A New Operator via Regular Open Sets in a New Topological Structure

Amani Rawshdeh[1,*], **Ahmad Al-Omari**[2]
[1]Department of Mathematics, Al-Balqa Applied University, Alsalt, Jordan
[2]Department of Mathematics, Faculty of Sciences, Al al-Bayt University, Mafraq, Jordan
Emails: amanirawshdeh@bau.edu.jo; omarimutah1@yahoo.com

Abstract

In this paper, we will use the family of regular open sets in a topological space (Z, τ) to define an operator $\Phi_R : 2^Z \to 2^Z$ by $\Phi_R(F) = \{s \in Z : \exists\ D \in \mathcal{RO}(Z, s) \text{ with } (D - F)^c \notin \mathcal{P}\}$ in frame of primal topological spaces. Then we introduce the notion of topology δ-compatible for a primal in a primal topological space and study some of its properties. Finally, we use the concept of δ-semi-open sets to provide additional properties for the operators $(^{\diamond}_{R})$ and $\Phi_R(F)$, and we add many illustrative examples that help clarify the relationships between the concepts that are presented.

Keywords: Primal; Primal topological spaces; The operator $\Phi_R(F)$; $\tau^{\diamond}_{R}$-topology; δ-compatible

1 Introduction

Topology is one of the fundamental areas of mathematics. Because of its many universal uses in both science and social science, many related structures have been proposed, such as filter,[13] ideal,[12] gril.[9] Kuratowski used filters to define and explore the notion of ideal,[13] and other topologies have investigated this notion in various directions. Studying the notion of rough sets via topology and ideals is one of these important directions (see,[1011]), where the importance of this research came from the importance of the concept of rough set which was recently defined by Pawlak[19] and its various applications in the literature. In addition, the concept of a grill is a universal classical topological structure that is comparable to the ideal. Numerous authors have examined it since Chpqut[9] first proposed it in 1947 (see[15,16,20]).

In,[1] Acharjee et al. introduced the concept of primal, which is a dual construction of grill. They constructed the concept of primal topological space and explored numerous basic features of this new structure. Since the introduction of the concept of primal, many other new studies have been developed regarding primal topological spaces (see[2–6,8,14]). Recently, the authors in [7] introduced the concept of a single-valued neutrosophic primal, which creates a wider structure that includes fuzzy primal and intuitionistic fuzzy primal. Moreover, they presented the notion of a single-valued neutrosophic open local function for a single-valued neutrosophic topological space.

In this work, we will rely mainly on the results presented in[17] by Özkoç et al., where they presented two new operators $(.)^{\diamond}_{\mathcal{R}}$ and $Cl^{\diamond}_{\mathcal{R}}(.)$ through primal, and they introduced a new topology on Z, denoted by $\tau^{\diamond}_{\mathcal{R}}$, which was obtained by $Cl^{\diamond}_{\mathcal{R}}(.)$. In addition, they proved many fundamental results regarding this new structure.

Throughout this paper, (Z, τ) is a topological space. For any subset $F \subseteq Z$, $cl(F)$ and $int(F)$ denote the closure and the interior of F, respectively in (Z, τ). A subset $F \subseteq Z$ is called regular open[21] in (Z, τ) if $F = int(cl(F))$ and the complement of a regular open subset is called regular closed.[21] Furthermore, $int_\delta(F) = \cup\{D : D \text{ is regular open in } (Z, \tau) \text{ and } D \subseteq F\}$ and $cl_\delta(F) = \cap\{D : D$ is regular closed in

(Z, τ) and $F \subseteq D\}$. A subset $F \subseteq Z$ is called δ-open[22] in (Z, τ) if $F = int_{\delta}(F)$, and the complement of a δ-open set in (Z, τ) is called δ-closed. The family of all δ-open sets in (Z, τ) forms a topology τ_{δ} that is weaker than τ. The family $\mathcal{RO}(Z)$ (resp., $\mathcal{RC}(Z), \delta C(Z)$) will represent the family of all regular open (resp., regular closed, δ-closed) subsets in (Z, τ). Also, $\mathcal{RO}(Z, s)$(resp., $\mathcal{RC}(Z, s)$) will be used to present the family of $\mathcal{RO}(Z)$ (resp., $\mathcal{RC}(Z)$) that contains a point $s \in Z$.

Definition 1.1. [9] A family $\mathcal{H}$ of 2^Z is called a grill on Z if the following holds:

1. $\emptyset \notin \mathcal{H}$,
2. $F \in \mathcal{H}$ or $D \in \mathcal{H}$ whenever $F \cup D \in \mathcal{H}$,
3. $D \in \mathcal{H}$ whenever $F \in \mathcal{H}$ and $F \subseteq D$.

Definition 1.2. [1] A family $\mathcal{P}$ of 2^Z is said to be a primal on Z if the following holds:

1. $Z \notin \mathcal{P}$,
2. $F \in \mathcal{P}$ or $D \in \mathcal{P}$ whenever $F \cap D \in \mathcal{P}$,
3. $D \in \mathcal{P}$ whenever $F \in \mathcal{P}$ and $D \subseteq F$.

A primal topological space, represented by $(Z, \tau, \mathcal{P})$, is a topological space (Z, τ) with a primal $\mathcal{P}$ on Z.

Definition 1.3. [17] Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then:

1. a function $(.)^{\diamond}_{R} : 2^Z \to 2^Z$ is defined by: $F^{\diamond}_{R} = \{s \in Z : \forall\ D \in \mathcal{RO}(Z, s), F^c \cup D^c \in \mathcal{P}\}$ for every $F \subseteq Z$.
2. a function $cl^{\diamond}_{R} : 2^Z \to 2^Z$ is defined by: $cl^{\diamond}_{R}(F) = F^{\diamond}_{R} \cup F$ for every $F \subseteq Z$.
3. $\tau^{\diamond}_{R}$ is a topology on Z induced by τ and $\mathcal{P}$ such that $\tau^{\diamond}_{R} = \{F \subseteq Z : cl^{\diamond}_{R}(F^c) = F^c\}$. Moreover, in[17] they proved that $\tau_{\delta} \subseteq \tau^{\diamond}_{R}$.

Theorem 1.4. [17] *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F, D \subseteq Z$. Then the following holds:*

1. *if $F \in \delta C(Z)$, then $F^{\diamond}_{R} \subseteq F$,*
2. *$F^{\diamond}_{R} \in \delta C(Z)$,*
3. *$(F^{\diamond}_{R})^{\diamond}_{R} \subseteq F^{\diamond}_{R}$,*
4. *if $F \subseteq D$, then $F^{\diamond}_{R} \subseteq D^{\diamond}_{R}$,*
5. *$(F \cup D)^{\diamond}_{R} = F^{\diamond}_{R} \cup D^{\diamond}_{R}$.*

Theorem 1.5. [17] *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F, D \subseteq Z$. If $F \in \tau_{\delta}$, then $F \cap D^{\diamond}_{R} \subseteq (F \cap D)^{\diamond}_{R}$.*

Theorem 1.6. [17] *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F, D \subseteq Z$. Then $F^{\diamond}_{R} - D^{\diamond}_{R} = (F - D)^{\diamond}_{R} - D^{\diamond}_{R}$.*

Theorem 1.7. [17] *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F, D \subseteq Z$. If $D^c \notin \mathcal{P}$, then $(F \cup D)^{\diamond}_{R} = F^{\diamond}_{R} = (F - D)^{\diamond}_{R}$.*

Corollary 1.8. [17] *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F \subseteq Z$. If $F^c \notin \mathcal{P}$, then $F^{\diamond}_{R} = \emptyset$.*

Theorem 1.9. [17] *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then the family $\beta = \{F \cap D : F \in \mathcal{RO}(Z)$ and $D \notin \mathcal{P}\}$ is a base for $\tau^{\diamond}_{R}$.*

2 A New operator via regular open sets

In this section, we will use the family of regular open sets to define an operator $\Phi_R : 2^Z \to 2^Z$ in frame of primal topological spaces and study some of its basic properties.

Definition 2.1. Let $(Z, \tau, \mathcal{P})$ be a primal topological space. For every $F \subseteq Z$ define an operator $\Phi_R : 2^Z \to 2^Z$ by: $\Phi_R(F) = \{s \in Z : \exists\ D \in \mathcal{RO}(Z, s) \text{ with } (D - F)^c \notin \mathcal{P}\}$.

Example 2.2. Consider $Z = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, Z, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Then, $\mathcal{RO}(Z) = \{\emptyset, Z, \{1\}, \{2, 3\}\}$ and hence:

$$\Phi_R(F) = \begin{cases} Z & : F = \{1, 2\} \\ \{2, 3\} & : F = \{3\} \end{cases}$$

Theorem 2.3. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F \subseteq Z$. Then:*

1. $\Phi_R(F)$ *is δ-open,*
2. $\Phi_R(F) = \bigcup\{D \in \mathcal{RO}(Z) : (D - F)^c \notin \mathcal{P}\}$.

