Neutrosophic Alpha Logarithm Exponential Distribution

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Abstract

The probability distribution holds considerable importance within the realm of probability theory, a concept that permeates nearly all scientific disciplines. Nevertheless, the principal aim of the present research endeavor is to introduce a novel distribution referred to as the neutrosophic Alpha logarithm Exponential, abbreviated as NALE. Various mathematical attributes that elucidate life survival and associated characteristics such as hazard rates, moment's functions, moment-generating functions, and additional metrics including mean and variance are also examined. Two methods were used to estimate the parameters; the Monte Carlo simulation has been employed to evaluate the efficacy of the NALE distribution estimation and to compare the two estimation methods. Therefore, the outcomes from the simulation executed in this research imply that a satisfactory level of precision in estimation is feasible only when the sample size is notably large. The real data has been utilized to demonstrate the specific manner in which the proposed NALE distribution has been recommended for application. Based on the analyses presented in the preceding sections, it can be inferred that the NALE distribution possesses a broad applicability since it is capable of accommodating of neutrosophic data; it does not differentiate between certainty, probabilities of uncertainty, ambiguities, or imprecisions.

Keywords: Alpha logarithm Exponential Distribution; Neutrosophic Distributions; Survival function; Hazard function; Neutrosophic statistics; Parameter Estimation

1. Introduction

A multitude of scholars has commenced the development of diverse investigations predicated on Neutrosophic statistics in the recent period. Smarandache [1] pioneered the foundational research pertaining to neutrosophic statistics. The foundation of neutrosophic statistics rests upon the premise that frequently within datasets, there exists information that is inherently vague and cannot be quantified through conventional methodologies, thus rendering it unsuitable for effective processing within the confines of traditional statistical approaches. Neutrosophic statistics serves as a methodological framework to address and conduct a more thorough analysis of such complex data. The application of fuzzy logic has been broadened, facilitating the representation of uncertainty, ambiguity, and contradictions inherent in the data [2-5].

The theory of neutrosophic probability is essential and possesses numerous applications. This domain of inquiry has not garnered substantial scholarly attention. In recent years, certain researchers have concentrated predominantly on the Neutrosophic statistical methodology and its applications across diverse disciplines. Patro and Smarandache [6] introduced the neutrosophic distributions, along with additional its properties. In their research, Alhabib et al. [3] investigated diverse neutrosophic distribution by extending particular classical distribution, like the uniform, exponential, and Poisson distributions, into the neutrosophic structure. A new neutrosophic model was crafted by Nayana et al. [7] employing the transformation of DUS-Weibull, in contrast, Alhasan and Smarandache [8] delved into the distribution of neutrosophic Weibull. Zeina and Hatip [9] advanced the concept of random variables in neutrosophic statistics. Sherwani [10] analyzed the distribution of neutrosophic beta. In conventional mathematics, clarity is regarded as the paramount requisite; however, in real-world scenarios, one often encounters ambiguous data. To address these challenges, mathematical frameworks predicated on

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uncertainty must be employed. Uncertainty modeling has become a focal interest for numerous scientists and engineers, as it aids in articulating and elucidating the valuable insights concealed within uncertain data. An escalating amount of scholarly work has turned its attention in the last few years to various dimensions of neutrosophic statistics, covering correlation, regression analysis, testing methods, probability distributions, and more.

Neutrosophic statistics comprise three distinct metrics that, in various dimensions, encapsulate the subtleties of the propositions under investigation: truth membership, indeterminacy membership, and falsity membership. Collectively, these metrics elucidate the extent of truth, ambiguity, or falsehood that correlates with a specific hypothesis or observation. Such extents are represented in a manner analogous to a fuzzy set through the application of membership functions.

The examination of survival statistics constitutes a fundamental component of neutrosophic information that warrants thorough investigation. Survival analysis, which many people consider event-time analysis or time-to-event observation, is concerned with identifying the progression leading to a particular event that captures attention. This methodology is widely employed across various disciplines, including social sciences, engineering, medical research, and other sectors where temporal outcomes are relevant. In numerous instances, when engaging in research where the temporal sequence is ambiguous or where participants may not receive uniform follow-up assessments, survival analysis demonstrates substantial utility. Numerous statistical distributions are extensively employed in survival analysis for the examination of time-to-event data. This indicates that the properties of the data, alongside the assumptions regarding the fundamental survival process, influence the selection of the distribution. Different kinds of distributions might be utilized depending on the unique traits of the data put forth by the researchers, along with the relevant hypotheses tied to the study. The exponential distribution and its extensions demonstrate significant relevance and application within these domains [11], [12].

