Some Einstein Operations on Rough Neutrosophic Sets with their Properties

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Abstract

Algebraic operations, which include addition, subtraction, division, scalar multiplication, and exponentiation, are the fundamental mathematical operations utilised in decision-making analysis. When performing on numbers, the algebraic operations are commonly referred to as arithmetic operations. Another alternative for algebraic operations, known as Einstein operations, has gained recognition for its smooth approximation and utilisation of Archimedean norms. However, it is crucial to note that Einstein operations are not designed to effectively address issues of indeterminacy, uncertainty, and lower-upper approximation. Thus, this paper defines some rough neutrosophic-based Einstein operations known as RNS Einstein addition, RNS Einstein multiplication, RNS Einstein scalar multiplication, and RNS Einstein exponentiation. By adopting rough neutrosophic sets (RNS), which incorporate neutrosophic lower and upper approximations, the proposed RNS Einstein operations offer a practical approach for handling uncertain situations. Some examples are provided to demonstrate the applicability of the RNS Einstein operations. Several desirable properties related to the defined RNS Einstein operations are investigated. Finally, the proposed RNS Einstein operations are applied in solving multi-criteria decision-making problems within a rough neutrosophic environment.

Keywords: Einstein operations; Rough neutrosophic set; Rough Neutrosophic Set Einstein operations; Multi-criteria decision-making

1. Introduction

Over the years, various theories have been developed to address the challenges posed by uncertainty in decision-making and analysis. Zadeh [1] introduced fuzzy sets along with membership functions and logic. Later, Pawlak [2] introduced rough sets, which utilize upper and lower approximations of a set. In general, rough sets are recognized as a valuable method for clarifying vague data by employing a range of approximation levels, which include both lower and upper approximations. The lower approximation within Pawlak's approximation framework encompasses sets of elements that are part of the object. In contrast, the upper approximation includes sets of elements that may potentially be part of the object. It connected the ideas of fuzzy and rough sets, as a means to deal with uncertainty and imprecision in mathematics. To address the issue of ambiguous information, Smarandache [3] introduced the notion of a neutrosophic set (NS). An NS is composed of three membership functions: a falsity membership function, a truth membership function, and an indeterminacy membership function.

Neutrosophic set (NS) theory and rough set (RS) theory are valuable approaches for effectively handling the data that is incomplete, ambiguous, uncertain, or inaccurate. Broumi et al. [4] introduced the concept of rough neutrosophic sets (RNS), which represent a combination of intelligent structures that are NS and RS. They explain basic neutrosophic sets and their related operations in a clear and detailed way, which helps people, understand these ideas more easily. In recent years, researchers have dedicated their efforts to the fusion of neutrosophic sets with rough sets. Alias et al. [5] proposed the rough neutrosophic multiset, which combines a rough set

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with single-valued neutrosophic multisets theory. Jin et al. [6] explored the theory and topological structures of rough neutrosophic sets, which are constructed from rough sets and neutrosophic sets. It introduces algebraic concepts, such as operations and identity relations, and discusses the properties of inverse, reflexive, symmetric, and transitive relations in rough neutrosophic sets. The study investigated the relationship between patient data, symptoms, and illness using the roughness-Cosine similarity score. Martina and Deepa [7] integrated rough set theory with neutrosophic matrix theory, leading to the introduction of the rough neutrosophic matrix.

Various of RNS based uncertainty theories have been proposed in these include the rough neutrosophic set like multigranulation rough neutrosophic set Bo et al. [8], hesitant fuzzy rough neutrosophic set Zhao and Zhang [9], VIKOR rough neutrosophic set Rogulj et al. [10], rough hesitant bipolar neutrosophic set Kumari and Thirucheran [11] rough pentapartitioned neutrosophic set Das et al. [12], and many more. The application of a rough neutrosophic set in solving problems, especially involving multiple criteria, has been widely investigated. For example, Samuel and Narmadhagnanam [13] employed a rough neutrosophic set to diagnose the disease affecting the patient. Donbosco and Ganesan [14] used rough neutrosophic for selecting the best building construction site. Praba et al. [15] created a double-bound rough neutrosophic set to detect facial expressions with and without masks. Recently, Myvizh [16] used a rough neutrosophic matric to decide on the finest hospital in the local area.

The exploration of rough neutrosophic sets in the relevant study is commendable; however, it is noteworthy that the current data analysis operations are solely reliant on algebraic operations. For example, Shazib Hameed et al. [17] discussed algebraic properties in a neutrosophic environment. Then, Abdel-Basset and Mohamed [18] proposed a rough neutrosophic set with the proposed algebraic algorithm, which aims to enhance the quality of services and decision-making in smart cities. Moreover, rough neutrosophic sets employ algebraic operations along with a distance-based similarity measure. It is clear that the other operations mentioned earlier rely on the fundamental algebraic product and sum.

Nevertheless, these are not the sole methods available to denote the intersection and union of RNS. Besides algebraic operations, there are more operations such as Frank operation, Dombi operation, and others, but this study is interested in using Einstein operations. It is necessary to explore and develop new operations or approaches that are better tailored to the specific challenges presented by the upper and lower approximation spaces environment. While algebraic methods do not consider t-norms and t-conorms, the absence of these norms may limit flexibility in certain scenarios. However, the simplicity and effectiveness of the Einstein operation make it a valuable tool for many practical applications. The Einstein operation consists of the Einstein product and Einstein sum, both serving as t-norm and t-conorm operations. Einstein operation can leverage the operation without the need for complex additional parameters, streamlining the computational process. Khan et al. [19] proposed to handle the air pollution model by applying Einstein operations with a neutrosophic cubic environment. Other than that, Einstein's operation can also be applied with neutrosophic enthalpy for multi-criteria decision-making Ye et al. [20]. This operation enables the flexible blending of uncertain or imprecise data, featuring distinctive examples of the Archimedean t-norm and t-conorm, celebrated for their continuous nature, strict monotonicity, and adherence to the Archimedean property Farid and Riaz [21].

Past research on the Einstein operator has given positive and successful outcomes. For instance, Einstein under bipolar neutrosophic environment Jamil et al. [22], single-valued neutrosophic hesitant fuzzy Einstein Kamran et al. [23], Einstein operation laws of spherical fuzzy sets Ajay et al. [24], neutrosophic triangular fuzzy number under Einstein aggregation operator Li et al. [25], interval-valued bipolar neutrosophic Einstein fuzzy aggregation operator Fahmi et al. [26] and to mention a few. There are the particular functions in the Einstein operator, such as it can handle a wide range of data types and relationships, making it adaptable to various scenarios, also the Einstein operator is often easy to interpret, aiding decision-making processes in any environment and can be applied in diverse fields such as decision analysis, fuzzy logic, and artificial intelligence. This study aims to define the neutrosophic rough Einstein operations to cover this wide range. To achieve this objective, the paper is structured as follows. In Section 2, we present some definitions of rough neutrosophic sets and Einstein operations. In Section 3, we propose Einstein operations for RNS, and in Section 4, the detailed properties of RNS based on Einstein operations is discussed. Section 5 presents a numerical example to demonstrate the usage of the proposed rough neutrosophic Einstein operations. Lastly, Section 6 presents the conclusion of the study.

2. Preliminaries

Section 2 presents the essential definitions used in this study.

Definition 2.1. ([2]) Let P be an equivalence relation on the universal set Z. The collection of all equivalence classes of Z under P is called Pawlak approximation space and it is defined as $R = Z/P$. Let X be a subset of Z. Let $\underline{R}(x)$ and $\overline{R}(x)$ be a lower and upper approximation of X in R, which are defined as follows:

$$\underline{R}(x) = \left\{ x \in Z \middle| [x]_P \subseteq X \right\}$$
$$\overline{R}(x) = \left\{ x \in Z \middle| [x]_P \cap X \neq \varnothing \right\}$$

Where $[x]_P$ denotes the equivalence class of P containing an element a. The sets $P(x) = \left(\underline{P}(x), \overline{P}(x) \right)$ is called the rough set of X in R.