Proof. (1) Follows from part (2) of Theorem 1.4.

(2) The proof is a consequence of the definition of Φ_R-operator.

□

The following theorem offers several fundamental facts about the operator Φ_R.

Theorem 2.4. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F, S, D \subseteq Z$. Then the following properties hold:*

1. $\Phi_R(F) = Z - (Z - F)^\diamond_R$,
2. $D \subseteq \Phi_R(D)$ *whenever* $D \in \tau^\diamond_R$,
3. $\Phi_R(F) \subseteq \Phi_R(\Phi_R(F))$,
4. $\Phi_R(F) = \Phi_R(\Phi_R(F))$ *iff* $(Z - F)^\diamond_R = ((Z - F)^\diamond_R)^\diamond_R$,
5. $\Phi_R(F) \cap F = int^\diamond_R(F)$,
6. $\Phi_R(F) \subseteq \Phi_R(S)$ *whenever* $F \subseteq S$,
7. $\Phi_R(F \cap S) = \Phi_R(F) \cap \Phi_R(S)$,
8. $\Phi_R(F) = Z - Z^\diamond_R$ *whenever* $F^c \notin \mathcal{P}$,
9. $\Phi_R(F - K) = \Phi_R(F)$ *and* $\Phi_R(F \cup K) = \Phi_R(F)$ *whenever* $K^c \notin \mathcal{P}$,
10. $\Phi_R(F) = \Phi_R(S)$ *whenever* $[(F - S) \cup (S - F)]^c \notin \mathcal{P}$.

Proof. (1) Let $s \in \Phi_R(F)$. Then there is $D \in \mathcal{RO}(Z, s)$ with $D^c \cup F = (D \cap (Z - F))^c = (D - F)^c \notin \mathcal{P}$, which means that $s \notin (Z - F)^\diamond_R$ and hence $s \in Z - (Z - F)^\diamond_R$. Conversely, assume that $s \in Z - (Z - F)^\diamond_R$, then $s \notin (Z - F)^\diamond_R$. Thus, there is $D \in \mathcal{RO}(Z, s)$ with $D^c \cup (Z - F)^c = (D - F)^c \notin \mathcal{P}$ and consequently, $s \in \Phi_R(F)$. Therefore, $\Phi_R(F) = Z - (Z - F)^\diamond_R$.

(2) If $D \in \tau^\diamond_R$, then $(Z - D)^\diamond_R \subseteq Z - D$. Thus, $D \subseteq Z - (Z - D)^\diamond_R = \Phi_R(D)$.
(3) It follows from part (2) and Theorem 2.3 part(1) and the fact that $\tau_\delta \subseteq \tau^\diamond_R$.

(4) It follows from part (1) and the fact that:

$\Phi_R(\Phi_R(F)) = Z - [Z - (Z - (Z - F)^{\diamond}_R)]^{\diamond}_R = Z - ((Z - F)^{\diamond}_R)^{\diamond}_R$.
(5) Assume that $s \in \Phi_R(F) \cap F$, then there is $D \in \mathcal{RO}(Z, s)$ with $(D - F)^c \notin \mathcal{P}$. By Theorem 1.9, $D \cap (D - F)^c$ is a $\tau^{\diamond}_R$-open neighborhood of s and hence $s \in int^{\diamond}_R(F)$. To show the reverse inclusion, suppose $s \in int^{\diamond}_R(F)$, then by Theorem 1.9 there is a basic $\tau^{\diamond}_R$-open neighborhood $W \cap K$ of s, where $W \in \mathcal{RO}(Z, s)$ and $K \notin \mathcal{P}$ with $s \in W \cap K \subseteq F$. It follows that $K \subseteq (W - F)^c$ and hence $(W - F)^c \notin \mathcal{P}$.Therefore, $s \in \Phi_R(F) \cap F$.

(6) Follows from part (1) and Theorem 1.4 part (4).
(7) From part (6) it is always true that $\Phi_R(F \cap S) \subseteq \Phi_R(F)$ and $\Phi_R(F \cap S) \subseteq \Phi_R(S)$. Thus, $\Phi_R(F \cap S) \subseteq \Phi_R(F) \cap \Phi_R(S)$. To show that $\Phi_R(F) \cap \Phi_R(S) \subseteq \Phi_R(F \cap S)$, let $s \in \Phi_R(F) \cap \Phi_R(S)$. Then there are $D, W \in \mathcal{RO}(Z, s)$ with $(D - F)^c \notin \mathcal{P}$ and $(W - S)^c \notin \mathcal{P}$. Set $G = D \cap W$. Then $G \in \mathcal{RO}(Z, s)$ such that $(G - F)^c \notin \mathcal{P}$ and $(G - S)^c \notin \mathcal{P}$ by heredity. Hence $[G - (F \cap S)]^c = (G - F)^c \cap (G - S)^c \notin \mathcal{P}$ and so $s \in \Phi_R(F \cap S)$.

(8) By Theorem 1.7 if $F^c \notin \mathcal{P}$, then $(Z - F)^{\diamond}_R = Z^{\diamond}_R$ and hence $\Phi_R(F) = Z - (Z - F)^{\diamond}_R = Z - Z^{\diamond}_R$.

(9) By Theorem 1.7 and the fact that $\Phi_R(F - K) = Z - [Z - (F - K)]^{\diamond}_R = Z - [(Z - F) \cup K]^{\diamond}_R = Z - (Z - F)^{\diamond}_R = \Phi_R(F)$. Moreover, $\Phi_R(F \cup K) = Z - [Z - (F \cup K)]^{\diamond}_R = Z - [(Z - F) - K]^{\diamond}_R = Z - (Z - F)^{\diamond}_R = \Phi_R(F)$.

(10) Set $K = F - S$ and $L = S - F$. From the hypothesis and heredity property of $\mathcal{P}$, we get $K^c, L^c \notin \mathcal{P}$. Since $S = (F - K) \cup L$, then by part (9) we have $\Phi_R(F) = \Phi_R(F - K) = \Phi_R[(F - K) \cup L] = \Phi_R(S)$. □

Theorem 2.5. *If $(Z, \tau, \mathcal{P})$ is a primal topological space, then $int_\delta(F) \subseteq \Phi_R(F)$ for every $F \subseteq Z$.*

Proof. Let $F \subseteq Z$. Suppose that there is $s \in Z$ and $s \notin \Phi_R(F)$. Then $s \in (Z - F)^{\diamond}_R$, which implies that $[D \cap (Z - F)]^c \in \mathcal{P}$ for every $D \in \mathcal{RO}(Z, s)$. Therefore, $D \cap (Z - F) \neq \emptyset$ for every $D \in \mathcal{RO}(Z, s)$ and hence $s \notin int_\delta(F)$. Thus, $int_\delta(F) \subseteq \Phi_R(F)$ for every $F \subseteq Z$. □

Corollary 2.6. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If F is a δ-open set, then $F \subseteq \Phi_R(F)$.*

Proof. Assume that F is a δ-open set. Then $int_\delta(F) = F$ and hence by Theorem 2.5 $F \subseteq \Phi_R(F)$. □

Theorem 2.7. *If $(Z, \tau, \mathcal{P})$ be a primal topological space and $F, S \subseteq Z$, then $int_\delta(F) \cap S^{\diamond}_R = int_\delta(F) \cap (F \cap S)^{\diamond}_R \subseteq (F \cap S)^{\diamond}_R$.*

Proof. Let $s \in int_\delta(F) \cap S^{\diamond}_R$. Since $s \in int_\delta(F)$, then there is $D \in \mathcal{RO}(Z, s)$ with $s \in D \subseteq F$. For every $W \in \mathcal{RO}(Z, s)$, $D \cap W \in \mathcal{RO}(Z, s)$. As $s \in S^{\diamond}_R$, then $[(D \cap W)]^c \cup S^c \in \mathcal{P}$. From the definition of primal we have $[W \cap (F \cap S)]^c \subseteq [(D \cap W) \cap S]^c = [(D \cap W)]^c \cup S^c \in \mathcal{P}$. Hence, $[W \cap (F \cap S)]^c \in \mathcal{P}$ and so we get $s \in (F \cap S)^{\diamond}_R$. That is $int_\delta(F) \cap S^{\diamond}_R \subseteq (F \cap S)^{\diamond}_R$. Now, since $(F \cap S)^{\diamond}_R \subseteq S^{\diamond}_R$, then $int_\delta(F) \cap (F \cap S)^{\diamond}_R \subseteq int_\delta(F) \cap S^{\diamond}_R$. Therefore, $int_\delta(F) \cap S^{\diamond}_R = int_\delta(F) \cap (F \cap S)^{\diamond}_R \subseteq (F \cap S)^{\diamond}_R$. □