Applications of the Alpha Logarithm Exponential distribution (ALE) [13] are prevalent across various fields, notably within the context of survival analysis. In the present study, we have broadened the scope of the Alpha logarithm Exponential distribution to encompass neutrosophical data represented in interval form, characterized by a degree of indeterminacy. A multitude of attributes is examined under the newly introduced distribution, and their practical implications are elucidated through both simulated datasets and real-world applications.

2. The Data in the Neutrosophical Approach

Neutrosophic statistics represents an extension of traditional statistical methodologies. In classical statistics, we utilize precise or defined values; conversely, in neutrosophic statistics, the random sample is derived from a neutrosophic population characterized by an environment of uncertainty. When delving into neutrosophic statistics, one might describe the information as indistinct, not exact, hard to interpret, questionable, partially available, or even wholly mysterious. Neutrosophic numbers possess a standardized format that is grounded in classical statistical principles, as delineated below.

$$X_N = E + I$$

Data may be segmented into two unique components, identified as E and I, where E illustrates the accurate or validated information, while I encompasses the unclear, ambiguous, or uncertain aspects of the information. This classification, the neutrosophic random variable, is analogous to the notation $X_N \in [X_L, X_U]$.

3. Neutrosophic Alpha logarithm Exponential Distribution

Let X_N denote a neutrosophic continuous random variable that adheres to a neutrosophic Alpha logarithm Exponential (NALE) distribution; consequently, its neutrosophic cumulative distribution function (CDF) and neutrosophic probability density function (pdf) are represented by Equation (1) and Equation (2), respectively.

$$F_{NALE}(x_N, \lambda_N, \alpha_N) = \frac{log\left(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1\right)}{log(\alpha_N + 1)}, \qquad x_N > 0;\ \alpha_N, \lambda_N > 0. \tag{1}$$

$$f_{NALE}(x_N, \lambda_N, \alpha_N) = \frac{\alpha_N e^{\frac{-x_N}{\lambda_N}}}{\lambda_N log(\alpha_N + 1)\left(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1\right)}. \tag{2}$$

The NALE distribution demonstrates superior proficiency in modeling intricate datasets compared to conventional distribution models. The NEPL distribution enhances traditional models, thereby offering sufficient capabilities to encapsulate diverse data patterns observed in practical applications. Figure 1 and Figure 1 display the C.D.F and the p.d.f of the NALE distribution corresponding to different parameter values, respectively.

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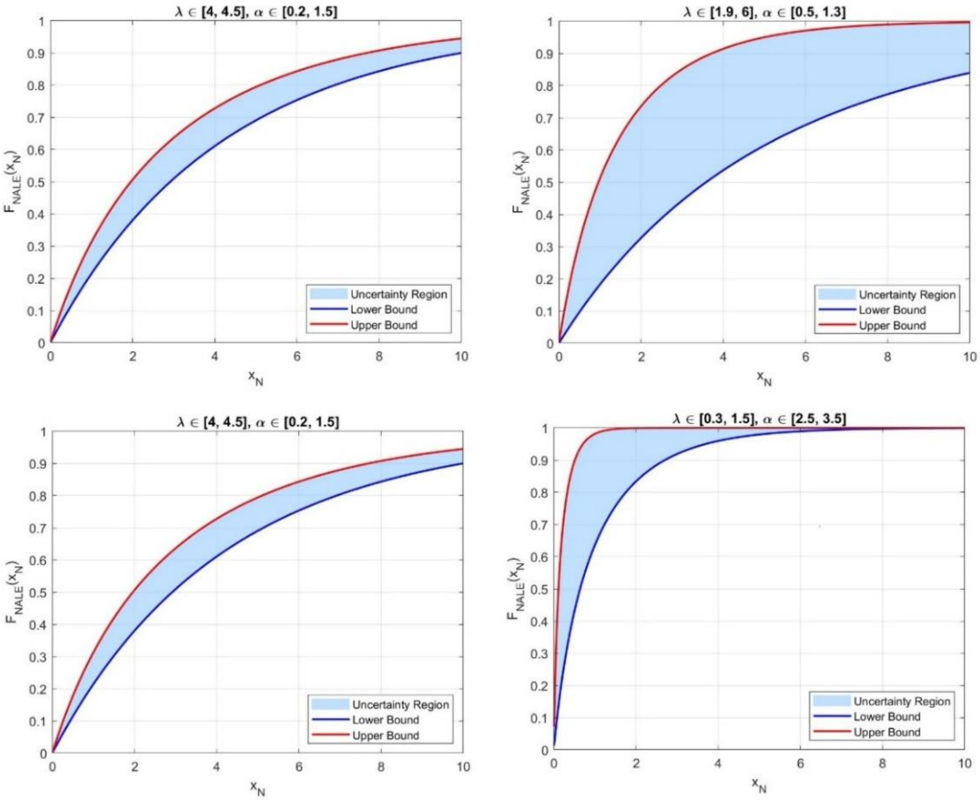