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Definition 2.2. ([27]) Let Z be a non-null set and P be an equivalence relation on Z. Let R be a neutrosophic set in Z with the membership function T_R , indeterminacy function I_R and non-membership function F_R. The lower and upper approximations of R in the approximation (Z,P) denoted $\underline{N}(R)$ by $\overline{N}(R)$ and respectively referred as:

$$\underline{N}(R)=\left\{< x, T_{\underline{N}(R)}(x), I_{\underline{N}(R)}(x), F_{\underline{N}(R)}(x) >: Z \in [x]_P, x \in Z\right\}$$
$$\overline{N}(R)=\left\{< x, T_{\overline{N}(R)}(x), I_{\overline{N}(R)}(x), F_{\overline{N}(R)}(x) >: Z \in [x]_P, x \in Z\right\}$$

where,

$$T_{\underline{N}(R)}(x)=\wedge_{z\in[x]_P} T_R(Z), I_{\underline{N}(R)}(x)=\vee_{z\in[x]_P} I_R(Z), F_{\underline{N}(R)}(x)=\vee_{z\in[x]_P} F_R(Z),$$
$$T_{\overline{N}(R)}(x)=\vee_{z\in[x]_P} T_R(Z), I_{\overline{N}(R)}(x)=\wedge_{z\in[x]_P} I_R(Z), F_{\overline{N}(R)}(x)=\wedge_{z\in[x]_P} F_R(Z),$$

such that,

$$0\le T_{\underline{N}(R)}(x)+I_{\underline{N}(R)}(x)+F_{\underline{N}(R)}(x)\le 3$$
$$0\le T_{\overline{N}(R)}(x)+I_{\overline{N}(R)}(x)+F_{\overline{N}(R)}(x)\le 3$$

The notations $\vee$ and $\wedge$ indicate “max“ and “min” operators respectively, $T_R(x)$, $I_R(x)$ and $F_R(x)$ are the membership, indeterminacy, and non-membership functions of on neutrosophic set R. It becomes evident that $\underline{N}(R)$ and $\overline{N}(R)$ are two neutrosophic sets in Z. The pair $\left(\underline{N}(R),\overline{N}(R)\right)$ is called the rough neutrosophic set in (Z,P).

Definition 2.3. ([15]) The t-operators are essentially union and intersection operators in fuzzy sets theory, denoted by t-conorm $\left(\Gamma^*\right)$ and t-norm $\left(\Gamma\right)$, respectively. T-operators play an important role in fuzzy theory and its application.

Definition 2.4. ([15]) $\Gamma^*:[0,1]\times[0,1]\to[0,1]$ is called t-conorm if it meets the following axioms:

Axioms 1. $\Gamma^*(1,u)=1$ and $\Gamma^*(0,u)=0$;

Axioms 2. $\Gamma^*(u,v)=\Gamma^*(v,u)$ for all a and b;

Axioms 3. $\Gamma^*\left(u,\Gamma^*(v,w)\right)=\Gamma^*\left(\Gamma^*(u,v),w\right)$ for all a,b and c;

Axioms 4. If $u\le u'$ and $v\le v'$, then $\Gamma^*(u,v)\le\Gamma^*(u',v')$.

The most recognized t-conorms include:

1. The default t-conorm: $\Gamma^*_{\max}(u,v)=\max(u,v)$.

2. The bounded t-conorm: $\Gamma^*_{bounded}(u,v)=\min(1,u+v)$.

3. The algebraic t-conorm: $\Gamma^*_{algebraic}(u,v)=u+v-uv$.

Definition 2.5. ([15]) $\Gamma:[0,1]\times[0,1]\to[0,1]$ is called t-norm if it meets the following axioms:

Axioms 5. $\Gamma(1,u)=u$ and $\Gamma(0,u)=0$;

Axioms 6. $\Gamma(u,v)=\Gamma(v,u)$ for all a and b;

Axioms 7. $\Gamma\left(u,\Gamma(v,w)\right)=\Gamma\left(\Gamma(u,v),w\right)$ for all a,b and c;

Axioms 8. If $u\le u'$ and $v\le v'$, then $\Gamma(u,v)\le\Gamma(u',v')$.

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The most recognized t-norms include:

1. The default t-norm: $\Gamma_{\min}(u,v)=\min(u,v)$.

2. The bounded t-norm: $\Gamma_{bounded}(u,v)=\max(0,u+v-1)$.

3. The algebraic t-norm: $\Gamma_{algebraic}(u,v)=uv$.

If $\Gamma^*(u,v)$ and $\Gamma(u,v)$ are continuous and $\Gamma^*(u,v)>u, \Gamma(u,u)<u$, then Γ^* and Γ are termed Archimedean t-conorm and t-norm, respectively. Any pair of dual t-conorm (Γ^*) and t-norm (Γ) can be utilized. It is known that t-norm and t-conorm operators satisfy the conditions of conjunction and disjunction operators, respectively. However, algebraic operations, such as algebraic sums and products, are not unique and may correspond to union and intersection. The families of t-conorms and t-norms cover a wide range, corresponding to unions and intersections. Among these, the Einstein sum and Einstein product are excellent choices as they provide smooth approximations similar to the algebraic sum and product, respectively. The Einstein sum $\oplus_E$ and the Einstein product $\otimes_E$ serve as examples of a t-conorm and a t-norm, respectively:

$$\Gamma^*_E(u,v)=\frac{u+v}{1+uv}$$

$$\Gamma_E(u,v)=\frac{uv}{1+(1-u)(1-v)}.$$

Definition 2.6. ([28]) the cosine function for criteria weights of a rough neutrosophic number $N(R_{ij})=\langle(\underline{T}_{ij},\underline{I}_{ij},\underline{F}_{ij}),(\overline{T}_{ij},\overline{I}_{ij},\overline{F}_{ij})\rangle (i=1,2,\ldots,m: j=1,2,\ldots,n)$ is defined as follows:

$$COS_j[N(R)]=\frac{1}{n}\sum_{i=1}^{n}\cos\frac{\pi}{2}\left(\frac{\left|4-2\left(\underline{T}_{ij}+\overline{T}_{ij}\right)+\underline{I}_{ij}+\overline{I}_{ij}+\underline{F}_{ij}+\overline{F}_{ij}\right|}{8}\right)$$

$$w_j=\frac{COS_j[N(R)]}{\sum_j^n COS_j[N(R)]}; j=1,2,\ldots,n \;\&\; \sum_{j=1}^{n}w_j=1.$$

Definition 2.7. ([28]) Let that $N(R)=\langle(\underline{N}(R)),(\overline{N}(R))\rangle=\langle(\underline{T},\underline{I},\underline{F}),(\overline{T},\overline{I},\overline{F})\rangle$ be a rough neutrosophic number. The score and accuracy functions of $N(R)$ such that $S:N(R)\to[0,1]$ and $A:N(R)\to[-1,1]$ are defined as follows:

$$\text{Score, } S[N(R)]=\frac{4+\underline{T}+\overline{T}-\underline{I}-\overline{I}-\underline{F}+\overline{F}}{6}$$

$$\text{Accuracy, } A[N(R)]=\frac{\underline{T}+\overline{T}-\underline{F}+\overline{F}}{2}.$$

3. Operations on Rough Neutrosophic Set

This section consists of two parts: introduce several new operations on RNS and examples of the operations, which will be further utilized throughout the article.

3.1. Einstein Sum, Product, and Scalar Multiplication

Considering both the dual t-conorm (Γ^*) and t-norm (Γ), the Einstein operations of union, intersection, sum, product, and scalar multiplication are defined within the RNS framework. Leveraging these definitions, a significant outcome of Einstein exponential multiplication is established.

