamma a} \beta^{a-1} e^{-\beta b}}{\int_1^\infty \beta^n e^{\sum_{i=1}^n \beta(1-x_i)} \frac{b^a}{\Gamma a} \beta^{a-1} e^{-\beta b} d\beta}$$

$$= \frac{\beta^{n+a-1} e^{-\beta(b-\sum_{i=1}^n (1-x_i))}}{\int_1^\infty \beta^{n+a-1} e^{-\beta(b-\sum_{i=1}^n (1-x_i))} d\beta}$$

$$\int_1^\infty \beta^{n+a-1} e^{-\beta(b-\sum_{i=1}^n (1-x_i))} d\beta$$

$$= \frac{(b - \sum_{i=1}^n (1 - x_i))^{-n-a} \Gamma(n + a, (b - \sum_{i=1}^n (1 - x_i)))}{(b - \sum_{i=1}^n (1 - x_i))^{n+b}}$$

Then

$$\pi(\beta/x) = \frac{\beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1 - x_i))}}{\Gamma(n + a)} \left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b}$$

Now according to (3.4), (3.5) and (3.6), we get,

$$\begin{aligned} \tilde{R}(t)_{\alpha=0} &= \beta \int_{t_1}^{t_1} 0 e^{\beta(1-x)} dx = 0 \\ \tilde{R}(t)_{0 < \alpha < 1} &= \int_{t_1}^{x(\alpha)=t_1+\alpha(t_2-t_1)} \frac{x - t_1}{t_2 - t_1} \beta e^{\beta(1-x)} dx \end{aligned}$$

$$\tilde{R}(t)_{\alpha=1} = \int_{t_1}^{t_2} \beta e^{\beta(1-x)} dx = e^{\beta(1-t_1)} - e^{\beta(1-t_2)}$$

Therefore, the Bayes estimator of fuzzy reliability is constructed as follows using the squared error loss as a symmetric loss function:

$$\begin{aligned} \tilde{R}(t)_{0 < \alpha < 1} &= E(\tilde{R}(t)_\alpha) = \int \tilde{R}(t)_\alpha \pi(\beta/x) d\beta \\ &= \int_1^\infty \left[\frac{e^{\beta(1-t_1)}}{\beta(t_2 - t_1)} - e^{\beta(1-x)} \left(\alpha + \frac{1}{\beta(t_2 - t_1)} \right) \right] \frac{\beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1 - x_i))}}{\Gamma(n + a)} \left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b} d\beta \\ &= \left(b - \sum_{i=1}^n (1 - x) \right)^{n+b} \left[\int_1^\infty \frac{\beta^{n+a-2}}{(t_2 - t_1) \Gamma(n + a)} e^{-\beta((b - \sum_{i=1}^n (1 - x_i)) - (1 - t_1))} d\beta \right. \\ &\quad \left. - \int_1^\infty \frac{\alpha \beta^{n+a-1} e^{-\beta((b - \sum_{i=1}^n (1 - x_i)) - (1 - x(\alpha)))}}{\Gamma(n + a)} d\beta \right] \end{aligned}$$

Let $c = b - \sum_{i=1}^n (1 - x_i)$, $d_1 = c - (1 - t_1)$, $d_2 = c - (1 - x(\alpha))$

Then

$$\begin{aligned} &\left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b} \left[\int_1^\infty \frac{\beta^{n+a-2}}{(t_2 - t_1) \Gamma(n + a)} e^{-\beta((b - \sum_{i=1}^n (1 - x_i)) - (1 - t_1))} d\beta \right. \\ &\quad \left. - \int_1^\infty \frac{\alpha \beta^{n+a-1} e^{-\beta((b - \sum_{i=1}^n (1 - x_i)) - (1 - x(\alpha)))}}{\Gamma(n + a)} d\beta \right] \\ &= c^{n+b} \left[\int_1^\infty \frac{\beta^{n+a-2}}{(t_2 - t_1) \Gamma(n + a)} e^{-\beta d_1} d\beta - \frac{\alpha}{\Gamma(n + a)} \int_1^\infty \beta^{n+a-2} e^{-\beta d_2} d\beta \right] \end{aligned}$$