Figure 1. The C.D.F of the NALE distribution to different parameter values

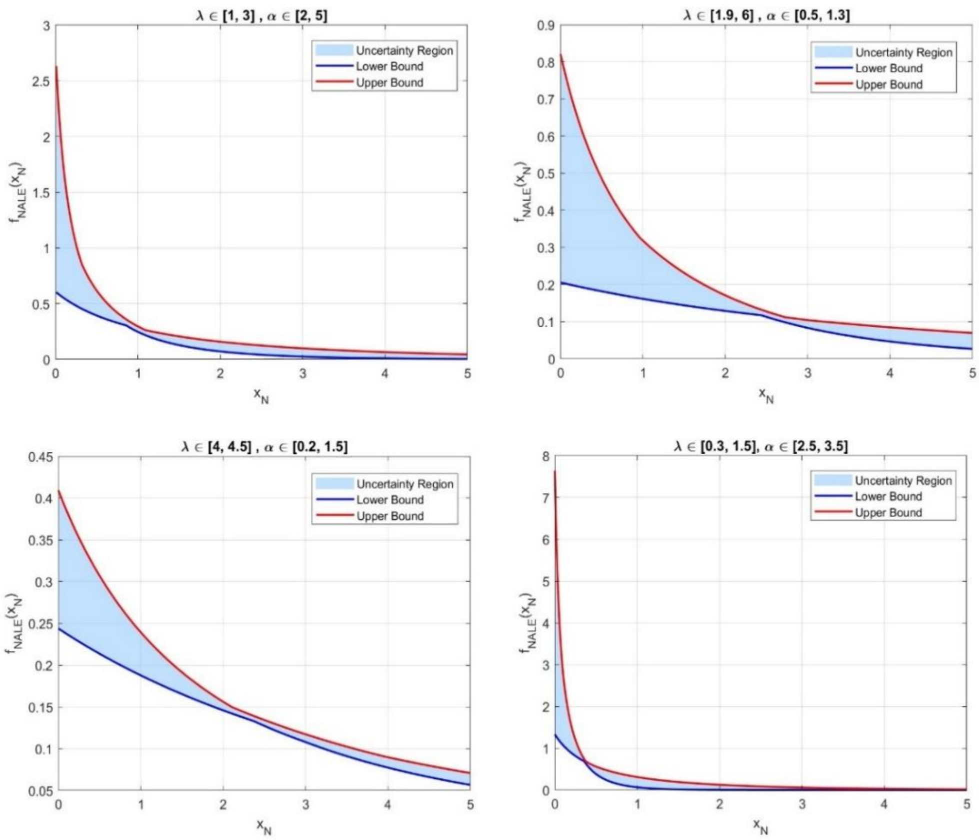

Figure 2. The p.d.f of the NALE distribution to different parameter values

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4. The Statistical Functions of NALE distribution

We introduce the reliability (survival) function $\bar{F}_{NALE}(x_N, \lambda_N, \alpha_N)$, the hazard rate function $h_{NALE}(x_N, \lambda_N, \alpha_N)$, the reversed hazard function $r_{NALE}(x_N, \lambda_N, \alpha_N)$ and cumulative hazard function $H_{NALE}(x_N, \lambda_N, \alpha_N)$ of NALE distribution.

A. Survival function of NALE distribution

The survival function $\bar{F}_{NALE}(x_N, \lambda_N, \alpha_N)$ of $X_N \sim NALE(\alpha_N, \lambda_N)$ is defined as follows

$$\bar{F}_{NALE}(x_N, \lambda_N, \alpha_N) = 1 - F_{NALE}(x_N, \lambda_N, \alpha_N) = 1 - \frac{log(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)}{log(\alpha_N + 1)}. \quad (3)$$

B. Hazard function of NALE distribution

The hazard function $h_{NALE}(x_N, \lambda_N, \alpha_N)$ of the NALE distributing is defined as follows

$$h_{NALE}(x_N, \lambda_N, \alpha_N) = \frac{f_{NALE}(x_N, \lambda_N, \alpha_N)}{\bar{F}_{NALE}(x_N, \lambda_N, \alpha_N)}$$

$$= \frac{\alpha_N e^{\frac{-x_N}{\lambda_N}}}{\lambda_N (log(\alpha_N + 1) - 1)(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)\, log(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)}. \quad (4)$$