Corollary 2.8. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F \subseteq Z$. If $D \subseteq int(D^{\diamond}_R)$, then $int_\delta(F) \cap D \subseteq int((F \cap D)^{\diamond}_R)$.*

Proof. Using Theorem 2.7,

$$\begin{aligned} int_\delta(F) \cap D &\subseteq int_\delta(F) \cap int(D^{\diamond}_R) \\ &= int[int_\delta(F) \cap D^{\diamond}_R] \\ &\subseteq int((F \cap D)^{\diamond}_R). \end{aligned}$$

□

Theorem 2.9. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If $\beta = \{F \subseteq Z : F \subseteq \Phi_R(F)\}$, then β is a topology on Z and $\beta = \tau_R^{\diamond}$.*

Proof. At first we notice that, $\emptyset \subseteq \Phi_R(\emptyset)$ and $Z \subseteq \Phi_R(Z) = Z$ and hence $\emptyset$, $Z \in \beta$. Secondly, if $F, D \in \beta$, then by Theorem 2.4 part (7), $F \cap D \subseteq \Phi_R(F) \cap \Phi(D) = \Phi_R(F \cap D)$ and hence $F \cap D \in \beta$. Finally, let $\{F_\alpha : \alpha \in \Delta\} \subseteq \beta$. Then by Theorem 2.4 part (6), for every $\alpha \in \Delta$, $F_\alpha \subseteq \Phi_R(F_\alpha) \subseteq \Phi_R(\bigcup_{\alpha\in\Delta} F_\alpha)$ and hence $\bigcup F_\alpha \subseteq \Phi_R(\bigcup_{\alpha\in\Delta} F_\alpha)$. Therefore, β is a topology on Z. To prove $\beta = \tau_R^{\diamond}$, let $S \in \tau_R^{\diamond}$ with $s \in S$. Then by Theorem 1.9, there is $W \in \mathcal{RO}(Z, s)$ and $K \notin \mathcal{P}$ with $s \in W \cap K \subseteq S$. Obviously, $K \subseteq (W - S)^c$and so by heredity we have $(W - S)^c \notin \mathcal{P}$, and thus $s \in \Phi_R(S)$. Hence, $S \subseteq \Phi_R(S)$ and so we have $\tau_R^{\diamond} \subseteq \beta$. Now, assume that $F \in \beta$, then $F \subseteq \Phi_R(F)$, which means $F \subseteq Z - (Z - F)_R^{\diamond}$ and $(Z - F)_R^{\diamond} \subseteq Z - F$. This proves that $Z - F$ is $\tau_R^{\diamond}$-closed and thus $F \in \tau_R^{\diamond}$. Therefore, $\beta \subseteq \tau_R^{\diamond}$ and hence $\beta = \tau_R^{\diamond}$. □

The following example shows that the topology β exists.

Example 2.10. Consider $Z = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, Z, \{1\}, \{2\}, \{1, 2\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. It is clear that $\mathcal{RO}(Z) = \{\emptyset, Z, \{1\}, \{2\}\}$ and $\beta = \{\emptyset, Z, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, \{1, 2\}\}$, as shown by the following table. If $F \subseteq Z$, then:

F	$(Z - F)_R^{\diamond}$	$\Phi_R(F)$
$\emptyset$	$\{3\}$	$\{1, 2\}$
Z	$\emptyset$	Z
$\{1\}$	$\{3\}$	$\{1, 2\}$
$\{2\}$	$\{3\}$	$\{1, 2\}$
$\{3\}$	$\emptyset$	Z
$\{1, 2\}$	$\{3\}$	$\{1, 2\}$
$\{1, 3\}$	$\emptyset$	Z
$\{2, 3\}$	$\emptyset$	Z

Corollary 2.11. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then:*

1. *a set F is closed in (Z, β) iff $F_R^{\diamond} \subseteq F$.*
2. *for every $F \subseteq Z$, $cl_\beta(F) = F \cup F_R^{\diamond}$, where $cl_\beta(F)$ is the closure of F in (Z, β).*

Proof. The proof follows from Theorem 2.9. □

Theorem 2.12. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $F \subseteq Z$. If $F \subseteq F_R^{\diamond}$, then $cl_\delta(F) = cl_R^{\diamond}(F) = cl_\delta(F_R^{\diamond}) = F_R^{\diamond}$.*

Proof. Since $\tau_\delta \subseteq \tau_R^{\diamond}$, then $cl_R^{\diamond}(F) \subseteq cl_\delta(F)$ for every $F \subseteq Z$. Now $s \notin cl_R^{\diamond}(F)$, then there is $W \in \mathcal{RO}(Z)$ and $S \notin \mathcal{P}$ with $s \in W \cap S$ and $(W \cap S) \cap F = \emptyset$. It follows that $[(W \cap S) \cap F]_R^{\diamond} = \emptyset$ and hence $[(W \cap F) - S^c]_R^{\diamond} = \emptyset$. Thus by Theorem 1.7, $(W \cap F)_R^{\diamond} = \emptyset$ and so by Theorem 1.5 we get $W \cap F_R^{\diamond} = \emptyset$ and $W \cap F = \emptyset$ (as $F \subseteq F_R^{\diamond}$). Therefore, $s \notin cl_\delta(F)$ and hence $cl_\delta(F) = cl_R^{\diamond}(F)$. Now, by Theorem 1.4 part (2) $F_R^{\diamond} = cl_\delta(F_R^{\diamond})$. Now, to prove that $F_R^{\diamond} \subseteq cl_\delta(F)$, let $s \notin cl_\delta(F)$. Then there is $D \in \mathcal{RO}(Z, s)$ with $D \cap F = \emptyset$ and so $(D \cap F)^c = D^c \cup F^c = Z \notin \mathcal{P}$. Hence $s \notin F_R^{\diamond}$. Again as $F_R^{\diamond} \subseteq cl_\delta(F)$, so we have $cl_\delta(F_R^{\diamond}) \subseteq cl_\delta(cl_\delta(F)) = cl_\delta(F)$. Also since $F \subseteq F_R^{\diamond}$, then $cl_\delta(F) \subseteq cl_\delta(F_R^{\diamond})$. Therefore, $cl_\delta(F) = cl_\delta(F_R^{\diamond}) = F_R^{\diamond}$. □

In,[17] they proved that $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$ iff $F \subseteq F_R^{\diamond}$ for every $F \in \mathcal{RO}(Z)$. We need this result to introduce the following Theorem.

Theorem 2.13. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$. If W is $\tau_R^\diamond$-open set such that $W = D \cap F$ with $D \in \mathcal{RO}(Z)$ and $F \notin \mathcal{P}$, then $cl_\delta(W) = cl_R^\diamond(W) = W_R^\diamond = D_R^\diamond = cl_\delta(D) = cl_R^\diamond(D)$.*

Proof. Let $W = D \cap F$ with $D \in \mathcal{RO}(Z)$ and $F \notin \mathcal{P}$. Since $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$, then $D \subseteq D_R^\diamond$. Thus, by Theorem 2.12, $D_R^\diamond = cl_\delta(D) = cl_R^\diamond(D)$. Now, let W be $\tau_R^\diamond$-open. We claim that $W \subseteq W_R^\diamond$. Since $cl_R^\diamond(Z - W) = Z - W$, then $(Z - W)_R^\diamond \subseteq Z - W$. By Theorem 1.6 and since $Z_R^\diamond = Z$, then $Z - W_R^\diamond = Z_R^\diamond - W_R^\diamond \subseteq (Z - W)_R^\diamond \subseteq Z - W$. Thus, $W \subseteq W_R^\diamond$ and hence by Theorem 2.12, $W_R^\diamond = cl_\delta(W) = cl_R^\diamond(W)$.

Moreover, $W \subseteq D$ and so we have $W_R^\diamond \subseteq D_R^\diamond$. Also, by Theorems [1.8 and 1.6] we have $W_R^\diamond = (D \cap F)_R^\diamond = (D - F^c)_R^\diamond \supseteq D_R^\diamond - (F^c)_R^\diamond = D_R^\diamond$ as $F \notin \mathcal{P}$. Therefore, $D_R^\diamond = W_R^\diamond$ and hence $cl_\delta(W) = cl_R^\diamond(W) = W_R^\diamond = D_R^\diamond = cl_\delta(D) = cl_R^\diamond(D)$. □

3 δ-compatible on a primal topological space

In this section, we study δ-compatible in a primal topological space and investigate some properties.

Definition 3.1. Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then τ is said to be δ-compatible for the primal $\mathcal{P}$ if $F^c \cup F_R^\diamond \notin \mathcal{P}$ for every $F \subseteq Z$.