Definition 3.1. Let $A=N(R_1)=\langle(\underline{T}_1,\underline{I}_1,\underline{F}_1),(\overline{T}_1,\overline{I}_1,\overline{F}_1)\rangle$ and $B=N(R_2)=\langle(\underline{T}_2,\underline{I}_2,\underline{F}_2),(\overline{T}_2,\overline{I}_2,\overline{F}_2)\rangle$ be rough neutrosophic sets. Then, the RNS Einstein operations are defined as

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a. RNS Einstein sum

$$A \oplus_E B = \left\langle \left(\frac{\underline{T}_1 + \underline{T}_2}{1 + \left(\left(\underline{T}_1\right)\left(\underline{T}_2\right) \right)}, \frac{\left(\underline{I}_1\right)\left(\underline{I}_2\right)}{1 + \left(1 - \underline{I}_1\right)\left(1 - \underline{I}_2\right)}, \frac{\left(\underline{F}_1\right)\left(\underline{F}_2\right)}{1 + \left(1 - \underline{F}_1\right)\left(1 - \underline{F}_2\right)} \right), \left(\frac{\overline{T}_1 + \overline{T}_2}{1 + \left(\left(\overline{T}_1\right)\left(\overline{T}_2\right) \right)}, \frac{\left(\overline{I}_1\right)\left(\overline{I}_2\right)}{1 + \left(1 - \overline{I}_1\right)\left(1 - \overline{I}_2\right)}, \frac{\left(\overline{F}_1\right)\left(\overline{F}_2\right)}{1 + \left(1 - \overline{F}_1\right)\left(1 - \overline{F}_2\right)} \right) \right\rangle$$

b. RNS Einstein product

$$A \otimes_E B = \left\langle \left(\frac{\left(\underline{T}_1\right)\left(\underline{T}_2\right)}{1 + \left(1 - \underline{T}_1\right)\left(1 - \underline{T}_2\right)}, \frac{\underline{I}_1 + \underline{I}_2}{1 + \left(\left(\underline{I}_1\right)\left(\underline{I}_2\right) \right)}, \frac{\underline{F}_1 + \underline{F}_2}{1 + \left(\left(\underline{F}_1\right)\left(\underline{F}_2\right) \right)} \right), \left(\frac{\left(\overline{T}_1\right)\left(\overline{T}_2\right)}{1 + \left(1 - \overline{T}_1\right)\left(1 - \overline{T}_2\right)}, \frac{\overline{I}_1 + \overline{I}_2}{1 + \left(\left(\overline{I}_1\right)\left(\overline{I}_2\right) \right)}, \frac{\overline{F}_1 + \overline{F}_2}{1 + \left(\left(\overline{F}_1\right)\left(\overline{F}_2\right) \right)} \right) \right\rangle$$

c. RNS Einstein scalar multiplication

$$\lambda_E A = \left\langle \left(\frac{\left(1 + \underline{T}_1\right)^{\lambda} - \left(1 - \underline{T}_1\right)^{\lambda}}{\left(1 + \underline{T}_1\right)^{\lambda} + \left(1 - \underline{T}_1\right)^{\lambda}}, \frac{2\left(\underline{I}_1\right)^{\lambda}}{\left(2 - \underline{I}_1\right)^{\lambda}\left(\underline{I}_1\right)^{\lambda}}, \frac{2\left(\underline{F}_1\right)^{\lambda}}{\left(2 - \underline{F}_1\right)^{\lambda}\left(\underline{F}_1\right)^{\lambda}} \right), \left(\frac{\left(1 + \overline{T}_1\right)^{\lambda} - \left(1 - \overline{T}_1\right)^{\lambda}}{\left(1 + \overline{T}_1\right)^{\lambda} + \left(1 - \overline{T}_1\right)^{\lambda}}, \frac{2\left(\overline{I}_1\right)^{\lambda}}{\left(2 - \overline{I}_1\right)^{\lambda}\left(\overline{I}_1\right)^{\lambda}}, \frac{2\left(\overline{F}_1\right)^{\lambda}}{\left(2 - \overline{F}_1\right)^{\lambda}\left(\overline{F}_1\right)^{\lambda}} \right) \right\rangle$$

After defining scalar multiplication over the RNS, the following results are established, concerning Einstein exponential multiplication on RNS values.

Theorem 3.1. *Let* $A = N(R_1) = \left\langle \left(\underline{T}_1, \underline{I}_1, \underline{F}_1\right), \left(\overline{T}_1, \overline{I}_1, \overline{F}_1\right) \right\rangle$ *and* $B = N(R_2) = \left\langle \left(\underline{T}_2, \underline{I}_2, \underline{F}_2\right), \left(\overline{T}_2, \overline{I}_2, \overline{F}_2\right) \right\rangle$ *be a rough neutrosophic set value and* λ_E *be a positive real number* $\left(\lambda_E > 0\right)$. *Then, the exponential operation defined by*

d. RNS Einstein exponential multiplication

$$A^{E^{\lambda}} = \left\langle \left(\frac{2\left(\underline{T}_1\right)^{\lambda}}{\left(2 - \underline{T}_1\right)^{\lambda} + \left(\underline{T}_1\right)^{\lambda}}, \frac{\left(1 + \underline{I}_1\right)^{\lambda} - \left(1 - \underline{I}_1\right)^{\lambda}}{\left(1 + \underline{I}_1\right)^{\lambda} + \left(1 - \underline{I}_1\right)^{\lambda}}, \frac{\left(1 + \underline{F}_1\right)^{\lambda} - \left(1 - \underline{F}_1\right)^{\lambda}}{\left(1 + \underline{F}_1\right)^{\lambda} + \left(1 - \underline{F}_1\right)^{\lambda}} \right), \left(\frac{2\left(\overline{T}_1\right)^{\lambda}}{\left(2 - \overline{T}_1\right)^{\lambda}\left(\overline{T}_1\right)^{\lambda}}, \frac{\left(1 + \overline{I}_1\right)^{\lambda} - \left(1 - \overline{I}_1\right)^{\lambda}}{\left(1 + \overline{I}_1\right)^{\lambda} + \left(1 - \overline{I}_1\right)^{\lambda}}, \frac{\left(1 + \overline{F}_1\right)^{\lambda} - \left(1 - \overline{F}_1\right)^{\lambda}}{\left(1 + \overline{F}_1\right)^{\lambda} + \left(1 - \overline{F}_1\right)^{\lambda}} \right) \right\rangle$$

where $A^{E^{\lambda}} = A \otimes_E A \otimes_E \ldots \otimes_E A\left(\lambda - times\right)$.

3.2. Example Rough Neutrosophic Set Einstein Operations

Some examples to demonstrate the defined RNS Einstein operations are as follows:

Let $N(R_1) = \left\langle \left(0.3, 0.4, 0.7\right), \left(0.4, 0.2, 0.5\right) \right\rangle$ and $N(R_2) = \left\langle \left(0.2, 0.5, 0.3\right), \left(0.4, 0.2, 0.3\right) \right\rangle$ be two RNS and $\lambda = 2$. Based on Definition 3.1, (a) to (d) the RNS Einstein sum, RNS Einstein product, RNS Einstein scalar multiplication, and RNS Einstein exponentiation on $N\left(R_1\right)$ and $N\left(R_2\right)$ are illustrated as follows

(a) Based on the RNS Einstein sum, we get

$$A \oplus_E B = \left\langle \left(\frac{0.3 + 0.2}{1 + \left((0.3)(0.2) \right)}, \frac{(0.4)(0.5)}{1 + (1 - 0.4)(1 - 0.5)}, \frac{(0.7)(0.3)}{1 + (1 - 0.7)(1 - 0.3)} \right), \left(\frac{0.4 + 0.4}{1 + \left((0.4)(0.4) \right)}, \frac{(0.2)(0.2)}{1 + (1 - 0.2)(1 - 0.2)}, \frac{(0.5)(0.3)}{1 + (0.5)(1 - 0.3)} \right) \right\rangle$$

$$= \left\langle \left(0.47, 0.15, 0.17\right), \left(0.68, 0.02, 0.11\right) \right\rangle.$$

(b) Based on the definition of RNS Einstein product as follows

$$A \otimes_E B = \left\langle \left(\frac{(0.3)(0.2)}{1 + (1 - 0.3)(1 - 0.2)}, \frac{0.4 + 0.5}{1 + \left((0.4)(0.5) \right)}, \frac{0.7 + 0.3}{1 + \left((0.7)(0.3) \right)} \right), \left(\frac{(0.4)(0.4)}{1 + (1 - 0.4)(1 - 0.4)}, \frac{0.2 + 0.2}{1 + \left((0.2)(0.2) \right)}, \frac{0.5 + 0.3}{1 + \left((0.5)(0.3) \right)} \right) \right\rangle$$

$$= \left\langle \left(0.03, 0.75, 0.82\right), \left(0.11, 0.38, 0.69\right) \right\rangle.$$