C. The reverse hazard

The reverse hazard function $r_{NALE}(x_N, \lambda_N, \alpha_N)$ of NALE distribution is defined follows

$$r_{NALE}(x_N, \lambda_N, \alpha_N) = \frac{f_{NALE}(x_N, \lambda_N, \alpha_N)}{F_{NALE}(x_N, \lambda_N, \alpha_N)} = \frac{\alpha_N e^{\frac{-x_N}{\lambda_N}}}{\lambda_N log(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)}. \quad (5)$$

D. The cumulative hazard function

The cumulative hazard $H_{NALE}(x_N, \lambda_N, \alpha_N)$ of NALE distribution is defined as

$$H_{NALE}(x_N, \lambda_N, \alpha_N) = -log[1 - F_{NALE}(x_N, \lambda_N, \alpha_N)] = -log\left[1 - \frac{log(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)}{log(\alpha_N + 1)}\right]. (6)$$

E. The Moments

Let x_N be neutrosophic variable of the NALE distribution then neutrosophic *r-th* moment (at the origin point) is as following.

$$E_{NALE}(x_N{}^r) = \frac{(r!)}{\lambda_N\, log(\alpha_N + 1)} \sum_{i=1}^{\infty} \frac{\left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-\left(\frac{1}{\lambda_N}\right)^{(r+1)}}}{i^{r+1}}. \quad (7)$$

The proof of formula in equation (7) is almost similar to proving the moments for the Alpha Logarithm Exponential distribution [13], but with the neutrosophic parameters, as following

$$E_{NALE}(x_N{}^r) = \int_0^{\infty} x_N{}^r\, f_{NALE}(x_N, \lambda_N, \alpha_N)\, dx_N, \qquad r = 1,2,3, \ldots$$

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$$= \int_0^\infty {x_N}^r \frac{\alpha_N e^{\frac{-x_N}{\lambda_N}}}{\lambda_N log(\alpha_N + 1)(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)} \, dx_N$$

$$= \frac{\alpha_N}{\lambda_N log(\alpha_N + 1)} \int_0^\infty \frac{{x_N}^r \, e^{\frac{-x_N}{\lambda_N}}}{(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)} \, dx_N$$

$$= \frac{\alpha_N}{\lambda_N log(\alpha_N + 1)} \int_0^\infty \frac{{x_N}^r}{(\alpha_N e^{\frac{x_N}{\lambda_N}} - \alpha_N + e^{\frac{x_N}{\lambda_N}})} \, dx_N$$

$$= \frac{\alpha_N}{\lambda_N log(\alpha_N + 1)(\alpha_N + 1)} \int_0^\infty \frac{{x_N}^{(r+1)-1}}{e^{\frac{x_N}{\lambda_N}} - \frac{\alpha_N}{(\alpha_N + 1)}} \, dx_N. \tag{8}$$

Since $\frac{\alpha_N}{(\alpha_N+1)} < 0$. The integral in Equation (8) can be solved using the formula BI(83)(5)(p.356) in the reference [14], thus the *r-th* moment is:

$$E_{NALE}({x_N}^r) = \frac{(r!)}{\lambda_N \, log(\alpha_N + 1)} \sum_{i=1}^{\infty} \frac{\left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-(\frac{1}{\lambda_N})^{(r+1)}}}{i^{r+1}}.$$

The first neutrosophic moment $E_{NALE}(x_N)$ and second neutrosophic moment $E_{NALE}({x_N}^2)$ of the NALE distribution are given by

$$E_{NALE}(x_N) = \frac{1}{\lambda_N \, log(\alpha_N + 1)} \sum_{i=1}^{\infty} \frac{\left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-(\frac{1}{\lambda_N})^2}}{i^2}. \tag{9}$$

$$E_{NALE}({x_N}^2) = \frac{2}{\lambda_N \, log(\alpha_N + 1)} \sum_{i=1}^{\infty} \frac{\left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-(\frac{1}{\lambda_N})^3}}{i^3}. \tag{10}$$