Example 3.2. Consider $Z = \{1, 2, 3\}$ with topology $\tau = \tau_\delta = \{\emptyset, Z, \{1\}, \{2\}, \{1, 2\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. It is clear that $\mathcal{RO}(Z) = \{\emptyset, Z, \{1\}, \{2\}\}$ and τ is δ-compatible for the primal $\mathcal{P}$, as shown by the following table, if $F \subseteq Z$, then:

$F \subseteq Z$	F^c	$F_R^\diamond$	$F^c \cup F_R^\diamond$	$\in \mathcal{P}$ or $\notin \mathcal{P}$
$\emptyset$	Z	$\emptyset$	Z	$\notin \mathcal{P}$
Z	$\emptyset$	$\{3\}$	$\{3\}$	$\notin \mathcal{P}$
$\{1\}$	$\{2,3\}$	$\emptyset$	$\{2,3\}$	$\notin \mathcal{P}$
$\{2\}$	$\{1,3\}$	$\emptyset$	$\{1,3\}$	$\notin \mathcal{P}$
$\{3\}$	$\{1,2\}$	$\{3\}$	$\{Z\}$	$\notin \mathcal{P}$
$\{1,2\}$	$\{3\}$	$\emptyset$	$\{3\}$	$\notin \mathcal{P}$
$\{1,3\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\notin \mathcal{P}$
$\{2,3\}$	$\{1\}$	$\{3\}$	$\{1,3\}$	$\notin \mathcal{P}$

Theorem 3.3. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then the following are equivalent:*

1. *τ is δ-compatible for the primal $\mathcal{P}$,*
2. *for every $\tau_R^\diamond$-closed subset F of Z, $F^c \cup F_R^\diamond \notin \mathcal{P}$,*
3. *whenever for any $F \subseteq Z$ and each $s \in F$ there corresponds some $D \in \mathcal{RO}(Z, s)$ with $D^c \cup F^c \notin \mathcal{P}$, it follows that $F^c \notin \mathcal{P}$,*
4. *for every $F \subseteq Z$ with $F \cap F_R^\diamond = \emptyset$, it follows that $F^c \notin \mathcal{P}$.*

Proof. (1) ⇒ (2) It is trivial.

(2) ⇒ (3) Let $F \subseteq Z$. Suppose that for every $s \in F$ there is $D \in \mathcal{RO}(Z, s)$ with $D^c \cup F^c \notin \mathcal{P}$. Then, $s \notin F_R^\diamond$ and hence $F \cap F_R^\diamond = \emptyset$. Now, since $F \cup F_R^\diamond = cl_R^\diamond(F)$ is $\tau_R^\diamond$-closed, then by part (2), $(F \cup F_R^\diamond)^c \cup (F \cup F_R^\diamond)_R^\diamond \notin \mathcal{P}$ and hence by Theorem 1.4 part (5), $(F \cup F_R^\diamond)^c \cup (F_R^\diamond \cup (F_R^\diamond)_R^\diamond) \notin \mathcal{P}$. By Theorem 1.4 part (3), $(F \cup F_R^\diamond)^c \cup F_R^\diamond \notin \mathcal{P}$. Since $F \cap F_R^\diamond = \emptyset$, then $(F \cup F_R^\diamond)^c \cup F^c \notin \mathcal{P}$ and so we have $F^c \notin \mathcal{P}$.

(3) ⇒ (4) Suppose that $F \subseteq Z$ with $F \cap F^{\diamond}_{R} = \emptyset$, then $F \subseteq Z - F^{\diamond}_{R}$. Let $s \in F$. Then $s \notin F^{\diamond}_{R}$ and hence there is $D \in \mathcal{RO}(Z, s)$ with $D^c \cup F^c \notin \mathcal{P}$ and so by part (3), $F^c \notin \mathcal{P}$.

(4) ⇒ (1) Let $F \subseteq Z$. To show that $(F - F^{\diamond}_{R}) \cap (F - F^{\diamond}_{R})^{\diamond}_{R} = \emptyset$, let $s \in (F - F^{\diamond}_{R}) \cap (F - F^{\diamond}_{R})^{\diamond}_{R}$. Then $s \in F - F^{\diamond}_{R}$ and hence $s \in F$ with $s \notin F^{\diamond}_{R}$. So there is $D \in \mathcal{RO}(Z, s)$ with $D^c \cup F^c \notin \mathcal{P}$. Now since $D^c \cup F^c \subseteq D^c \cup (F - F^{\diamond}_{R})^c$, then $D^c \cup (F - F^{\diamond}_{R})^c \notin \mathcal{P}$. Hence $s \notin (F - F^{\diamond}_{R})^{\diamond}_{R}$, which is a contradiction. Therefore, by (4) $(F - F^{\diamond}_{R})^c = F^c \cup F^{\diamond}_{R} \notin \mathcal{P}$ and thus τ is δ-compatible for the primal $\mathcal{P}$. □

Theorem 3.4. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then the following are equivalent:*

1. *for every $F \subseteq Z$, if $F \cap F^{\diamond}_{R} = \emptyset$, then $F^{\diamond}_{R} = \emptyset$,*
2. *for every $F \subseteq Z$, $(F - F^{\diamond}_{R})^{\diamond}_{R} = \emptyset$,*
3. *for every $F \subseteq Z$, $(F \cap F^{\diamond}_{R})^{\diamond}_{R} = F^{\diamond}_{R}$.*

Proof. (1) ⇒ (2) It follows from the fact that $(F - F^{\diamond}_{R}) \cap (F - F^{\diamond}_{R})^{\diamond}_{R} = \emptyset$, for every $F \subseteq Z$.

(2) ⇒ (3): Since $F = (F - (F \cap F^{\diamond}_{R})) \cup (F \cap F^{\diamond}_{R})$, then $F^{\diamond}_{R} = (F - (F \cap F^{\diamond}_{R}))^{\diamond}_{R} \cup (F \cap F^{\diamond}_{R})^{\diamond}_{R} = (F - F^{\diamond}_{R})^{\diamond}_{R} \cup (F \cap F^{\diamond}_{R})^{\diamond}_{R} = \emptyset \cup (F \cap F^{\diamond}_{R})^{\diamond}_{R} = (F \cap F^{\diamond}_{R})^{\diamond}_{R}$.

(3) ⇒ (1): Assume that $F \subseteq Z$ with $F \cap F^{\diamond}_{R} = \emptyset$. Then by part (3), $F^{\diamond}_{R} = (F \cap F^{\diamond}_{R})^{\diamond}_{R} = \emptyset^{\diamond}_{R} = \emptyset$. □

Note that, from Theorem 3.3, any one of the three conditions in Theorem 3.4 is necessary for τ to be δ-compatible for the primal $\mathcal{P}$.

Corollary 3.5. *If $(Z, \tau, \mathcal{P})$ be a primal topological space and τ is δ-compatible for $\mathcal{P}$, then the operator $(^{\diamond}_{R})$ is an idempotent operator i.e., $F^{\diamond}_{R} = (F^{\diamond}_{R})^{\diamond}_{R}$ for every $F \subseteq Z$.*

Proof. By Theorem 1.4 part (3) we obtain $(F^{\diamond}_{R})^{\diamond}_{R} \subseteq F^{\diamond}_{R}$. Also, by Theorem 3.4 and Theorem 1.4 part (4), we have $F^{\diamond}_{R} = (F \cap F^{\diamond}_{R})^{\diamond}_{R} \subseteq (F^{\diamond}_{R})^{\diamond}_{R}$. □

Theorem 3.6. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then τ is δ-compatible for the primal $\mathcal{P}$ iff $[\Phi_R(F) - F]^c \notin \mathcal{P}$ for every $F \subseteq Z$.*

Proof. Assume that $F \subseteq Z$ and τ is δ-compatible for the primal $\mathcal{P}$. Let $s \in \Phi_R(F) - F$. Then $s \notin (Z - F)^{\diamond}_{R}$ and hence there is $D \in \mathcal{RO}(Z, s)$ with $(D - F)^c = (D)^c \cup F \notin \mathcal{P}$. Note that, for every $s \in \Phi_R(F) - F$ there corresponds some $D \in \mathcal{RO}(Z, s)$ with $[D \cap (\Phi_R(F) - F)]^c = [D]^c \cup [(\Phi_R(F) - F)]^c \notin \mathcal{P}$ by heredity. Therefore, by Theorem 3.3 part (3) we have $[\Phi_R(F) - F]^c \notin \mathcal{P}$.

Conversely, let $F \subseteq Z$. Since $\Phi_R(F^c) - (F^c) = F - F^{\diamond}_{R}$ and $F^c \cup F^{\diamond}_{R} = [F - F^{\diamond}_{R}]^c = [\Phi_R(F^c) - (F^c)]^c \notin \mathcal{P}$, then τ is δ-compatible for the primal $\mathcal{P}$. □

Theorem 3.7. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If τ is δ-compatible for the primal $\mathcal{P}$, then $\beta = \{\Phi_R(F) - K : F \subseteq Z,\ K^c \notin \mathcal{P}\}$.*

Proof. By using Theorem 2.4, for $K^c \notin \mathcal{P}$ we have $\Phi_R[\Phi_R(F) - K] = \Phi_R[\Phi_R(F)] \supseteq \Phi_R(F) \supseteq \Phi_R(F) - K$. Thus, according to Theorem 2.9 all sets of the form $\Phi_R(F) - K$ are in β.