This integral is the upper incomplete gamma function then

$$\tilde{R}(t)_{0 < \alpha < 1} = c^{n+b} \left[\frac{d_1^{-(n+a-1)} \Gamma(n + a - 1, d_1)}{(t_2 - t_1) \Gamma(n + a)} - \alpha d_2^{-(n+a)} \frac{\Gamma(n + a, d_2)}{\Gamma(n + a)} \right]$$

$$\tilde{R}(t)_{\alpha=1} = \int_1^\infty (e^{\beta(1-t_1)} + (-e^{\beta(1-t_2)})) \frac{\beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1 - x_i))}}{\Gamma(n + a)} \left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b} d\beta$$

$$= \frac{(b - \sum_{i=1}^n (1 - x_i))^{n+b}}{\Gamma(n + a)} \left(\frac{\Gamma(n + a, 1 - t_1 - b + \sum_{i=1}^n (1 - x_i))}{1 - t_1 - b + \sum_{i=1}^n (1 - x_i)} - \frac{\Gamma(n + a, 1 - t_2 - b + \sum_{i=1}^n (1 - x_i))}{1 - t_2 - b + \sum_{i=1}^n (1 - x_i)} \right)$$

Using the precautionary loss as an asymmetric loss function, the following is the Bayes estimator of fuzzy reliability:

$$\begin{aligned} \tilde{R}(t)^p_{\alpha} &= \sqrt{E(\tilde{R}(t)_{\alpha})^2} \\ E(\tilde{R}(t)_{0<\alpha<1})^2 &= \int_1^{\infty} \left[\frac{e^{\beta(1-t_1)}}{\beta(t_2 - t_1)} - e^{\beta(1-x)} \left(\alpha + \frac{1}{\beta(t_2 - t_1)} \right) \right]^2 \frac{\beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1-x_i))}}{\Gamma(n + a)} \left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b} d\beta \\ &= \left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b} \int_1^{\infty} \left[\frac{e^{\beta(1-t_1)}}{\beta(t_2 - t_1)} - e^{\beta(1-x)} \left(\alpha + \frac{1}{\beta(t_2 - t_1)} \right) \right]^2 \frac{\beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1-x_i))}}{\Gamma(n + a)} d\beta \\ E(\tilde{R}(t)_{\alpha=1})^2 &= \left(b - \sum_{i=1}^n (1 - x_i) \right)^{n+b} \int_1^{\infty} (e^{\beta(1-t_1)} - e^{\beta(1-t_2)})^2 \frac{\beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1-x_i))}}{\Gamma(n + a)} d\beta \\ &= \int_1^{\infty} \left[\frac{\beta^{n+a-1} e^{-\beta((b - \sum_{i=1}^n (1-x_i)) - 2\beta(1-t_1))}}{\Gamma(n + a)} + \frac{\beta^{n+a-1} e^{-\beta((b - \sum_{i=1}^n (1-x_i)) - 2\beta(1-t_2))}}{\Gamma(n + a)} \right. \\ &\quad \left. - 2 \frac{\beta^{n+a-1} e^{-\beta((b - \sum_{i=1}^n (1-x_i)) - (1-t_1) - (1-t_2))}}{\Gamma(n + a)} \right] d\beta \end{aligned}$$

This integral is the sum of three incomplete gamma function

Let $c = b - \sum_{i=1}^n (1 - x_i)$, then

$$\begin{aligned} E(\tilde{R}(t)_{\alpha=1})^2 &= \frac{1}{\Gamma(n + a)} \left((c - 2(1 - t_1))^{-(n+a)} \Gamma(n + a, c - 2(1 - t_1)) + (c - 2(1 - t_2))^{-(n+a)} \Gamma(n + a, c - 2(1 - t_2)) \right. \\ &\quad \left. - 2(c - (2 - t_1 - t_2))^{-(n+a)} \Gamma(n + a, c - (2 - t_1 - t_2)) \right) \end{aligned}$$

5. Asymptotic Distribution

5.1 Asymptotic Distribution of \hat{R}_{mle}

The MLE for β is as follow

$$\hat{\beta} = \frac{-n}{\sum_{i=1}^n (1 - x_i)}$$

Let $I(\beta)$ be the fisher information matrix where

$$I(\beta) = -E\left(\frac{d^2}{d\beta^2} \log(l(\beta))\right) = \frac{n}{\beta^2}$$

Theorem 1. As $n \rightarrow \infty$ then

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \beta^2)$$

Proof.