Now, from equations (9) and (10), we find the neutrosophic variance as following

$$V_{NALE}(x_N) = \frac{2}{\lambda_N \, log(\alpha_N + 1)} \sum_{i=1}^{\infty} \frac{\left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-(\frac{1}{\lambda_N})^3}}{i^3} - \frac{1}{{\lambda_N}^2 (log(\alpha_N + 1))^2} \left(\sum_{i=1}^{\infty} \frac{\left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-(\frac{1}{\lambda_N})^2}}{i^2} \right)^2. \tag{11}$$

F. The moment generating function

The moment generating $M_{x_N}(t)$ function of the NALE distribution is given by:

$$M_{x_N}(t) = \frac{1}{\lambda_N \, log(\alpha_N + 1)} \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \frac{t^j \left(\frac{\alpha_N}{1 + \alpha_N}\right)^{i+1-(\frac{1}{\lambda_N})^{(j+1)}}}{i^{j+1}}. \tag{12}$$

The proof of this formula can be found by defining the moment generating $M_{x_N}(t)$ function.

$$M_{x_N}(t) = E_{NALE}(e^{tx_N}) = \int_0^\infty e^{tx_N} \, f_{NALE}(x_N, \lambda_N, \alpha_N) \, dx_N$$

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$$= \int_{0}^{\infty} e^{tx_N} \frac{\alpha_N e^{\frac{-x_N}{\lambda_N}}}{\lambda_N log(\alpha_N + 1)(\alpha_N - \alpha_N e^{\frac{-x_N}{\lambda_N}} + 1)} dx_N. \qquad (13)$$

By the Maclaurin series the e^{tx_N} function.

$$e^{tx_N} = \sum_{j=0}^{\infty} \frac{t^j x_N{}^j}{j!}. \qquad (14)$$

By substituting Equation (14) into Equation (13), then:

$$M_{x_N}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_{0}^{\infty} x_N{}^j \; f_{NALE}(x_N, \lambda_N, \alpha_N) \; dx_N,$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} E_{NALE}(x_N{}^j). \qquad (15)$$

Where $E_{NALE}(x_N{}^j)$ is the *j-th* moment in Equation (7), and by substituting it into Equation (15), we get $M_{x_N}(t)$ Equation (12).

5. Parameter Estimation of NALE Distribution

Two methods for estimating the NALE distribution parameters are described: (1) the maximum likelihood Method (MLE), (2) Moment method (ME).

A. The maximum likelihood method (MLE)

Now we find the maximum likelihood estimator of the two neutrosophic parameters, if $x_{N_1}, x_{N_2}, \dots, x_{N_n}$ denote a neutrosophic random from the NALE distribution, then the Likelihood function is provided by.

$$f_{NALE}(x_N, \lambda_N, \alpha_N) = \frac{\alpha_N{}^n e^{\frac{-1}{\lambda_N}\sum_{i=1}^{n} x_{Ni}}}{\lambda_N{}^n (log(\alpha_N + 1))^n \prod_{i=1}^{n}(\alpha_N - \alpha_N e^{\frac{-x_{Ni}}{\lambda_N}} + 1)}.$$

$$L(\lambda_N, \alpha_N; x_{Ni}) = \prod_{i=1}^{n} \frac{\alpha_N e^{\frac{-x_{Ni}}{\lambda_N}}}{\lambda_N log(\alpha_N + 1)(\alpha_N - \alpha_N e^{\frac{-x_{Ni}}{\lambda_N}} + 1)}$$

$$= \frac{\alpha_N{}^n e^{\frac{-1}{\lambda_N}\sum_{i=1}^{n} x_{Ni}}}{\lambda_N{}^n (log(\alpha_N + 1))^n \prod_{i=1}^{n}(\alpha_N - \alpha_N e^{\frac{-x_{Ni}}{\lambda_N}} + 1)}. \qquad (16)$$

The national log-likelihood function is

$$log\, L(\lambda_N, \alpha_N; x_{Ni}) = \text{n}\, log\, \alpha_N + n\, log(log(\alpha_N + 1)) - \text{n}\, log\, \lambda_N - \frac{1}{\lambda_N}\sum_{i=1}^{n} x_{Ni}$$

$$-\sum_{i=1}^{n} log(\alpha_N - \alpha_N e^{\frac{-x_{Ni}}{\lambda_N}} + 1). \qquad (17)$$

by the derivative of $log\, L(\lambda_N, \alpha_N; x_{Ni})$ w.r.t. the neutrosophic parameters λ_N and also α_N, as following.