(c) Example for RNS Einstein scalar multiplication as follows

$$\lambda\ (A) = \left\langle \left(\frac{(1 + 0.3)^2 - (1 - 0.3)^2}{(1 + 0.3)^2 + (1 - 0.3)^2}, \frac{2(0.4)^2}{(2 - 0.4)^2 (0.4)^2}, \frac{2(0.7)^2}{(2 - 0.7)^2 (0.7)^2} \right), \left(\frac{(1 + 0.4)^2 - (1 - 0.4)^2}{(1 + 0.4)^2 + (1 - 0.4)^2}, \frac{2(0.2)^2}{(2 - 0.2)^2 (0.2)^2}, \frac{2(0.5)^2}{(2 - 0.5)^2 (0.5)^2} \right) \right\rangle$$

$$= \left\langle \left(0.55, 0.11, 0.44\right), \left(0.68, 0.02, 0.20\right) \right\rangle.$$

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(d) Based on Theorem 3.1 RNS Einstein exponentiation, we get

$$A^{\lambda}=\left\langle\left(\frac{2(0.3)^2}{(2-0.3)^2+(0.3)^2},\frac{(1+0.4)^2-(1-0.4)^2}{(1+0.4)^2+(1-0.4)^2},\frac{(1+0.7)^2-(1-0.7)^2}{(1+0.7)^2+(1-0.7)^2}\right),\left(\frac{2(0.4)^2}{(2-0.4)^2(0.4)^2},\frac{(1+0.2)^2-(1-0.2)^2}{(1+0.2)^2+(1-0.2)^2},\frac{(1+0.5)^2-(1-0.5)^2}{(1+0.5)^2+(1-0.5)^2}\right)\right\rangle$$

$$=\langle(0.30,0.40,0.70),(0.40,0.20,0.50)\rangle.$$

4. Properties of Rough Neutrosophic Set Based on Einstein Operations

In this section, we investigate some properties of the proposed RNS based on Einstein operations. The corresponding operational relations are derived and examined.

Proposition 4.1. Let $A=N(R_1)=\langle(\underline{T}_1,\underline{I}_1,\underline{F}_1),(\overline{T}_1,\overline{I}_1,\overline{F}_1)\rangle$, $B=N(R_2)=\langle(\underline{T}_2,\underline{I}_2,\underline{F}_2),(\overline{T}_2,\overline{I}_2,\overline{F}_2)\rangle$ and $C=N(R_3)=\langle(\underline{T}_3,\underline{I}_3,\underline{F}_3),(\overline{T}_3,\overline{I}_3,\overline{F}_3)\rangle$ be rough neutrosophic sets. Then, the RNS Einstein operations are defined as

(a) $A\oplus_E B=B\oplus_E A$

(b) $A\otimes_E B=B\otimes_E A$

(c) $\lambda(A\oplus_E B)=\lambda A\oplus_E \lambda B$

(d) $(A\otimes_E B)^{\lambda}=A^{\lambda}\otimes_E B^{\lambda}$

(e) $\lambda_1 A\oplus_E \lambda_2 A=(\lambda_1\oplus_E\lambda_2)A$

(f) $A^{\lambda_1}\otimes_E A^{\lambda_2}=A^{\lambda_1+\lambda_2}$

(g) $(A\oplus_E B)\oplus_E C=A\oplus_E(B\oplus_E C)$

(h) $(A\otimes_E B)\otimes_E C=A\otimes_E(B\otimes_E C)$

Proof. For three RNS A,B and C by Proposition 4.1 are presented as follows

(a) $A\oplus_E B=B\oplus_E A$

$$=\langle(\underline{T}_1,\underline{I}_1,\underline{F}_1),(\overline{T}_1,\overline{I}_1,\overline{F}_1)\rangle\oplus_E\langle(\underline{T}_2,\underline{I}_2,\underline{F}_2),(\overline{T}_2,\overline{I}_2,\overline{F}_2)\rangle$$

$$=\left\langle\left(\frac{\underline{T}_1+\underline{T}_2}{1+((\underline{T}_1)(\underline{T}_2))},\frac{(\underline{I}_1)(\underline{I}_2)}{1+(1-\underline{I}_1)(1-\underline{I}_2)},\frac{(\underline{F}_1)(\underline{F}_2)}{1+(1-\underline{F}_1)(1-\underline{F}_2)}\right),\left(\frac{\overline{T}_1+\overline{T}_2}{1+((\overline{T}_1)(\overline{T}_2))},\frac{(\overline{I}_1)(\overline{I}_2)}{1+(1-\overline{I}_1)(1-\overline{I}_2)},\frac{(\overline{F}_1)(\overline{F}_2)}{1+(1-\overline{F}_1)(1-\overline{F}_2)}\right)\right\rangle$$

$$=\left\langle\left(\frac{\underline{T}_2+\underline{T}_1}{1+((\underline{T}_2)(\underline{T}_1))},\frac{(\underline{I}_2)(\underline{I}_1)}{1+(1-\underline{I}_2)(1-\underline{I}_1)},\frac{(\underline{F}_2)(\underline{F}_1)}{1+(1-\underline{F}_2)(1-\underline{F}_1)}\right),\left(\frac{\overline{T}_2+\overline{T}_1}{1+((\overline{T}_2)(\overline{T}_1))},\frac{(\overline{I}_2)(\overline{I}_1)}{1+(1-\overline{I}_2)(1-\overline{I}_1)},\frac{(\overline{F}_2)(\overline{F}_1)}{1+(1-\overline{F}_2)(1-\overline{F}_1)}\right)\right\rangle$$

$$=B\oplus_E A$$ □

(b) $A\otimes_E B=B\otimes_E A$

$$=\langle(\underline{T}_1,\underline{I}_1,\underline{F}_1),(\overline{T}_1,\overline{I}_1,\overline{F}_1)\rangle\otimes_E\langle(\underline{T}_2,\underline{I}_2,\underline{F}_2),(\overline{T}_2,\overline{I}_2,\overline{F}_2)\rangle$$

$$=\left\langle\left(\frac{(\underline{T}_1)(\underline{T}_2)}{1+(1-\underline{T}_1)(1-\underline{T}_2)},\frac{\underline{I}_1+\underline{I}_2}{1+((\underline{I}_1)(\underline{I}_2))},\frac{\underline{F}_1+\underline{F}_2}{1+((\underline{F}_1)(\underline{F}_2))}\right),\left(\frac{(\overline{T}_1)(\overline{T}_2)}{1+(1-\overline{T}_1)(1-\overline{T}_2)},\frac{\overline{I}_1+\overline{I}_2}{1+((\overline{I}_1)(\overline{I}_2))},\frac{\overline{F}_1+\overline{F}_2}{1+((\overline{F}_1)(\overline{F}_2))}\right)\right\rangle$$