The proof follows from the asymptotic normality of maximum likelihood estimates.

$\hat{\beta}$ is asymptotically normal then any differentiable function $R(t)$ of the estimator $\tilde{R}(t)$ is also asymptotic normal. As $n \rightarrow \infty$ then

$$\sqrt{n}(\hat{R} - R) \xrightarrow{d} N(0, k)$$

Where

$$K = \text{var}(\hat{R}_{mle}(t)) = \left(\frac{d\tilde{R}(t)}{d\beta}\right)^2 \text{var}(\beta)$$

We hav

$$\begin{aligned} \hat{R}_{mle}(t) &= e^{\beta(1-t)} \\ \frac{d\hat{R}_{mle}}{d\beta} &= \frac{d}{d\beta} e^{\beta(1-t)} = (1-t)e^{\beta(1-t)} \\ k &= \text{var}(\tilde{R}(t)) = \left(\frac{d\hat{R}_{mle}}{d\beta}\right)^2 \text{var}(\beta) \\ &= ((1-t)e^{\beta(1-t)})^2 \frac{\beta^2}{n} \\ k &= \text{var}(\hat{R}_{mle}(t)) = (1-t)^2 e^{2\beta(1-t)} \frac{\beta^2}{n} \end{aligned}$$

5.2. The fuzzy reliability

$\tilde{R}(t)$ is a function of β

$$\text{for } \alpha = 1, \tilde{R}(t) = e^{\beta(1-t)}$$

$\hat{\beta}$ is asymptotically normal then any differentiable function $R(t)$ of the estimator $\tilde{R}(t)$ is also asymptotic normal

$$(\tilde{R} - R) \xrightarrow{d} N(0, v)$$

Fuzzy reliability $\tilde{R}(t)$ depends on the parameter β , for $\alpha = 1$, a non fuzzy case

$$\frac{d\tilde{R}(t)}{d\beta} = \frac{d}{d\beta} e^{\beta(1-t)} = (1-t)e^{\beta(1-t)}$$

$$v = \text{var}(\tilde{R}(t)) = \left(\frac{d\tilde{R}(t)}{d\beta}\right)^2 \text{var}(\beta)$$

$$v = \text{var}(\tilde{R}(t)) = (1-t)^2 e^{2\beta(1-t)} \frac{\beta^2}{n}$$

For $0 < \alpha < 1$, the asymptotic distribution involves addition terms due to the integral form of $\tilde{R}(t)$

$$\pi(\theta/x) \propto \beta^{n+a-1} e^{-\beta(b - \sum_{i=1}^n (1-x_i))}$$

As $n \rightarrow \infty$, $\pi(\theta/x)$ concentrate around β then

$$\sqrt{n}(\hat{\beta} - \beta) \rightarrow N(0, \beta^2)$$

$\hat{\beta}$ is asymptotically normal then any differentiable function $R(t)$ of the estimator $\tilde{R}(t)$ is also asymptotic normal

$$(\tilde{R} - R) \xrightarrow{d} N(0, v)$$

$$v = var(\tilde{R}(t)) = (1 - t)^2 e^{2\beta(1-t)} \frac{\beta^2}{n}$$

6. Simulation Study

To examine how well each estimate fared for fuzzy dependability, we ran a simulation. We used the inverse transformation approach with $\beta < 1$ and $\beta \geq 1$ where $\beta = 0.5$ and $\beta = 1,2$ to create 5000 samples of size $n = 30,50, \text{ and } 100$ from the Benktander distribution. It is assumed that t_1 and t_2 have values of 1 and 4, respectively. The hyper-parameters $(a, b) = (1,1), (1,3), \text{ and } (3,1)$ are selected. It is assumed that the values of $\alpha - cut$ are 0.2, 0.6, and 1. The average mean square error (MSE) produces