$$\frac{\partial\, log\, L(\lambda_N, \alpha_N; x_{Ni})}{\partial \lambda_N} = \frac{-n}{\lambda_N} + \frac{1}{\lambda_N{}^2}\sum_{i=1}^{n} x_{Ni} + \frac{\alpha_N}{\lambda_N{}^2}\sum_{i=1}^{n} \frac{x_{Ni} e^{\frac{-x_{Ni}}{\lambda_N}}}{(\alpha_N - \alpha_N e^{\frac{-x_{Ni}}{\lambda_N}} + 1)}. \qquad (18)$$

By equal this to zero, then we have the following equation

$$-\hat{\lambda}_N + \frac{1}{n}\sum_{i=1}^{n} x_{Ni} - \frac{\hat{\alpha}_N}{n}\sum_{i=1}^{n} \frac{x_{Ni} e^{\frac{-x_{Ni}}{\lambda_N}}}{(\hat{\alpha}_N - \hat{\alpha}_N e^{\frac{-x_{Ni}}{\hat{\lambda}_N}} + 1)} = 0. \qquad (19)$$

And,

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$$\frac{\partial \log L(\lambda_N,\alpha_N;x_{Ni})}{\partial \alpha_N}=\frac{n}{\alpha_N}-\frac{n}{(\alpha_N+1)\log(\alpha_N+1)}-\sum_{i=1}^{n}\frac{1-e^{\frac{-x_{Ni}}{\lambda_N}}}{(\alpha_N-\alpha_N e^{\frac{-x_{Ni}}{\lambda_N}}+1)}. \quad (20)$$

Also, by equal this to zero, then we have the following equation

$$\frac{n}{\hat{\alpha}_N}-\frac{n}{(\hat{\alpha}_N+1)\log(\hat{\alpha}_N+1)}-\sum_{i=1}^{n}\frac{1-e^{\frac{-x_{Ni}}{\hat{\lambda}_N}}}{(\hat{\alpha}_N-\hat{\alpha}_N e^{\frac{-x_{Ni}}{\hat{\lambda}_N}}+1)}=0\ . \quad (21)$$

By solving the equation (19) and the equation (21) using method of the Newton–Raphson, then we obtain the MLE of the parameter λ_N and the MLE of the parameter α_N .

B. The Moment method (ME)

The method of moments typically requires the computation of the first two population moments and their corresponding sample counterparts. By setting the theoretical mean and variance equal to the sample mean and variance, and to find the estimator of the two neutrosophic parameters by using this method, if $x_{N_1}, x_{N_2}, \dots, x_{N_n}$ denote a neutrosophic random sample taken from the NALE distribution, then a system of two equations is obtained, as following.

$$\frac{1}{\hat{\lambda}_N\log(\hat{\alpha}_N+1)}\sum_{i=1}^{\infty}\frac{\left(\frac{\hat{\alpha}_N}{1+\hat{\alpha}_N}\right)^{i+1-\left(\frac{1}{\hat{\lambda}_N}\right)^2}}{i^2}=\sum_{i=1}^{n}\frac{x_{Ni}}{n}. \quad (22)$$

$$\frac{2}{\hat{\lambda}_N\log(\hat{\alpha}_N+1)}\sum_{i=1}^{\infty}\frac{\left(\frac{\hat{\alpha}_N}{1+\hat{\alpha}_N}\right)^{i+1-\left(\frac{1}{\hat{\lambda}_N}\right)^3}}{i^3}=\sum_{i=1}^{n}\frac{{x_{Ni}}^2}{n}. \quad (23)$$

By solving the equation (22) and the equation (23) using method of the Newton–Raphson, then we obtain the moments estimator of the parameter λ_N and the moments estimator of the parameter α_N.

6. The simulation

To evaluate the efficacy of the proposed NALE distribution, a comprehensive simulation study has been conducted. This simulation methodology is based on Monte Carlo simulation by generating neutrosophic data by using the inverse of the CDF on the number intervals. MATLAB was used to implement this simulation. The results pertaining to the estimated parameters of the NALE distribution and their efficacy are articulated as the average neutrosophic biases (Avg. NBiases) for the neutrosophic parameters λ_N and α_N, as well as the average neutrosophic mean square errors (Avg. NMSEs) for the neutrosophic parameters λ_N and α_N, employing a rigorous Monte Carlo simulation analysis using different sample sizes (50, 100, 250, 500) for 1000 repetitions.

The Monte Carlo simulation results of the average neutrosophic Bias and the average neutrosophic MSE are summarized in the Tables 1-3. It is observed from these tables that Avg. NBiases and Avg. NMSEs exhibit a declining trend as the sample size increases, which aligns with anticipated outcomes.