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$$=\left\langle\left(\frac{(\underline{T}_2)(\underline{T}_1)}{1+(1-\underline{T}_2)(1-\underline{T}_1)},\frac{\underline{I}_2+\underline{I}_1}{1+((\underline{I}_2)(\underline{I}_1))},\frac{\underline{F}_2+\underline{F}_1}{1+((\underline{F}_2)(\underline{F}_1))}\right),\left(\frac{(\overline{T}_2)(\overline{T}_1)}{1+(1-\overline{T}_2)(1-\overline{T}_1)},\frac{\overline{I}_2+\overline{I}_1}{1+((\overline{I}_2)(\overline{I}_1))},\frac{\overline{F}_2+\overline{F}_1}{1+((\overline{F}_2)(\overline{F}_1))}\right)\right\rangle$$

$$=B\otimes_E A$$ ∎

(c) $\lambda(A\oplus_E B)=\lambda A\oplus_E \lambda B$

Note that, $\lambda(A\oplus_E B)$ is

$$=\lambda\left(\left\langle\left(\frac{\underline{T}_1+\underline{T}_2}{1+((\underline{T}_1)(\underline{T}_2))},\frac{(\underline{I}_1)(\underline{I}_2)}{1+(1-\underline{I}_1)(1-\underline{I}_2)},\frac{(\underline{F}_1)(\underline{F}_2)}{1+(1-\underline{F}_1)(1-\underline{F}_2)}\right),\left(\frac{\overline{T}_1+\overline{T}_2}{1+((\overline{T}_1)(\overline{T}_2))},\frac{(\overline{I}_1)(\overline{I}_2)}{1+(1-\overline{I}_1)(1-\overline{I}_2)},\frac{(\overline{F}_1)(\overline{F}_2)}{1+(1-\overline{F}_1)(1-\overline{F}_2)}\right)\right\rangle\right)$$

$$=\left\langle\left(\frac{\left(1+\frac{\underline{T}_1+\underline{T}_2}{1+((\underline{T}_1)(\underline{T}_2))}\right)^{\lambda}-\left(1-\frac{\underline{T}_1+\underline{T}_2}{1+((\underline{T}_1)(\underline{T}_2))}\right)^{\lambda}}{\left(1+\frac{\underline{T}_1+\underline{T}_2}{1+((\underline{T}_1)(\underline{T}_2))}\right)^{\lambda}+\left(1-\frac{\underline{T}_1+\underline{T}_2}{1+((\underline{T}_1)(\underline{T}_2))}\right)^{\lambda}},\frac{2\left(\frac{(\underline{I}_1)(\underline{I}_2)}{1+(1-\underline{I}_1)(1-\underline{I}_2)}\right)^{\lambda}}{\left(2-\frac{(\underline{I}_1)(\underline{I}_2)}{1+(1-\underline{I}_1)(1-\underline{I}_2)}\right)^{\lambda}+\left(\frac{(\underline{I}_1)(\underline{I}_2)}{1+(1-\underline{I}_1)(1-\underline{I}_2)}\right)^{\lambda}},\right.\right.$$

$$\left.\frac{2\left(\frac{(\underline{F}_1)(\underline{F}_2)}{1+(1-\underline{F}_1)(1-\underline{F}_2)}\right)^{\lambda}}{\left(2-\frac{(\underline{F}_1)(\underline{F}_2)}{1+(1-\underline{F}_1)(1-\underline{F}_2)}\right)^{\lambda}+\left(\frac{(\underline{F}_1)(\underline{F}_2)}{1+(1-\underline{F}_1)(1-\underline{F}_2)}\right)^{\lambda}}\right),\left(\frac{\left(1+\frac{\overline{T}_1+\overline{T}_2}{1+((\overline{T}_1)(\overline{T}_2))}\right)^{\lambda}-\left(1-\frac{\overline{T}_1+\overline{T}_2}{1+((\overline{T}_1)(\overline{T}_2))}\right)^{\lambda}}{\left(1+\frac{\overline{T}_1+\overline{T}_2}{1+((\overline{T}_1)(\overline{T}_2))}\right)^{\lambda}+\left(1-\frac{\overline{T}_1+\overline{T}_2}{1+((\overline{T}_1)(\overline{T}_2))}\right)^{\lambda}},\right.$$

$$\left.\left.\frac{2\left(\frac{(\overline{I}_1)(\overline{I}_2)}{1+(1-\overline{I}_1)(1-\overline{I}_2)}\right)^{\lambda}}{\left(2-\frac{(\overline{I}_1)(\overline{I}_2)}{1+(1-\overline{I}_1)(1-\overline{I}_2)}\right)^{\lambda}+\left(\frac{(\overline{I}_1)(\overline{I}_2)}{1+(1-\overline{I}_1)(1-\overline{I}_2)}\right)^{\lambda}},\frac{2\left(\frac{(\overline{F}_1)(\overline{F}_2)}{1+(1-\overline{F}_1)(1-\overline{F}_2)}\right)^{\lambda}}{\left(2-\frac{(\overline{F}_1)(\overline{F}_2)}{1+(1-\overline{F}_1)(1-\overline{F}_2)}\right)^{\lambda}+\left(\frac{(\overline{F}_1)(\overline{F}_2)}{1+(1-\overline{F}_1)(1-\overline{F}_2)}\right)^{\lambda}}\right)\right\rangle$$

$$=\left\langle\left(\frac{(1+\underline{T}_1)^{\lambda}(1+\underline{T}_2)^{\lambda}-(1-\underline{T}_1)^{\lambda}(1-\underline{T}_2)^{\lambda}}{(1+\underline{T}_1)^{\lambda}(1+\underline{T}_2)^{\lambda}+(1-\underline{T}_1)^{\lambda}(1-\underline{T}_2)^{\lambda}},\frac{2(\underline{I}_1)^{\lambda}(\underline{I}_2)^{\lambda}}{(2-\underline{I}_1)^{\lambda}(2-\underline{I}_2)^{\lambda}+(\underline{I}_1)^{\lambda}(\underline{I}_2)^{\lambda}},\frac{2(\underline{F}_1)^{\lambda}(\underline{F}_2)^{\lambda}}{(2-\underline{F}_1)^{\lambda}(2-\underline{F}_2)^{\lambda}+(\underline{F}_1)^{\lambda}(\underline{F}_2)^{\lambda}}\right),\right.$$

$$\left.\left(\frac{(1+\overline{T}_1)^{\lambda}(1+\overline{T}_2)^{\lambda}-(1-\overline{T}_1)^{\lambda}(1-\overline{T}_2)^{\lambda}}{(1+\overline{T}_1)^{\lambda}(1+\overline{T}_2)^{\lambda}+(1-\overline{T}_1)^{\lambda}(1-\overline{T}_2)^{\lambda}},\frac{2(\overline{I}_1)^{\lambda}(\overline{I}_2)^{\lambda}}{(2-\overline{I}_1)^{\lambda}(2-\overline{I}_2)^{\lambda}+(\overline{I}_1)^{\lambda}(\overline{I}_2)^{\lambda}},\frac{2(\overline{F}_1)^{\lambda}(\overline{F}_2)^{\lambda}}{(2-\overline{F}_1)^{\lambda}(2-\overline{F}_2)^{\lambda}+(\overline{F}_1)^{\lambda}(\overline{F}_2)^{\lambda}}\right)\right\rangle.$$

On the other hand, $\lambda A\oplus_E \lambda B$ as follows

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$$=\left\langle\left(\left(\frac{\left(\frac{(1+\underline{T}_1)^\lambda-(1-\underline{T}_1)^\lambda}{(1+\underline{T}_1)^\lambda+(1-\underline{T}_1)^\lambda}\right)+\left(\frac{(1+\underline{T}_2)^\lambda-(1-\underline{T}_2)^\lambda}{(1+\underline{T}_2)^\lambda+(1-\underline{T}_2)^\lambda}\right)}{1+\left(\frac{(1+\underline{T}_1)^\lambda-(1-\underline{T}_1)^\lambda}{(1+\underline{T}_1)^\lambda+(1-\underline{T}_1)^\lambda}\right)\left(\frac{(1+\underline{T}_2)^\lambda-(1-\underline{T}_2)^\lambda}{(1+\underline{T}_2)^\lambda+(1-\underline{T}_2)^\lambda}\right)},\frac{\left(\frac{2(\underline{I}_1)^\lambda}{(2-\underline{I}_1)^\lambda+(\underline{I}_1)^\lambda}\right)\left(\frac{2(\underline{I}_2)^\lambda}{(2-\underline{I}_2)^\lambda+(\underline{I}_2)^\lambda}\right)}{1+\left(1-\frac{2(\underline{I}_1)^\lambda}{(2-\underline{I}_1)^\lambda+(\underline{I}_1)^\lambda}\right)\left(1-\frac{2(\underline{I}_2)^\lambda}{(2-\underline{I}_2)^\lambda+(\underline{I}_2)^\lambda}\right)},\right.\right.\right.$$