To prove the reverse inclusion, let $F \in \beta$. Therefore, $F \subseteq \Phi_R(F)$. Since τ is δ-compatible for the primal $\mathcal{P}$, then by Theorem 3.6 we get $[\Phi_R(F) - F]^c \notin \mathcal{P}$. Now set $K = \Phi_R(F) - F$. Then $F = \Phi_R(F) - K$ with $K^c \notin \mathcal{P}$. Hence $F \in \{\Phi_R(F) - K : F \subseteq Z,\ K^c \notin \mathcal{P}\} = \beta$. □

Theorem 3.8. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space such that τ is δ-compatible for $\mathcal{P}$. Then a subset F of Z is $\tau_R^\diamond$-closed iff $F = D \cup S$ with $D \in \delta C(Z)$ and $S^c \notin \mathcal{P}$.*

Proof. Assume that F is a $\tau_R^\diamond$-closed subset of Z. Then, $F_R^\diamond \subseteq F$. Since τ is δ-compatible for $\mathcal{P}$, then by Theorem 3.3, $F^c \cup F_R^\diamond = (F - F_R^\diamond)^c \notin \mathcal{P}$ and by Theorem 1.4 part (2), $F_R^\diamond \in \delta C(Z)$. Now, since $F = F_R^\diamond \cup (F - F_R^\diamond)$, then the result follows.

Conversely, let $F = D \cup S$ with $D \in \delta C(Z)$ and $S^c \notin \mathcal{P}$. Then by Theorem 1.4 parts (2 and 5), $F_R^\diamond = (D \cup S)_R^\diamond = D_R^\diamond = cl_\delta(D_R^\diamond) \subseteq cl_\delta(D) = D \subseteq F$. Therefore, F is $\tau_R^\diamond$-closed. □

Theorem 3.9. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $S \subseteq Z$. If τ is δ-compatible for $\mathcal{P}$, then $(F \cap S)_R^\diamond = (F \cap S_R^\diamond)_R^\diamond = cl_\delta(F \cap S_R^\diamond)$ for every $F \in \mathcal{RO}(Z)$.*

Proof. Assume that $F \in \mathcal{RO}(Z)$. First, we will show that $(F \cap S)_R^\diamond = (F \cap S_R^\diamond)_R^\diamond$. By Theorem 1.5, $F \cap S_R^\diamond \subseteq (F \cap S)_R^\diamond$ and thus by Theorem 1.4 part (4) and by Corollary 3.5 we get $(F \cap S_R^\diamond)_R^\diamond \subseteq [(F \cap S)_R^\diamond]_R^\diamond = (F \cap S)_R^\diamond$. Now by Theorem 1.4 part (4) and by Theorem 3.4 we get $[F \cap (S - S_R^\diamond)]_R^\diamond \subseteq (S - S_R^\diamond)_R^\diamond = \emptyset$.

Also, by Theorem1.6, $(F \cap S)_R^\diamond - (F \cap S_R^\diamond)_R^\diamond \subseteq [(F \cap S) - (F \cap S_R^\diamond)]_R^\diamond = [F \cap (S - S_R^\diamond)]_R^\diamond = \emptyset$. Hence, $(F \cap S)_R^\diamond \subseteq (F \cap S_R^\diamond)_R^\diamond$ and so we get $(F \cap S)_R^\diamond = (F \cap S_R^\diamond)_R^\diamond$.

Again $(F \cap S)_R^\diamond = (F \cap S_R^\diamond)_R^\diamond \subseteq cl_\delta(F \cap S_R^\diamond)$, since $\tau_\delta \subseteq \tau_R^\diamond$. Now, by using Theorem 1.5, $F \cap S_R^\diamond \subseteq (F \cap S)_R^\diamond$, and hence $cl_\delta(F \cap S_R^\diamond) \subseteq cl_\delta((F \cap S)_R^\diamond) = (F \cap S)_R^\diamond$. Therefore, $(F \cap S)_R^\diamond = cl_\delta(F \cap S_R^\diamond)$. □

Corollary 3.10. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space such that τ is δ-compatible for $\mathcal{P}$. If $F \in \mathcal{RO}(Z)$ and $F^c \notin \mathcal{P}$, then $F \subseteq Z - Z_R^\diamond$.*

Proof. Set $S = Z$ in Theorem 3.9, then $F_R^\diamond = (F \cap Z)_R^\diamond = cl_\delta(F \cap Z_R^\diamond)$. Since $F^c \notin \mathcal{P}$, then $F_R^\diamond = \emptyset$ and hence $\emptyset = F_R^\diamond = (F \cap Z)_R^\diamond = cl_\delta(F \cap Z_R^\diamond)$. Thus, $F \cap Z_R^\diamond = \emptyset$ and hence $F \subseteq Z - Z_R^\diamond$. □

Proposition 3.11. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space such that τ is δ-compatible for the primal $\mathcal{P}$ and $F \subseteq Z$. If $D \subseteq F_R^\diamond \cap \Phi_R(F)$ and D is nonempty regular open, then $[D - F]^c \notin \mathcal{P}$ and $[D \cap F]^c \in \mathcal{P}$.*

Proof. Assume that $D \subseteq F_R^\diamond \cap \Phi_R(F)$, then $[\Phi_R(F) - F]^c \subseteq [D - F]^c$. By Theorem 3.6, $[\Phi_R(F) - F]^c \notin \mathcal{P}$ and hence by heredity$[D - F]^c \notin \mathcal{P}$. Since $D \in \mathcal{RO}(Z) - \{\emptyset\}$ and $D \subseteq F_R^\diamond$, then $D^c \cup F^c = [D \cap F]^c \in \mathcal{P}$ by the definition of $F_R^\diamond$. □

The following results from the aforementioned theorems.

Corollary 3.12. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If τ is δ-compatible for the primal $\mathcal{P}$, then $\Phi_R(\Phi_R(F)) = \Phi_R(F)$ for every $F \subseteq Z$.*

Proof. From part (3) of Theorem 2.4 we have $\Phi_R(F) \subseteq \Phi_R(\Phi_R(F))$. Since τ is δ-compatible for the primal $\mathcal{P}$, then by using Theorem 3.6 we get $[\Phi_R(F) - F]^c \notin \mathcal{P}$ for every $F \subseteq Z$. Set $K = \Phi_R(F) - F$, Then $\Phi_R(F) \subseteq F \cup K$ for some $K^c \notin \mathcal{P}$ and hence by parts (6 and 9) of Theorem 2.4 we have $\Phi_R(\Phi_R(F)) \subseteq \Phi_R(F \cup K) = \Phi_R(F)$.Therefore, $\Phi_R(\Phi_R(F)) = \Phi_R(F)$. □

Theorem 3.13. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If τ is δ-compatible for the primal $\mathcal{P}$, then $\Phi_R(F) = \cup\{\Phi_R(D) : D \in \mathcal{RO}(Z), [\Phi_R(D) - F]^c \notin \mathcal{P}\}$.*

Proof. Let $\mathcal{H} = \cup\{\Phi_R(D) : D \in \mathcal{RO}(Z), [\Phi_R(D) - F]^c \notin \mathcal{P}\}$ and let $s \in \mathcal{H}$. Then there is $D \in \mathcal{RO}(Z)$ with $[\Phi_R(D) - F]^c \notin \mathcal{P}$ and $s \in \Phi_R(D)$. By using Theorem 2.3 part (2) there is $W \in \mathcal{RO}(Z, s)$ with $[W - D]^c \notin \mathcal{P}$. By Corollary 2.6, $D \subseteq \Phi_R(D)$ and $D - F \subseteq \Phi_R(D) - F$ and since $[\Phi_R(D) - F]^c \notin \mathcal{P}$, then $[D - F]^c \notin \mathcal{P}$. Now, since $(W - D)^c \cap (D - F)^c \notin \mathcal{P}$ and $(W - D)^c \cap (D - F)^c \subseteq [W - F]^c$, then $[W - F]^c \notin \mathcal{P}$. Also, as $W \in \mathcal{RO}(Z, s)$ and $[W - F]^c \notin \mathcal{P}$, then $s \in \Phi_R(F)$. Thus, $\mathcal{H} \subseteq \Phi_R(F)$. Now, let $s \in \Phi_R(F)$. Then there is $D \in \mathcal{RO}(Z, s)$ with $[D - F]^c \notin \mathcal{P}$. Now, by Corollary 2.6, $D \subseteq \Phi_R(D)$ and so $[\Phi_R(D) - D]^c \cap [D - F]^c \subseteq [\Phi_R(D) - F]^c$. Now, by using Theorem 3.6, $[\Phi_R(D) - D]^c \notin \mathcal{P}$ and hence $[\Phi_R(D) - F]^c \notin \mathcal{P}$. Thus, $s \in \mathcal{H}$ and $\Phi_R(F) \subseteq \mathcal{H}$. Therefore, $\mathcal{H} = \Phi_R(F)$. □

Definition 3.14. Let $(Z, \tau, \mathcal{P})$ be a primal topological space. A subset F of Z is called a regular baire set with respect to τ and $\mathcal{P}$, if there is a regular open set D with $(F - D)^c \cap (D - F)^c \notin \mathcal{P}$.