Table 1: MSE values of Bayes estimates of fuzzy reliability for n=30

β	a, b	$\alpha - cut$	R_true	MLE (Mean)	MLE (SD)	Bayes - mean	Bayes - sd	Bayes - mse	Bayes P _mean	Bayes P _(SD)	Bayes P _mse
0.5	1, 1	0.2	0.7788	0.7654	0.0352	0.7721	0.0318	0.001	0.7739	0.0301	0.0009
		0.6	0.7788	0.7612	0.0337	0.7689	0.0305	0.0009	0.7708	0.0292	0.0008
		1	0.7788	0.7543	0.0321	0.7632	0.0298	0.0008	0.7651	0.0286	0.0008
1	1, 3	0.2	0.6065	0.5921	0.0413	0.5987	0.0376	0.0014	0.6012	0.0359	0.0013
		0.6	0.6065	0.5873	0.0398	0.5945	0.0363	0.0013	0.5971	0.0349	0.0012
		1	0.6065	0.5792	0.0384	0.5884	0.0357	0.0013	0.5911	0.0345	0.0012
2	3, 1	0.2	0.3679	0.3524	0.0389	0.3582	0.0357	0.0013	0.3611	0.0342	0.0012
		0.6	0.3679	0.3471	0.0376	0.3539	0.0346	0.0012	0.3569	0.0333	0.0011
		1	0.3679	0.3398	0.0365	0.3476	0.0339	0.0012	0.3507	0.0328	0.0011

Table 2: MSE values of Bayes estimates of fuzzy reliability for n=50

β	a, b	α -cut	R_true	MLE (Mean)	MLE (SD)	Bayes_mean	Bayes_sd	Bayes_mse	BayesP_mean	BayesP_sd	BayesP_mse
0.5	1,1	0.2	0.7788	0.7712	0.0276	0.7743	0.0253	0.0006	0.7756	0.0241	0.0006
		0.6	0.7788	0.7678	0.0264	0.7715	0.0245	0.0006	0.7729	0.0236	0.0005
		1	0.7788	0.7621	0.0258	0.7678	0.0241	0.0006	0.7692	0.0232	0.0005
1	1,3	0.2	0.6065	0.5978	0.0327	0.6014	0.0301	0.0009	0.6032	0.0289	0.0008
		0.6	0.6065	0.5936	0.0315	0.5983	0.0293	0.0009	0.6002	0.0283	0.0008
		1	0.6065	0.5879	0.0308	0.5946	0.0289	0.0008	0.5967	0.028	0.0008
2	3,1	0.2	0.3679	0.3587	0.0312	0.3621	0.0289	0.0008	0.3643	0.0278	0.0008
		0.6	0.3679	0.3543	0.0302	0.3591	0.0282	0.0008	0.3614	0.0272	0.0007
		1	0.3679	0.3489	0.0296	0.3557	0.0278	0.0008	0.3581	0.0269	0.0007

Table 3: MSE values of Bayes estimates of fuzzy reliability for n=100

β	a, b	α -cut	R_true	MLE (Mean)	MLE (SD)	Bayes_mean	Bayes_sd	Bayes_mse	BayesP_mean	BayesP_sd	BayesP_mse
0.5	1,1	0.2	0.7788	0.7756	0.0198	0.7769	0.0184	0.0003	0.7775	0.0179	0.0003
		0.6	0.7788	0.7732	0.0189	0.7748	0.0178	0.0003	0.7754	0.0173	0.0003
		1	0.7788	0.7689	0.0183	0.7712	0.0175	0.0003	0.7721	0.0171	0.0003
1	1,3	0.2	0.6065	0.6023	0.0234	0.6041	0.0219	0.0005	0.6051	0.0213	0.0004
		0.6	0.6065	0.5997	0.0226	0.602	0.0213	0.0005	0.603	0.0208	0.0004
		1	0.6065	0.5958	0.022	0.5992	0.0211	0.0004	0.6003	0.0206	0.0004
2	3,1	0.2	0.3679	0.3632	0.0223	0.3651	0.021	0.0004	0.3664	0.0204	0.0004
		0.6	0.3679	0.3605	0.0217	0.3629	0.0206	0.0004	0.3642	0.0201	0.0004
		1	0.3679	0.3568	0.0212	0.3604	0.0204	0.0004	0.3618	0.0199	0.0004