$$\left.\frac{\left(\frac{2(\underline{F}_1)^\lambda}{(2-\underline{F}_1)^\lambda+(\underline{F}_1)^\lambda}\right)\left(\frac{2(\underline{F}_2)^\lambda}{(2-\underline{F}_2)^\lambda+(\underline{F}_2)^\lambda}\right)}{1+\left(1-\frac{2(\underline{F}_1)^\lambda}{(2-\underline{F}_1)^\lambda+(\underline{F}_1)^\lambda}\right)\left(1-\frac{2(\underline{F}_2)^\lambda}{(2-\underline{F}_2)^\lambda+(\underline{F}_2)^\lambda}\right)}\right),\left(\frac{\left(\frac{(1+\overline{T}_1)^\lambda-(1-\overline{T}_1)^\lambda}{(1+\overline{T}_1)^\lambda+(1-\overline{T}_1)^\lambda}\right)+\left(\frac{(1+\overline{T}_2)^\lambda-(1-\overline{T}_2)^\lambda}{(1+\overline{T}_2)^\lambda+(1-\overline{T}_2)^\lambda}\right)}{1+\left(\frac{(1+\overline{T}_1)^\lambda-(1-\overline{T}_1)^\lambda}{(1+\overline{T}_1)^\lambda+(1-\overline{T}_1)^\lambda}\right)\left(\frac{(1+\overline{T}_2)^\lambda-(1-\overline{T}_2)^\lambda}{(1+\overline{T}_2)^\lambda+(1-\overline{T}_2)^\lambda}\right)},\right.$$

$$\left.\left.\left.\frac{\left(\frac{2(\overline{I}_1)^\lambda}{(2-\overline{I}_1)^\lambda+(\overline{I}_1)^\lambda}\right)\left(\frac{2(\overline{I}_2)^\lambda}{(2-\overline{I}_2)^\lambda+(\overline{I}_2)^\lambda}\right)}{1+\left(1-\frac{2(\overline{I}_1)^\lambda}{(2-\overline{I}_1)^\lambda+(\overline{I}_1)^\lambda}\right)\left(1-\frac{2(\overline{I}_2)^\lambda}{(2-\overline{I}_2)^\lambda+(\overline{I}_2)^\lambda}\right)},\frac{\left(\frac{2(\overline{F}_1)^\lambda}{(2-\overline{F}_1)^\lambda+(\overline{F}_1)^\lambda}\right)\left(\frac{2(\overline{F}_2)^\lambda}{(2-\overline{F}_2)^\lambda+(\overline{F}_2)^\lambda}\right)}{1+\left(1-\frac{2(\overline{F}_1)^\lambda}{(2-\overline{F}_1)^\lambda+(\overline{F}_1)^\lambda}\right)\left(1-\frac{2(\overline{F}_2)^\lambda}{(2-\overline{F}_2)^\lambda+(\overline{F}_2)^\lambda}\right)}\right)\right)\right\rangle$$

$$=\left\langle\left(\left(\frac{(1+\underline{T}_1)^\lambda(1+\underline{T}_2)^\lambda-(1-\underline{T}_1)^\lambda(1-\underline{T}_2)^\lambda}{(1+\underline{T}_1)^\lambda(1+\underline{T}_2)^\lambda+(1-\underline{T}_1)^\lambda(1-\underline{T}_2)^\lambda},\frac{2(\underline{I}_1)^\lambda(\underline{I}_2)^\lambda}{(2-\underline{I}_1)^\lambda(2-\underline{I}_2)^\lambda+(\underline{I}_1)^\lambda(\underline{I}_2)^\lambda},\frac{2(\underline{F}_1)^\lambda(\underline{F}_2)^\lambda}{(2-\underline{F}_1)^\lambda(2-\underline{F}_2)^\lambda+(\underline{F}_1)^\lambda(\underline{F}_2)^\lambda}\right),\right.\right.$$

$$\left.\left.\left(\frac{(1+\overline{T}_1)^\lambda(1+\overline{T}_2)^\lambda-(1-\overline{T}_1)^\lambda(1-\overline{T}_2)^\lambda}{(1+\overline{T}_1)^\lambda(1+\overline{T}_2)^\lambda+(1-\overline{T}_1)^\lambda(1-\overline{T}_2)^\lambda},\frac{2(\overline{I}_1)^\lambda(\overline{I}_2)^\lambda}{(2-\overline{I}_1)^\lambda(2-\overline{I}_2)^\lambda+(\overline{I}_1)^\lambda(\overline{I}_2)^\lambda},\frac{2(\overline{F}_1)^\lambda(\overline{F}_2)^\lambda}{(2-\overline{F}_1)^\lambda(2-\overline{F}_2)^\lambda+(\overline{F}_1)^\lambda(\overline{F}_2)^\lambda}\right)\right)\right\rangle.$$

As such $\lambda(A\oplus B)=\lambda A\oplus\lambda B,\lambda>0$.

(d) $(A\otimes_E B)^\lambda=A^\lambda\otimes_E B^\lambda$

We have, $(A\otimes_E B)^\lambda=$

$$=\left(\left\langle\left(\frac{(\underline{T}_1)(\underline{T}_2)}{1+(1-\underline{T}_1)(1-\underline{T}_2)},\frac{\underline{I}_1+\underline{I}_2}{1+((\underline{I}_1)(\underline{I}_2))},\frac{\underline{F}_1+\underline{F}_2}{1+((\underline{F}_1)(\underline{F}_2))}\right),\left(\frac{(\overline{T}_1)(\overline{T}_2)}{1+(1-\overline{T}_1)(1-\overline{T}_2)},\frac{\overline{I}_1+\overline{I}_2}{1+((\overline{I}_1)(\overline{I}_2))},\frac{\overline{F}_1+\overline{F}_2}{1+((\overline{F}_1)(\overline{F}_2))}\right)\right\rangle\right)^\lambda$$

$$=\left\langle\left(\left(\frac{2\left(\frac{(\underline{T}_1)(\underline{T}_2)}{1+(1-\underline{T}_1)(1-\underline{T}_2)}\right)^\lambda}{\left(2-\frac{(\underline{T}_1)(\underline{T}_2)}{1+(1-\underline{T}_1)(1-\underline{T}_2)}\right)^\lambda\left(\frac{(\underline{T}_1)(\underline{T}_2)}{1+(1-\underline{T}_1)(1-\underline{T}_2)}\right)^\lambda},\frac{\left(1+\frac{\underline{I}_1+\underline{I}_2}{1+((\underline{I}_1)(\underline{I}_2))}\right)^\lambda-\left(1-\frac{\underline{I}_1+\underline{I}_2}{1+((\underline{I}_1)(\underline{I}_2))}\right)^\lambda}{\left(1+\frac{\underline{I}_1+\underline{I}_2}{1+((\underline{I}_1)(\underline{I}_2))}\right)^\lambda+\left(1-\frac{\underline{I}_1+\underline{I}_2}{1+((\underline{I}_1)(\underline{I}_2))}\right)^\lambda},\right.\right.\right.$$

$$\left.\frac{\left(1+\frac{\underline{F}_1+\underline{F}_2}{1+((\underline{F}_1)(\underline{F}_2))}\right)^\lambda-\left(1-\frac{\underline{F}_1+\underline{F}_2}{1+((\underline{F}_1)(\underline{F}_2))}\right)^\lambda}{\left(1+\frac{\underline{F}_1+\underline{F}_2}{1+((\underline{F}_1)(\underline{F}_2))}\right)^\lambda+\left(1-\frac{\underline{F}_1+\underline{F}_2}{1+((\underline{F}_1)(\underline{F}_2))}\right)^\lambda}\right),\in\left(\frac{2\left(\frac{(\overline{T}_1)(\overline{T}_2)}{1+(1-\overline{T}_1)(1-\overline{T}_2)}\right)^\lambda}{\left(2-\frac{(\overline{T}_1)(\overline{T}_2)}{1+(1-\overline{T}_1)(1-\overline{T}_2)}\right)^\lambda\left(\frac{(\overline{T}_1)(\overline{T}_2)}{1+(1-\overline{T}_1)(1-\overline{T}_2)}\right)^\lambda},\right.$$