Table 1: The simulation results $\lambda_N=[0.5,1.5], \alpha_N=[0.5,2]$

Size r.s.	Meth.	Avg.NBiases		Avg.NMSEs	
		$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$
50	MLE	[-0.8419,-0.2936]	[-1.0046, -0.9753]	[0.0923, 0.7472]	[1.4735, 6.3054]
	ME	[-0.8249,-0.2846]	[-0.9636, -0.2196]	[0.0868, 0.7307]	[2.5061, 36.216]

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100	MLE	[-0.8374,-0.2852]	[-1.1284,-1.1622]	[0.0837, 0.7205]	[1.3766, 2.9722]
	ME	[-0.8184,-0.2798]	[-1.0671,-0.7221]	[0.0812, 0.6947]	[1.3773, 5.4572]
250	MLE	[-0.8293,-0.2840]	[-1.1721,-1.2250]	[0.0815, 0.6959]	[1.4040, 2.0735]
	ME	[-0.8095,-0.2792]	[-1.1343,-0.9003]	[0.0790, 0.6671]	[1.3391, 2.3902]
500	MLE	[-0.8258,-0.2823]	[-1.2482,-1.1770]	[0.0801, 0.6860]	[1.3999, 1.8068]
	ME	[-0.8051,-0.2772]	[-1.1423,-0.9717]	[0.0774, 0.6539]	[1.3281, 1.5597]

Table 2: The simulation results $\lambda_N = [0.2,1.1], \alpha_N = [0.1,0.9]$

Size r.s.	Meth.	Avg.NBiases		Avg.NMSEs	
		$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$
50	MLE	[-0.6140,-0.1459]	[-0.6274,0.2745]	[0.0252, 0.3983]	[1.4707, 2.3985]
	ME	[-0.5989,-0.0959]	[-0.7456,-0.1065]	[0.0102, 0.3857]	[0.8879, 16.438]
100	MLE	[-0.6109,-0.1276]	[-0.7434,-0.0681]	[0.0194, 0.3842]	[1.2684,1.2999]
	ME	[-0.5947,-0.0947]	[-0.8187,-0.4775]	[0.0094, 0.3667]	[0.7510, 2.3337]
250	MLE	[-0.6037,-0.0988]	[-0.7781,-0.6854]	[0.0107, 0.3689]	[0.8536,0.8693]
	ME	[-0.5890,-0.0944]	[-0.8553,-0.5953]	[0.0091, 0.3526]	[0.7500, 0.8553]
500	MLE	[-0.5993,-0.0919]	[-0.8422,-0.7708]	[0.0086, 0.3614]	[0.7153,0.7588]
	ME	[-0.5850,-0.0940]	[-0.8635,-0.6137]	[0.0089, 0.3451]	[0.6179,0.7538]

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Table 3: The simulation results $\lambda_N = [2.2,3.1], \alpha_N = [2.6,3.4]$

Size r.s.	Meth.	Avg.NBiases		Avg.NMSEs	
		$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$
50	MLE	[-1.7657,-1.2358]	[-1.3287,-0.9543]	[1.6178, 3.3001]	[8.1411, 12.5591]
	ME	[-1.7525,-1.2329]	[-0.3215,-0.2551]	[1.6256, 3.2994]	[42.3104, 90.202]
100	MLE	[-1.7286,1.2170]	[-1.5130,-1.1200]	[1.5313, 3.0753]	[4.3367, 6.2815]
	ME	[-1.7075,-1.2071]	[-0.9073,-0.7666]	[1.5176, 3.0368]	[8.8151, 16.6129]
250	MLE	[-1.7033-1.1991]	[-1.5739,-1.1953]	[1.4562, 2.9383]	[2.4217, 3.8000]
	ME	[-1.6730,-1.1834]	[-1.6730,-0.9286]	[1.4252, 2.8502]	[3.2126, 4.8993]
500	MLE	[-1.7065,-1.1998]	[-1.6550,-1.2481]	[1.4487, 2.9297]	[2.0142, 3.3550]
	ME	[-1.6747,-1.1814]	[-1.3099,-1.0166]	[1.4081, 2.8301]	[1.9652, 3.2620]

To clarify the difference between the two estimation methods used in this study, we show the following:

In the first model ($\lambda_N = [0.5,1.5], \alpha_N = [0.5,2]$), according to the simulation results shown in Table 1, we note that the method of ME is better than the method of MLE for the the estimator of parameter λ_N, while for the estimator of parameter α_N, the method of MLE is better than the method of ME. The difference may not seem significant when changing the sample size for estimator of parameter λ_N. However, for estimator of parameter α_N, increasing the sample size makes the MLE method's preference decrease. At a sample size of 500, the advantage shifts to the ME method.