Lemma 3.15. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space with τ is δ-compatible for the primal $\mathcal{P}$. If F and $S \in \mathcal{RO}(Z)$ with $\Phi_R(F) = \Phi_R(S)$, then $(S - F)^c \cap (F - S)^c \notin \mathcal{P}$.*

Proof. Since $F \in \mathcal{RO}(Z)$, then $F \subseteq \Phi_R(F)$ and hence by Theorem 3.6 $[\Phi_R(F) - S]^c = [\Phi_R(S) - S]^c \notin \mathcal{P}$. Since $[\Phi_R(S) - S]^c \subseteq [F - S]^c$, then $[F - S]^c \notin \mathcal{P}$. Similarly, $[S - F]^c \notin \mathcal{P}$. Therefore, $(F - S)^c \cap (S - F)^c \notin \mathcal{P}$ by additivity. □

Theorem 3.16. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space with τ is δ-compatible for the primal $\mathcal{P}$. If F and S are regular baire sets with $\Phi_R(F) = \Phi_R(S)$, then $(F - S)^c \cap (S - F)^c \notin \mathcal{P}$.*

Proof. Since F and S are regular baire sets, then their are $D, W \in \mathcal{RO}(Z)$ with $(F - D)^c \cap (D - F)^c \notin \mathcal{P}$ and $(W - S)^c \cap (S - W)^c \notin \mathcal{P}$. By part (10) of Theorem 2.4 we get $\Phi_R(F) = \Phi_R(D)$ and $\Phi_R(S) = \Phi_R(W)$. Since $\Phi_R(F) = \Phi_R(S)$, then $\Phi_R(D) = \Phi_R(W)$, and hence by Lemma 3.15, $(D - W)^c \cap (W - D)^c \notin \mathcal{P}$. It follows that $(F \cup D \cup W)^c \cup (F \cap D \cap W) \notin \mathcal{P}$ and $(S \cup D \cup W)^c \cup (S \cap D \cap W) \notin \mathcal{P}$ and hence $[(F \cup D \cup W)^c \cup (F \cap D \cap W)] \cap [(S \cup D \cup W)^c \cup (S \cap D \cap W)] \subseteq [(F \cup D \cup W \cup S)^c \cup (F \cap D \cap W \cap S)] \notin \mathcal{P}$. Now since $[(F \cup D \cup W \cup S)^c \cup (F \cap D \cap W \cap S)] \subseteq (F \cup S)^c \cup (F \cap S)$, then $(F \cup S)^c \cup (F \cap S) \notin \mathcal{P}$. Therefore, $(F - S)^c \cap (S - F)^c \notin \mathcal{P}$. □

4 More properties for $(^{\diamond}_R)$ and for $\Phi_R(F)$

In this section we use the concept of δ-semi-open sets to study additional properties for the operators $(^{\diamond}_R)$ and $\Phi_R(F)$, where a subset S of a topological space (Z, τ) is called a δ-semiopen[18] if $S \subseteq cl(int_\delta(S))$, or equivalent to if there is a δ-open set F with $F \subseteq S \subseteq cl(F)$. Moreover, $cl(F) = cl_\delta(F)$ for every open set F of (Z, τ).

Theorem 4.1. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then the following are equivalent:*

1. *$\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$,*
2. *$F \subseteq F^{\diamond}_R$ for every $F \in \tau_\delta$,*
3. *$S \subseteq S^{\diamond}_R$ for every δ-semiopen set S,*
4. *$cl_\delta(F) = F^{\diamond}_R$ for every $F \in \tau_\delta$,*
5. *$cl_\delta(S) = S^{\diamond}_R$ for every δ-semiopen set S,*
6. *$int_\delta(F) \subseteq int_\delta(F^{\diamond}_R)$ for every subset F of Z.*

Proof. (1) ⇒ (2) Let $F \in \tau_\delta$ and $s \in F$. Then there is $D \in \mathcal{RO}(Z)$ with $s \in D \subseteq F$. Now, if $W \in \mathcal{RO}(Z, s)$, then $s \in D \cap W \in \mathcal{RO}(Z, s)$. Also, since $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$, then $[D \cap W]^c \in \mathcal{P}$ and $[F \cap W]^c \in \mathcal{P}$ and hence $s \in F^\diamond_R$. Thus $F \subseteq F^\diamond_R$.
(2) ⇒ (3) Let S be a δ-semiopen set, then there is $F \in \tau_\delta$ with $F \subseteq S \subseteq cl_\delta(F)$. By using Theorem 1.4 part (2) we have $S \subseteq cl_\delta(F) = F^\diamond_R$ and hence $S \subseteq F^\diamond_R \subseteq S^\diamond_R$.
(3) ⇒ (4) Let $F \in \tau_\delta$. Since every δ-open set is δ-semiopen, then by part (3) $F \subseteq F^\diamond_R$ and thus by Theorem 1.4 part (2) we get $cl_\delta(F) = F^\diamond_R$.
(4) ⇒ (5) Let S be δ-semiopen set. Then there is $F \in \tau_\delta$ with $F \subseteq S \subseteq F^\diamond_R$. Therefore, by Theorem 1.4 we get $cl_\delta(S) = cl_\delta(F) = F^\diamond_R \subseteq S^\diamond_R \subseteq cl_\delta(S)$ and hence $cl_\delta(S) = S^\diamond_R$.
(5) ⇒ (6) Let $F \subseteq Z$ with $s \in int_\delta(F)$. Then there is $D \in \tau_\delta$ with $s \in D \subseteq F$. Since every δ-open set is δ-semiopen, then by part (5) $D \subseteq cl_\delta(D) = D^\diamond_R \subseteq F^\diamond_R$ and hence $s \in int_\delta(F^\diamond_R)$. Therefore, $int_\delta(F) \subseteq int_\delta(F^\diamond_R)$
(6) ⇒ (1) Suppose that $F \in \mathcal{RC}(Z) \setminus \{Z\}$, then $\emptyset \neq F^c \in \mathcal{RO}(Z)$. Hence there is $s \in F^c$ with $F^c = int_\delta(F^c) \subseteq int_\delta[(F^c)^\diamond_R] \subseteq (F^c)^\diamond_R$ and so $(F^c)^\diamond_R \neq \emptyset$. By Corollary 1.8 we have $F \in \mathcal{P}$. Therefore, $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$. □

Proposition 4.2. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then:*

1. *if S is a regular baire set and $S^c \in \mathcal{P}$, then there is a nonempty set $F \in \mathcal{RO}(Z)$ with $(F - S)^c \cap (S - F)^c \notin \mathcal{P}$.*
2. *a set S is regular baire and $S^c \in \mathcal{P}$ iff there is a nonempty set $F \in \mathcal{RO}(Z)$ with $(F-S)^c \cap (S-F)^c \notin \mathcal{P}$ whenever $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$.*

Proof. (1) Assume that S is a regular baire set and $S^c \in \mathcal{P}$. Then there is $F \in \mathcal{RO}(X)$ with $(F - S)^c \cap (S - F)^c \notin \mathcal{P}$ and hence $[S - F]^c \notin \mathcal{P}$. Note that, if $F = \emptyset$, then $S^c \notin \mathcal{P}$ which is a contradiction.
(2) Assume that there is a nonempty set $F \in \mathcal{RO}(Z)$ with $(F-S)^c \cap (S-F)^c \notin \mathcal{P}$. Then $F^c = (S-L)^c \cap K^c$, where $L^c = [S - F]^c \notin \mathcal{P}$ and $K^c = [F - S]^c \notin \mathcal{P}$. Now, suppose that $S^c \notin \mathcal{P}$, then $(S - L)^c \notin \mathcal{P}$ and hence $F^c \notin \mathcal{P}$ by additivity, which is a contradiction since $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$. Therefore, S is a regular baire set with $S^c \in \mathcal{P}$ and also by using part (1) the proof is completed. □

Proposition 4.3. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$. If S is a regular baire set and $S^c \in \mathcal{P}$, then $\Phi_R(S) \cap int_\delta(S^\diamond_R) \neq \emptyset$.*