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$$\left.\left.\frac{\left(1+\frac{\overline{I}_1+\overline{I}_2}{1+\left(\left(\overline{I}_1\right)\left(\overline{I}_2\right)\right)}\right)^\lambda-\left(1-\frac{\overline{I}_1+\overline{I}_2}{1+\left(\left(\overline{I}_1\right)\left(\overline{I}_2\right)\right)}\right)^\lambda}{\left(1+\frac{\overline{I}_1+\overline{I}_2}{1+\left(\left(\overline{I}_1\right)\left(\overline{I}_2\right)\right)}\right)^\lambda+\left(1-\frac{\overline{I}_1+\overline{I}_2}{1+\left(\left(\overline{I}_1\right)\left(\overline{I}_2\right)\right)}\right)^\lambda},\frac{\left(1+\frac{\overline{F}_1+\overline{F}_2}{1+\left(\left(\overline{F}_1\right)\left(\overline{F}_2\right)\right)}\right)^\lambda-\left(1-\frac{\overline{F}_1+\overline{F}_2}{1+\left(\left(\overline{F}_1\right)\left(\overline{F}_2\right)\right)}\right)^\lambda}{\left(1+\frac{\overline{F}_1+\overline{F}_2}{1+\left(\left(\overline{F}_1\right)\left(\overline{F}_2\right)\right)}\right)^\lambda+\left(1-\frac{\overline{F}_1+\overline{F}_2}{1+\left(\left(\overline{F}_1\right)\left(\overline{F}_2\right)\right)}\right)^\lambda}\right)\right\rangle$$

$$=\left\langle\left(\frac{2\left(\underline{T}_1\right)^\lambda\left(\underline{T}_2\right)^\lambda}{\left(2-\underline{T}_1\right)^\lambda\left(2-\underline{T}_2\right)^\lambda+\left(\underline{T}_1\right)^\lambda\left(\underline{T}_2\right)^\lambda},\frac{\left(1+\underline{I}_1\right)^\lambda\left(1+\underline{I}_2\right)^\lambda-\left(1-\underline{I}_1\right)^\lambda\left(1-\underline{I}_2\right)^\lambda}{\left(1+\underline{I}_1\right)^\lambda\left(1+\underline{I}_2\right)^\lambda+\left(1-\underline{I}_1\right)^\lambda\left(1-\underline{I}_2\right)^\lambda},\frac{\left(1+\underline{F}_1\right)^\lambda\left(1+\underline{F}_2\right)^\lambda-\left(1-\underline{F}_1\right)^\lambda\left(1-\underline{F}_2\right)^\lambda}{\left(1+\underline{F}_1\right)^\lambda\left(1+\underline{F}_2\right)^\lambda+\left(1-\underline{F}_1\right)^\lambda\left(1-\underline{F}_2\right)^\lambda}\right),\right.$$

$$\left.\left(\frac{2\left(\overline{T}_1\right)^\lambda\left(\overline{T}_2\right)^\lambda}{\left(2-\overline{T}_1\right)^\lambda\left(2-\overline{T}_2\right)^\lambda+\left(\overline{T}_1\right)^\lambda\left(\overline{T}_2\right)^\lambda},\frac{\left(1+\overline{I}_1\right)^\lambda\left(1+\overline{I}_2\right)^\lambda-\left(1-\overline{I}_1\right)^\lambda\left(1-\overline{I}_2\right)^\lambda}{\left(1+\overline{I}_1\right)^\lambda\left(1+\underline{I}_2\right)^\lambda+\left(1-\overline{I}_1\right)^\lambda\left(1-\overline{I}_2\right)^\lambda},\frac{\left(1+\overline{F}_1\right)^\lambda\left(1+\overline{F}_2\right)^\lambda-\left(1-\overline{F}_1\right)^\lambda\left(1-\overline{F}_2\right)^\lambda}{\left(1+\overline{F}_1\right)^\lambda\left(1+\overline{F}_2\right)^\lambda+\left(1-\overline{F}_1\right)^\lambda\left(1-\overline{F}_2\right)^\lambda}\right)\right\rangle.$$

And $A^\lambda \otimes_E B^\lambda$ we get

$$=\left\langle\left(\frac{\left(\frac{2\left(\underline{T}_1\right)^\lambda}{\left(2-\underline{T}_1\right)^\lambda+\left(\underline{T}_1\right)^\lambda}\right)\left(\frac{2\left(\underline{T}_2\right)^\lambda}{\left(2-\underline{T}_2\right)^\lambda+\left(\underline{T}_2\right)^\lambda}\right)}{1+\left(1-\frac{2\left(\underline{T}_1\right)^\lambda}{\left(2-\underline{T}_1\right)^\lambda+\left(\underline{T}_1\right)^\lambda}\right)\left(1-\frac{2\left(\underline{T}_2\right)^\lambda}{\left(2-\underline{T}_2\right)^\lambda+\left(\underline{T}_2\right)^\lambda}\right)},\frac{\left(\frac{\left(1+\underline{I}_1\right)^\lambda-\left(1-\underline{I}_1\right)^\lambda}{\left(1+\underline{I}_1\right)^\lambda+\left(1-\underline{I}_1\right)^\lambda}\right)+\left(\frac{\left(1+\underline{I}_2\right)^\lambda-\left(1-\underline{I}_2\right)^\lambda}{\left(1+\underline{I}_2\right)^\lambda+\left(1-\underline{I}_2\right)^\lambda}\right)}{1+\left(\frac{\left(1+\underline{I}_1\right)^\lambda-\left(1-\underline{I}_1\right)^\lambda}{\left(1+\underline{I}_1\right)^\lambda+\left(1-\underline{I}_1\right)^\lambda}\right)\left(\frac{\left(1+\underline{I}_2\right)^\lambda-\left(1-\underline{I}_2\right)^\lambda}{\left(1+\underline{I}_2\right)^\lambda+\left(1-\underline{I}_2\right)^\lambda}\right)},\right.\right.$$

$$\left.\frac{\left(\frac{\left(1+\underline{F}_1\right)^\lambda-\left(1-\underline{F}_1\right)^\lambda}{\left(1+\underline{F}_1\right)^\lambda+\left(1-\underline{F}_1\right)^\lambda}\right)+\left(\frac{\left(1+\underline{F}_2\right)^\lambda-\left(1-\underline{F}_2\right)^\lambda}{\left(1+\underline{F}_2\right)^\lambda+\left(1-\underline{F}_2\right)^\lambda}\right)}{1+\left(\frac{\left(1+\underline{F}_1\right)^\lambda-\left(1-\underline{F}_1\right)^\lambda}{\left(1+\underline{F}_1\right)^\lambda+\left(1-\underline{F}_1\right)^\lambda}\right)\left(\frac{\left(1+\underline{F}_2\right)^\lambda-\left(1-\underline{F}_2\right)^\lambda}{\left(1+\underline{F}_2\right)^\lambda+\left(1-\underline{F}_2\right)^\lambda}\right)}\right),\left(\frac{\left(\frac{2\left(\overline{T}_1\right)^\lambda}{\left(2-\overline{T}_1\right)^\lambda+\left(\overline{T}_1\right)^\lambda}\right)\left(\frac{2\left(\overline{T}_2\right)^\lambda}{\left(2-\overline{T}_2\right)^\lambda+\left(\overline{T}_2\right)^\lambda}\right)}{1+\left(1-\frac{2\left(\overline{T}_1\right)^\lambda}{\left(2-\overline{T}_1\right)^\lambda+\left(\overline{T}_1\right)^\lambda}\right)\left(1-\frac{2\left(\overline{T}_2\right)^\lambda}{\left(2-\overline{T}_2\right)^\lambda+\left(\overline{T}_2\right)^\lambda}\right)},\right.$$