In the second model ($\lambda_N = [0.2,1.1], \alpha_N = [0.1,0.9]$) according to the simulation results shown in Table 2, we note that the method of ME is better than the method of MLE for the the estimator of parameter λ_N, also for the estimator of parameter α_N.

In the third model ($\lambda_N = [2.2,3.1], \alpha_N = [2.6,3.4]$), according to the simulation results shown in Table 3, we note that the method of ME is better than the method of MLE for the the estimator of parameter λ_N but with a smaller difference than in the first model, while for the estimator of parameter α_N, the method of MLE is better than the method of ME. The difference may not seem significant when changing the sample size for estimator of parameter λ_N. However, for estimator of parameter α_N, increasing the sample size makes the MLE method's preference decrease. At a sample size of 500, the advantage shifts to the ME method.

7. Application

A comprehensive examination of the NALE distribution model is conducted utilizing actual data within this section, for the neutrosophic normal (NN), gamma (NG), Weibull (NW), Rayleigh (NR), and neutrosophic exponential (NE) distributions. The AIC, BIC and S-K criteria were used in this comparison.

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To exemplify a real case, we have examined the approximate population density of several villages situated in rural areas of the United States. The data utilized in this analysis is derived from [15], which conducted an investigation concerning the neutrosophic W/S test predicated on the data that adheres to a neutrosophic distribution. This dataset encompasses the populations of the 17 villages across the United States along with their corresponding neutrosophic data, Which have been mentioned here for ease of reference: [4.13,4.14], [4.53,4.55], [4.69,4.70], [4.76,4.78], [4.77,4.79], [4.96,4.98], [4.97,4.99], [5.00,5.06], [5.04,5.06], [5.10,5.12], [5.25,5.27], [5.36,5.38], [5.94,5.96], [6.06,6.08], [6.19,6.21], [6.30,6.32], [7.73,7.98]. The findings presented in Table 4 further indicate that the NALE model demonstrates a superior capability in accommodating the data compared to the NR model and NE model.

Table 4: Goodness-of-fit statistics.

Model	LogL	AIC	BIC	KS
NN	[-21.1554,-21.9577]	[46.3107,47.9155]	[53.6436,55.2483]	[0.2007,0.2024]
NG	[-20.2481,-20.9136]	[44.4962,45.8272]	[51.8290,53.1600]	[0.1821,0.1816]
NW	[-23.0417,-23.9417]	[50.0834,51.8834]	[57.4162,59.2162]	[0.2097,0.2115]
NR	[-34.3024,-34.4533]	[72.6049,72.9067]	[79.9377,80.2395]	[0.4457,0.4439]
NE	[-45.4788,-45.5815]	[94.9577,95.1630]	[102.2905,102.4959]	[0.5386,0.5373]
NALE	[-35.4532,-35.9068]	[74.9064, 75.8137]	[76.5729, 77.4802]	[0.5584,0.5617]

The dataset was subjected to a comprehensive analysis employing the innovative NALE distribution. We computed the mean and variance via the implementation of moment functions or the moment-generating function; furthermore, by leveraging the functionalities embedded within the quantile function, we determined the Median, Kurtosis and Skewness, as illustrated in Table 5.

Table 5: Descriptive analyzes for the datasets

Mean	Median	Variance	Kurtosis	Skewness
[5.3400, 5.3747]	[5.0400, 5.0600]	[0.7494, 0.8220]	[4.4254, 5.0149]	[1.2506, 1.4093]

8. Conclusion

In this study, we explore an innovative version of the exponential distribution guided by neutrosophic statistical principles. A limited number of scholars have investigated probability distributions, in this study; we presented the neutrosophic Alpha Logarithm Exponential distribution with application in neutrosophic real statistics. The characteristics of the proposed Neutrosophic Alpha logarithm Exponential distribution are rigorously examined. The inherent properties of the distribution are analysed through a variety of neutrosophic parametric combinations. Parameter estimation is conducted utilizing both the maximum likelihood and moments methods. A simulation study is performed within a neutrosophic context. As anticipated, the average neutrosophic bias and the average

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neutrosophic mean squared error exhibit a decline as the sample size expands. To sum up, the real-life implementation of the suggested NALE distribution is evidenced through the scrutiny of actual data collections. A comparative analysis with alternative distributions is also executed based on real empirical data sets. Based on real data analysis, it is concluded that the NALE distribution demonstrates superior performance in comparison to some distributions that have been used for comparison.

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