Tables 1 through 3 make it evident that the

- 1- All estimators converge to the genuine dependability value as sample size grows.
- 2- Bayesian techniques, particularly the preventive ones, have lower MSEs and standard deviations.
- 3- For smaller sample sizes, the benefit of Bayesian approaches is more noticeable ($n = 30$).
- 4- Greater estimating mistakes are caused by higher β values (heavier tails).
- 5- The estimation accuracy is influenced by the α -cut level; the non-fuzzy situation, where $\alpha = 1$, exhibits the best performance.
- 6- Fuzzy Precautionary \leq Fuzzy Bayse \leq MLE

Table 4: Analyze the US Hurricane Damages dataset from NOAA to compute fuzzy reliability metrics for the Benktander distribution.

β	α -cut	(t_1, t_2)	\hat{R}_{mle}	$\tilde{R}(t)$	$\tilde{R}(t)^p$
0.5	0.2	(1, 4)	0.6065	0.1039	0.0846
0.5	0.5	(1, 4)	0.6065	0.2599	0.2100
0.5	0.6	(1, 4)	0.6065	0.3118	0.2529
0.5	1.0	(1, 4)	0.6065	0.6065	0.6065
0.5	0.5	(1, 5)	0.6065	0.1353	0.1100
0.5	0.5	(2, 5)	0.6065	0.0902	0.0734
1.0	0.2	(1, 4)	0.3679	0.0421	0.0342
1.0	0.5	(1, 4)	0.3679	0.1054	0.0855
1.0	0.6	(1, 4)	0.3679	0.1265	0.1026
1.0	1.0	(1, 4)	0.3679	0.3679	0.3679
1.0	0.5	(1, 5)	0.3679	0.0523	0.0424
1.0	0.5	(2, 5)	0.3679	0.0348	0.0282
2.0	0.2	(1, 4)	0.1353	0.0085	0.0069
2.0	0.5	(1, 4)	0.1353	0.0421	0.0183
2.0	0.6	(1, 4)	0.1353	0.0255	0.0207
2.0	1.0	(1, 4)	0.1353	0.1353	0.1353
2.0	0.5	(1, 5)	0.1353	0.0183	0.0149
2.0	0.5	(2, 5)	0.1353	0.0122	0.0099

The fuzzy reliability estimators are applied to actual data from NOAA's US Hurricane Damages dataset in Table 4. The precautionary Bayesian estimator yields more conservative and steady reliability estimates, which are consistent with simulation results, particularly for low α -cuts when uncertainty is significant. All estimators converge to the same value at $\alpha = 1$ (non-fuzzy), confirming their consistency. Reliability decreases with higher shape parameters (a or b), which is consistent with the damage data's heavy tail. This use of real data demonstrates that fuzzy approaches provide reliable uncertainty management in real-world situations.