Proof. Let S be a regular baire set and $S^c \in \mathcal{P}$. Then by Proposition 4.2 part (1) there is a nonempty set $F \in \mathcal{RO}(Z)$ with $(F - S)^c \cap (S - F)^c \notin \mathcal{P}$. It follows that $F \subseteq F^\diamond_R = ((S - K) \cup L)^\diamond_R = S^\diamond_R$, where $K^c = [S - F]^c \notin \mathcal{P}$ and $L^c = [F - S]^c \notin \mathcal{P}$ by using Theorems [4.1 and 1.7]. Hence, $F \subseteq int_\delta(S^\diamond_R)$. Also, since $F \subseteq \Phi_R(F) = \Phi_R(S)$ by Corollary 2.6 and Theorem 2.4 part (10), then $F \subseteq \Phi_R(S) \cap int_\delta(S^\diamond_R)$. Therefore, $\Phi_R(S) \cap int_\delta(S^\diamond_R) \neq \emptyset$. □

Theorem 4.4. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$, then $\Phi_R(F) \subseteq F^\diamond_R$ for every subset F.*

Proof. Let $s \in \Phi_R(F)$ and $s \notin F^\diamond_R$. Then there is $D \in \mathcal{RO}(Z, s)$ with $[D \cap F]^c \notin \mathcal{P}$. Since $s \in \Phi_R(F)$, then by Theorem 2.3 part (2), $s \in \cup\{D \in \mathcal{RO}(Z) : [D - F]^c \notin \mathcal{P}\}$ and hence there is $S \in \mathcal{RO}(Z, s)$ with $[S - F]^c \notin \mathcal{P}$. It follows that $D \cap S \in \mathcal{RO}(Z, s)$ with $[D \cap S \cap F]^c \notin \mathcal{P}$ and $[(D \cap S) - F]^c \notin \mathcal{P}$ by heredity. Now, by finite additivity $[D \cap S]^c = [D \cap S \cap F]^c \cap [(D \cap S) - F]^c \notin \mathcal{P}$ which is a contradiction since $\mathcal{RC}(X) - \{Z\} \subseteq \mathcal{P}$. Therefore, $s \in F^\diamond_R$ and hence $\Phi_R(F) \subseteq F^\diamond_R$. □

Corollary 4.5. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$, then $\Phi_R(F) \subseteq cl_\delta(F^\diamond_R)$ for every subset F of Z.*

Theorem 4.6. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$, then $\Phi_R(F) \cap \Phi_R(Z-F) = \emptyset$ for every subset F of Z.*

Proof. Assume that $s \in \Phi_R(F) \cap \Phi_R(Z-F)$ for some $s \in Z$. Then there are $S, D \in \mathcal{RO}(Z,s)$ with $[S-F]^c \notin \mathcal{P}$ and $[D \cap F]^c \notin \mathcal{P}$ respectively. Thus, $[(S \cap D) - F]^c \notin \mathcal{P}$ and $[(S \cap D) \cap F]^c \notin \mathcal{P}$ hence $[S \cap D]^c \notin \mathcal{P}$ with $[S \cap D]^c \in \mathcal{RC}(Z)$. Now, since $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$, then $S \cap D = \emptyset$, which is a contradiction. Thus, $\Phi_R(F) \cap \Phi_R(Z-F) = \emptyset$. □

Corollary 4.7. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$, then $F^{\diamond}_R \cup (Z-F)^{\diamond}_R = Z$ for every subset F of Z.*

Theorem 4.8. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then the following properties are equivalent:*

1. *$\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$,*
2. *$\Phi_R(\emptyset) = \emptyset$,*
3. *if $F \in \mathcal{RC}(Z)$, then $\Phi_R(F) - F = \emptyset$,*
4. *if $K^c \notin \mathcal{P}$, then $\Phi_R(K) = \emptyset$.*

Proof. (1) ⇒ (2) Since $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$, then by Theorem 2.3 $\Phi_R(\emptyset) = \cup\{D \in \mathcal{RO}(Z) : D^c \notin \mathcal{P}\} = \emptyset$.
(2) ⇒ (3) Assume that $s \in \Phi_R(F) - F$. Then there is $S \in \mathcal{RO}(Z,s)$ such that $[S-F]^c \notin \mathcal{P}$ and $S - F \in \mathcal{RO}(Z,s)$. But $S - F \subseteq \cup\{D \in \mathcal{RO}(Z) : D^c \notin \mathcal{P}\} = \Phi_R(\emptyset)$ which implies that $\Phi_R(\emptyset) \neq \emptyset$. Hence $\Phi_R(F) - F = \emptyset$.
(3) ⇒ (4) Let $K^c \notin \mathcal{P}$ and since $\emptyset \in \mathcal{RC}(Z)$, then $\Phi_R(K) = \Phi_R(K \cup \emptyset) = \Phi_R(\emptyset) = \emptyset$.
(4) ⇒ (1) Suppose that $\mathcal{RC}(Z) - \{Z\} \nsubseteq \mathcal{P}$. Then there is a nonempty set $F \in \mathcal{RO}(Z)$ with $F^c \notin \mathcal{P}$ and by part (4) $\Phi_R(F) = \emptyset$. Since $F \in \mathcal{RO}(Z)$, by Corollary 2.6 we have $F \subseteq \Phi_R(F) = \emptyset$. This is a contradiction. Hence, $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$. □

A subset F in a primal topological space $(Z, \tau, \mathcal{P})$ is said to be $\mathcal{P}_R$-dense if $F^{\diamond}_R = Z$.

Proposition 4.9. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. Then for $s \in Z$, $Z - \{s\}$ is $\mathcal{P}_R$-dense iff $\Phi_R(\{s\}) = \emptyset$.*

Proof. The proof follows from the definition of $\mathcal{P}_R$-dense sets, since $\Phi_R(\{s\}) = Z - (Z - \{s\})^{\diamond}_R = \emptyset$ iff $Z = (Z - \{s\})^{\diamond}_R$. □

Theorem 4.10. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space. If $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$ and $F \subseteq Z$, then F is $\mathcal{P}_R$-dense iff F is dense in $\tau^{\diamond}_R$.*

Proof. Assume that F is $\mathcal{P}_R$-dense, then $F^{\diamond}_R = Z$ and $cl^{\diamond}_R(F) = F \cup F^{\diamond}_R = Z$. Therefore, F is dense in $\tau^{\diamond}_R$.
Conversely, let F is dense in $\tau^{\diamond}_R$. Then $cl^{\diamond}_R(F) = F \cup F^{\diamond}_R = Z$. To prove that $F^{\diamond}_R = Z$, let $s \in Z$ with $s \notin F^{\diamond}_R$. Then there is $W \in \mathcal{RO}(Z,s)$ with $[W \cap F]^c \notin \mathcal{P}$. Since $\mathcal{RC}(Z) \setminus \{Z\} \subseteq \mathcal{P}$, then $W^c \in \mathcal{P}$. Now we want to show $W^c \cup F \in \mathcal{P}$. If $W^c \cup F \notin \mathcal{P}$, then $[W \cap F]^c \cap [W^c \cup F] \notin \mathcal{P}$ and hence $W^c \cup [F^c \cap F] \notin \mathcal{P}$ and so $W^c \notin \mathcal{P}$, which is a contradiction. Therefore, $[W \cap F^c]^c \in \mathcal{P}$ and hence $W \cap F^c \neq \emptyset$. Let $z \in W \cap F^c$. Then, $z \notin F$ and also $z \notin F^{\diamond}_R$. Because if $z \in F^{\diamond}_R$, then $[W \cap F]^c \in \mathcal{P}$ which is a contrary to $[W \cap F]^c \notin \mathcal{P}$. Thus, $z \notin F \cup F^{\diamond}_R = cl^{\diamond}_R(F) = Z$, which is a contradiction. Therefore, we obtain $F^{\diamond}_R = Z$ and hence F is $\mathcal{P}_R$-dense. □

Proposition 4.11. *Let $(Z, \tau, \mathcal{P})$ be a primal topological space and $\mathcal{RC}(Z) - \{Z\} \subseteq \mathcal{P}$. Then, $\Phi_R(F) \neq \emptyset$ iff F contains the nonempty $\tau^{\diamond}_R$-interior.*

Proof. Let $\Phi_R(F) \neq \emptyset$. By Theorem 2.3 part (2), $\Phi_R(F) = \cup\{D \in \mathcal{RO}(Z) : [D-F]^c \notin \mathcal{P}\}$ and hence there is a nonempty set $D \in \mathcal{RO}(Z)$ with $[D-F]^c \notin \mathcal{P}$. Let $D - F = K$, where $K^c \notin \mathcal{P}$. Now $D \cap K^c \subseteq F$ and hence by Theorem 1.9, $D \cap K^c \in \tau^{\diamond}_R$ and so F contains the nonempty $\tau^{\diamond}_R$-interior.
Conversely, suppose that F contains the nonempty $\tau^{\diamond}_R$-interior. Hence there is a nonempty $D \in \mathcal{RO}(Z)$ and $K \notin \mathcal{P}$ with $D \cap K \subseteq F$. So $D - F \subseteq K^c$. Since $K \subseteq [D-F]^c \notin \mathcal{P}$. Hence, $\cup\{D \in \mathcal{RO}(Z) : [D-F]^c \notin \mathcal{P}\} = \Phi_R(F) \neq \emptyset$. □