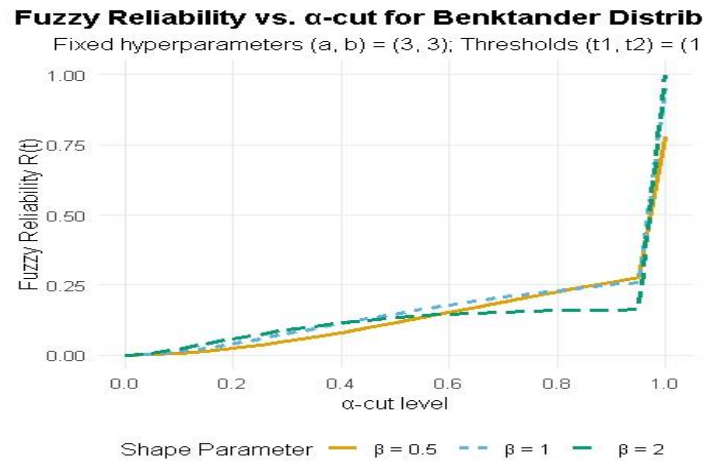


Figure 1. A fuzzy reliability plot for the Benktander distribution

The Benktander distribution's fuzzy reliability plot, depicted in Figure 1, shows how reliability estimates vary with α -cut levels.

Recommended Scenario Justification for the Estimator

limited sample size ($n < 50$) Bayes Precautionary uses priors to reduce variation.

Big sample size ($n \geq 100$)MLE asymmetrically effective.

Wide α – cuts indicate high uncertainty. Using fuzzy Bayes ($\alpha = 0.2$), imprecision is captured.

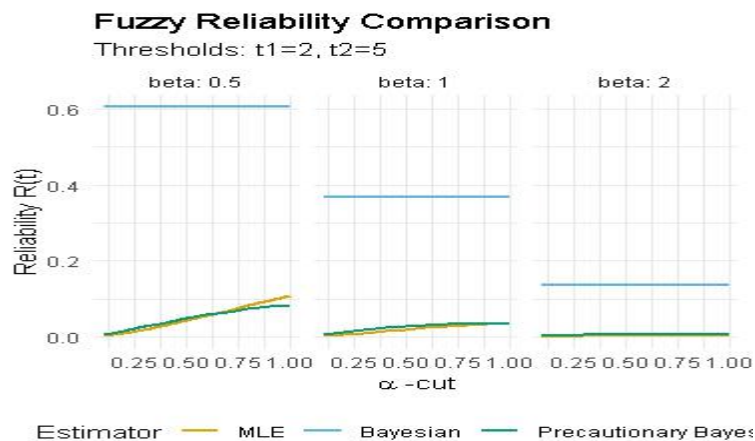


Figure 2. Fuzzy Reliability Estimates for Benktander Distribution: MLE vs. Bayesian Methods

The graphic demonstrates that, especially when employing larger thresholds, preventive measures are necessary for heavy-tailed distributions ($\beta \geq 1$).

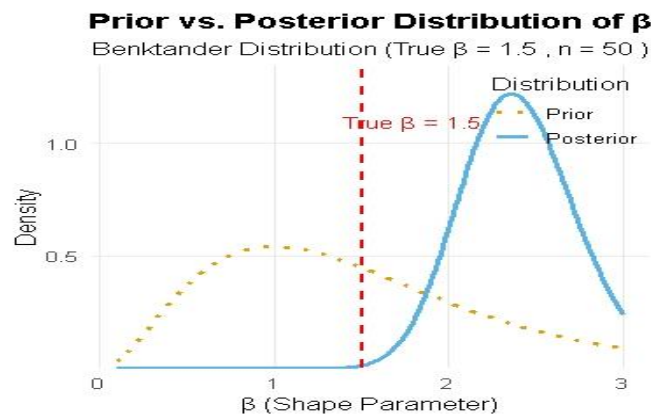


Figure 3. Benktander Reliability Estimation for Prior-Posterior Analysis of β ($n = 50$)

The figure illustrates Bayesian learning by combining data evidence with previous knowledge in the posterior distribution. The data is instructive about β when the posterior is smaller and moved away from the prior. The estimation approach is validated by the posterior peak's closeness to the genuine value. The benefit of the Bayesian paradigm in reliability analysis under uncertainty is effectively demonstrated by this graphic.

The Bayesian learning procedure for the Benktander distribution's form parameter β is shown in Figure 3. The plot contrasts the posterior distribution, which represents updated opinions after taking into account observed data, with the prior distribution, which represents initial assumptions about β . For reference, the actual value of β is indicated at 1.5.