The following examples illustrate the relations between the concepts:

Example 4.12. Let $Z = \{1,2,3\}$ with topology $\tau = \{\emptyset, Z, \{1\}, \{3\}, \{2,3\}, \{1,3\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. It is clear that $\mathcal{RO}(Z) = \tau_\delta = \{\emptyset, Z, \{1\}\}$ and as shown by the following table. If $F \subseteq Z$, then:

F	F^c	$F^\diamond_R$	$(Z-F)^\diamond_R$	$\Phi_R(F)$
$\emptyset$	Z	$\emptyset$	$\{2,3\}$	$\{1\}$
Z	$\emptyset$	$\{2,3\}$	$\emptyset$	Z
$\{1\}$	$\{2,3\}$	$\emptyset$	$\{2,3\}$	$\{1\}$
$\{2\}$	$\{1,3\}$	$\emptyset$	$\{2,3\}$	$\{1\}$
$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\emptyset$	Z
$\{1,2\}$	$\{3\}$	$\emptyset$	$\{2,3\}$	$\{1\}$
$\{1,3\}$	$\{2\}$	$\{2,3\}$	$\emptyset$	Z
$\{2,3\}$	$\{1\}$	$\{2,3\}$	$\emptyset$	Z

Example 4.13. Let $Z = \{1,2,3\}$ with topology $\tau = \{\emptyset, Z, \{1\}, \{3\}, \{2,3\}, \{1,3\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$. It is clear that $\mathcal{RO}(Z) = \tau_\delta = \{\emptyset, Z, \{1\}\}$ and as shown by the following table, if $F \subseteq Z$, then:

F	F^c	$F^\diamond_R$	$(Z-F)^\diamond_R$	$\Phi_R(F)$
$\emptyset$	Z	$\emptyset$	Z	$\emptyset$
Z	$\emptyset$	Z	$\emptyset$	Z
$\{1\}$	$\{2,3\}$	Z	$\{2,3\}$	$\{1\}$
$\{2\}$	$\{1,3\}$	$\{2,3\}$	Z	$\emptyset$
$\{3\}$	$\{1,2\}$	$\emptyset$	Z	$\emptyset$
$\{1,2\}$	$\{3\}$	Z	$\emptyset$	Z
$\{1,3\}$	$\{2\}$	Z	$\{2,3\}$	$\{1\}$
$\{2,3\}$	$\{1\}$	$\{2,3\}$	Z	$\emptyset$

Example 4.14. Consider $Z = \{1,2,3\}$ with topology $\tau = \tau_\delta = \{\emptyset, Z, \{1\}, \{2\}, \{1,2\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$. It is clear that $\mathcal{RO}(Z) = \{\emptyset, Z, \{1\}, \{2\}\}$ as shown by the following table. If $F \subseteq Z$, then:

F	F^c	$F^\diamond_R$	$(Z-F)^\diamond_R$	$\Phi_R(F)$
$\emptyset$	Z	$\emptyset$	Z	$\emptyset$
Z	$\emptyset$	Z	$\emptyset$	Z
$\{1\}$	$\{2,3\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$
$\{2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,3\}$	$\{2\}$
$\{3\}$	$\{1,2\}$	$\emptyset$	Z	$\emptyset$
$\{1,2\}$	$\{3\}$	Z	$\emptyset$	Z
$\{1,3\}$	$\{2\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$
$\{2,3\}$	$\{1\}$	$\{2,3\}$	$\{1,3\}$	$\{2\}$

Example 4.15. Let $(\mathbb{R}, \tau, \mathcal{P})$ be defined as follows: $D \in \tau$ iff $1 \in D$ or $D = \emptyset$. Moreover, $F \in \mathcal{P}$ iff $1 \notin F$. If $F \subseteq \mathbb{R}$, then we have two cases:

Case 1. $1 \notin F$. Then, $F^c \notin \mathcal{P}$ since $1 \in F^c$ and $\mathcal{RO}(\mathbb{R}) = \{\emptyset, \mathbb{R}\}$. Therefore, $F^\diamond_R = \emptyset$.

Case 2. $1 \in F$. Then, $F^c \in \mathcal{P}$ since $1 \notin F^c$ and $\mathcal{RO}(\mathbb{R}) = \{\emptyset, \mathbb{R}\}$. Therefore, $F^\diamond_R = \mathbb{R}$

Note that:

$$F^\diamond_R = \Phi_R(F) = \begin{cases} \emptyset & \text{if } \ 1 \notin F \\ \mathbb{R} & \text{if } \ 1 \in F \end{cases}$$

Author contributions

Both authors A. Rawshdeh and A. Al-Omari contributed equally to this work. Both authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

[1] S. Acharjee, M. Özkoç, F. Y. Issaka, Primal topological spaces, *Bol. Soc. Paran. Mat.* (3s.)v. 2025 (43) 1-9, http://dx.doi.org/10.5269/bspm.66792

[2] O. Alghamdi, A. Al-Omari, M. H. Alqahtani, Novel operators in the frame of primal topological spaces, *AIMS Mathematics*, **9** (2024), 25792–25808.

[3] M. Alotaibi, A. Alzahrani, Primal-A study of soft topological spaces and their applications, *Mathematics*, **9**(12) (2021), 1560.

[4] A. Bhattacharya, S. Choudhury, Neutrosophic soft set and its applications in decision-making, *Soft Computing*, **25**(9) (2021), 7071-7082.

[5] R. A. Mohammed, T. M. Al-Shami, Fuzzy soft sets and their applications in decision-making problems, *Information Sciences*, **582** (2022), 564-576.

[6] T. M. Al-Shami, Z. A. Ameen, R. Abu-Gdairi, A. Mhemdi, On primal soft topology, *Mathematics*, **11** (2023), 2329. https://doi.org/10.3390/math.11102329.

[7] F. Alsharari, H. Alohali, Y. Saber, F. Smarandache, An introduction to single-valued neutrosophic primal theory, *Symmetry*, **16**(4) (2024), 402. https://doi.org/10.3390/sym1604040.

[8] Z. A. Ameen, R. A. Mohammed, T. M. Al-shami, B. A. Asaad, Novel fuzzy topologies formed by fuzzy primal frameworks, *J. Intell. Fuzzy Sys.*, pre-press, 2024, 1–10. https://doi.org/10.3233/JIFS 238408.

[9] G. Choquet, Sur les notions de filtre et de grille, *Comptes Rendus Acad. Sci. Paris*, **224** (1947), 171–173.

[10] M. Hosny, Rough sets theory via new topological notions based on ideals and applications, *AIMS Mathematics*, **7** (2021), 869–902. https://doi.org/10.3934/math.2022052.

[11] R. A. Hosny, B. A. Asaad, A. A. Azzam, T. M. Al-shami, Various topologies generated from Ej-neighbourhoods via ideals, *Complexity*, **2021** (2021), 4149368. http://doi.org/10.1155/2021/4149368.

[12] D. Jankovic, T. R. Hamlett, New topologies from old via ideals, *Am. Math. Mon.*, **97** (1990), 295–310. https://doi.org/10.1080/00029890.1990.11995593.

[13] K. Kuratowski, *Topology*, American: Academic Press, 2014.

[14] J. A. Zadeh, The Topology Generalized fuzzy sets and their applications, *Journal of Fuzzy Logic and Intelligent Systems*. **12** (1) (2023), 1-10.

[15] S. Modak, Topology on grill-filter space and continuity, *Bol. Soc. Parana. Mat.*, **31** (2013), 219–230. http://doi.org/10.5269/bspm.v31i2.16603.

[16] S. Modak, Grill-filter space, *J. Indian Math. Soc.*, **80** (2013), 313–320.

[17] M. Özkoç, B. KÖstel. On the topology $\tau_R^\diamond$ of primal topological spaces, *AIMS Mathematics*, **9** (2024), 17171-17183. http://doi.org/10.3934/math.2024834.

[18] J.H. Park, B.Y. Lee and M.J. Son, On δ-semi-open sets in topological space, *J. Indian Acad. Math.*, **19** (1997), 59 - 67.

[19] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956.

[20] B. Roy, M. N. Mukherjee, On a typical topology induced by a grill, *Soochow J. Math.*, **33** (2007), 771–786.

[21] M. H. Stone, Applications of the theory of Boolean rings to general topology, *T. Am. Math. Soc.*, **41** (1937), 375-381.https://doi.org/10.2307/1989788.

[22] N. V. Velicko, H-Closed topological spaces, *Amer. Math. Soc. Transl.*, **78** (1968), 103–118.