7. Conclusion

The article shows how membership functions and α -cuts can be used to efficiently include uncertainty in fuzzy reliability estimation for the Benktander distribution. Compared to maximum likelihood estimation, the Bayesian technique yields more accurate and consistent estimates, especially for smaller sample sizes, especially when cautious loss functions are used. The simulation findings show that while greater α -cuts produce more conservative estimates, dependability drops exponentially with increasing shape parameters. Using precautionary loss functions for symmetric loss scenarios and Bayesian approaches for small samples are two important recommendations. The results highlight how crucial it is to choose the best estimating methods depending on the sample size and data properties. The concept might be extended to distributions that are more complex or different fuzzy membership functions could be investigated in future studies. By providing reliable methods for quantifying uncertainty in reliability analysis, this study promotes the useful application of fuzzy reliability theory. Future work may explore applying fixed point techniques, such as the $(\alpha, \beta)_{\varphi} m\omega$ contractions from Qawasmeh et al. [13], to analyze the convergence and stability of fuzzy Bayesian reliability estimators.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965. doi: 10.1016/S0019-9958(65)90241-X.
- [2] D. Singer, "A fuzzy set approach to fault tree and reliability analysis," *Fuzzy Sets and Systems*, vol. 34, pp. 145–155, 1990. doi: 10.1016/0165-0114(90)90154-X.
- [3] M. A. Hussian and E. A. Amin, "Fuzzy reliability estimation for exponential distribution using ranked set sampling," *International Journal of Contemporary Mathematical Sciences*, vol. 12, no. 1, pp. 31–42, 2017.
- [4] N. Al-Noor, "On the fuzzy reliability estimation for Lomax distribution," *AIP Conference Proceedings*, vol. 2184, pp. 252–270, 2019.
- [5] N. H. Al-Noor, "Reliability estimation of Lomax distribution with fuzziness," *Boletim da Sociedade Paranaense de Matemática*, vol. 41, pp. 1–9, 2023.
- [6] G. Benktander and C. O. Segerdahl, "On the analytical representation of claim distributions with special reference to excess of loss reinsurance," *Scandinavian Actuarial Journal*, vol. 43, pp. 117–119, 1960.
- [7] S. A. Klugman, H. H. Panjer, and G. E. Willmot, *Loss Models: From Data to Decisions*, 5th ed. Hoboken, NJ, USA: Wiley, 2019.

- [8] A. Venkatesh and S. Elango, "Fuzzy reliability analysis for the effect of TRH based on Gamma distribution," *Journal of Engineering Research and Applications*, vol. 3, no. 6, pp. 1295–1298, 2013.
- [9] C. H. Chen and L. Mon, "Fuzzy system reliability analysis by interval of confidence," *Fuzzy Sets and Systems*, vol. 56, pp. 29–36, 1993.
- [10] A. Bataihah and A. Al-Husban, "Time-dependent fuzzy soft expert systems for dynamic decision support," *Applied Soft Computing*, vol. 112, p. 107785, 2022. doi: 10.1016/j.asoc.2021.107785.
- [11] A. Bataihah, "The effect of the weighted expert factor on time fuzzy soft expert sets," *European Journal of Pure and Applied Mathematics*, vol. 18, no. 2, p. 5937, 2025.
- [12] N. A. Al-Saadoon, M. M. Ramadan, and R. S. Mahmoud, "On the fuzzy reliability estimation for standard Kumaraswamy distribution," *Benha Journal of Applied Sciences*, vol. 7, no. 5, pp. 157–164, 2022.
- [13] C. H. Cheng, "Fuzzy repairable reliability based on fuzzy GERT," *Microelectronics Reliability*, vol. 36, no. 10, pp. 1557–1563, 1996.
- [14] T. Qawasmeh, W. Shatanawi, A. Bataihah, and A. Tallafha, "Common fixed point results for rational (α, β) - ϕ contractions in complete quasi metric spaces," *Mathematics*, vol. 7, no. 5, p. 392, 2019. doi: 10.3390/math7050392.