$$\left.\left.\frac{\left(\frac{\left(1+\overline{I}_1\right)^\lambda-\left(1-\overline{I}_1\right)^\lambda}{\left(1+\overline{I}_1\right)^\lambda+\left(1-\overline{I}_1\right)^\lambda}\right)+\left(\frac{\left(1+\overline{I}_2\right)^\lambda-\left(1-\overline{I}_2\right)^\lambda}{\left(1+\underline{I}_2\right)^\lambda+\left(1-\overline{I}_2\right)^\lambda}\right)}{1+\left(\frac{\left(1+\overline{I}_1\right)^\lambda-\left(1-\overline{I}_1\right)^\lambda}{\left(1+\overline{I}_1\right)^\lambda+\left(1-\overline{I}_1\right)^\lambda}\right)\left(\frac{\left(1+\overline{I}_2\right)^\lambda-\left(1-\overline{I}_2\right)^\lambda}{\left(1+\underline{I}_2\right)^\lambda+\left(1-\overline{I}_2\right)^\lambda}\right)},\frac{\left(\frac{\left(1+\overline{F}_1\right)^\lambda-\left(1-\overline{F}_1\right)^\lambda}{\left(1+\overline{F}_1\right)^\lambda+\left(1-\overline{F}_1\right)^\lambda}\right)+\left(\frac{\left(1+\overline{F}_2\right)^\lambda-\left(1-\overline{F}_2\right)^\lambda}{\left(1+\overline{F}_2\right)^\lambda+\left(1-\overline{F}_2\right)^\lambda}\right)}{1+\left(\frac{\left(1+\overline{F}_1\right)^\lambda-\left(1-\overline{F}_1\right)^\lambda}{\left(1+\overline{F}_1\right)^\lambda+\left(1-\overline{F}_1\right)^\lambda}\right)\left(\frac{\left(1+\overline{I}_2\right)^\lambda-\left(1-\overline{I}_2\right)^\lambda}{\left(1+\underline{I}_2\right)^\lambda+\left(1-\overline{I}_2\right)^\lambda}\right)}\right)\right\rangle$$

$$=\left\langle\left(\frac{2\left(\underline{T}_1\right)^\lambda\left(\underline{T}_2\right)^\lambda}{\left(2-\underline{T}_1\right)^\lambda\left(2-\underline{T}_2\right)^\lambda+\left(\underline{T}_1\right)^\lambda\left(\underline{T}_2\right)^\lambda},\frac{\left(1+\underline{I}_1\right)^\lambda\left(1+\underline{I}_2\right)^\lambda-\left(1-\underline{I}_1\right)^\lambda\left(1-\underline{I}_2\right)^\lambda}{\left(1+\underline{I}_1\right)^\lambda\left(1+\underline{I}_2\right)^\lambda+\left(1-\underline{I}_1\right)^\lambda\left(1-\underline{I}_2\right)^\lambda},\frac{\left(1+\underline{F}_1\right)^\lambda\left(1+\underline{F}_2\right)^\lambda-\left(1-\underline{F}_1\right)^\lambda\left(1-\underline{F}_2\right)^\lambda}{\left(1+\underline{F}_1\right)^\lambda\left(1+\underline{F}_2\right)^\lambda+\left(1-\underline{F}_1\right)^\lambda\left(1-\underline{F}_2\right)^\lambda}\right),\right.$$

$$\left.\left(\frac{2\left(\overline{T}_1\right)^\lambda\left(\overline{T}_2\right)^\lambda}{\left(2-\overline{T}_1\right)^\lambda\left(2-\overline{T}_2\right)^\lambda+\left(\overline{T}_1\right)^\lambda\left(\overline{T}_2\right)^\lambda},\frac{\left(1+\overline{I}_1\right)^\lambda\left(1+\overline{I}_2\right)^\lambda-\left(1-\overline{I}_1\right)^\lambda\left(1-\overline{I}_2\right)^\lambda}{\left(1+\overline{I}_1\right)^\lambda\left(1+\underline{I}_2\right)^\lambda+\left(1-\overline{I}_1\right)^\lambda\left(1-\overline{I}_2\right)^\lambda},\frac{\left(1+\overline{F}_1\right)^\lambda\left(1+\overline{F}_2\right)^\lambda-\left(1-\overline{F}_1\right)^\lambda\left(1-\overline{F}_2\right)^\lambda}{\left(1+\overline{F}_1\right)^\lambda\left(1+\overline{F}_2\right)^\lambda+\left(1-\overline{F}_1\right)^\lambda\left(1-\overline{F}_2\right)^\lambda}\right)\right\rangle.$$

Hence, it is shown that $\left(A\otimes B\right)^\lambda = A^\lambda \otimes B^\lambda, \lambda > 0$.

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(e) $\lambda_1 A \oplus_E \lambda_2 A = (\lambda_1 \oplus_E \lambda_2) A$

When $\lambda_1 A \oplus_E \lambda_2 A$, we can see that

$$=\left\langle\left(\frac{\left(\frac{(1+\underline{T}_1)^{\lambda_1}-(1-\underline{T}_1)^{\lambda_1}}{(1+\underline{T}_1)^{\lambda_1}+(1-\underline{T}_1)^{\lambda_1}}\right)+\left(\frac{(1+\underline{T}_1)^{\lambda_2}-(1-\underline{T}_1)^{\lambda_2}}{(1+\underline{T}_1)^{\lambda_2}+(1-\underline{T}_1)^{\lambda_2}}\right)}{1+\left(\left(\frac{(1+\underline{T}_1)^{\lambda_1}-(1-\underline{T}_1)^{\lambda_1}}{(1+\underline{T}_1)^{\lambda_1}+(1-\underline{T}_1)^{\lambda_1}}\right)+\left(\frac{(1+\underline{T}_1)^{\lambda_2}-(1-\underline{T}_1)^{\lambda_2}}{(1+\underline{T}_1)^{\lambda_2}+(1-\underline{T}_1)^{\lambda_2}}\right)\right)},\frac{\left(\frac{2(\underline{I}_1)^{\lambda_1}}{(2-\underline{I}_1)^{\lambda_1}+(\underline{I}_1)^{\lambda_1}}\right)+\left(\frac{2(\underline{I}_1)^{\lambda_2}}{(2-\underline{I}_1)^{\lambda_2}+(\underline{I}_1)^{\lambda_2}}\right)}{1+\left(\left(1-\frac{2(\underline{I}_1)^{\lambda_1}}{(2-\underline{I}_1)^{\lambda_1}+(\underline{I}_1)^{\lambda_1}}\right)+\left(1-\frac{2(\underline{I}_1)^{\lambda_2}}{(2-\underline{I}_1)^{\lambda_2}+(\underline{I}_1)^{\lambda_2}}\right)\right)},\right.$$

$$\left.\frac{\left(\frac{2(\underline{F}_1)^{\lambda_1}}{(2-\underline{F}_1)^{\lambda_1}+(\underline{F}_1)^{\lambda_1}}\right)+\left(\frac{2(\underline{F}_1)^{\lambda_2}}{(2-\underline{F}_1)^{\lambda_2}+(\underline{F}_1)^{\lambda_2}}\right)}{1+\left(\left(1-\frac{2(\underline{F}_1)^{\lambda_1}}{(2-\underline{F}_1)^{\lambda_1}+(\underline{F}_1)^{\lambda_1}}\right)+\left(1-\frac{2(\underline{F}_1)^{\lambda_2}}{(2-\underline{F}_1)^{\lambda_2}+(\underline{F}_1)^{\lambda_2}}\right)\right)}\right),\left(\frac{\left(\frac{(1+\overline{T}_1)^{\lambda_1}-(1-\overline{T}_1)^{\lambda_1}}{(1+\overline{T}_1)^{\lambda_1}+(1-\overline{T}_1)^{\lambda_1}}\right)+\left(\frac{(1+\overline{T}_1)^{\lambda_2}-(1-\overline{T}_1)^{\lambda_2}}{(1+\overline{T}_1)^{\lambda_2}+(1-\overline{T}_1)^{\lambda_2}}\right)}{1+\left(\left(\frac{(1+\overline{T}_1)^{\lambda_1}-(1-\overline{T}_1)^{\lambda_1}}{(1+\overline{T}_1)^{\lambda_1}+(1-\overline{T}_1)^{\lambda_1}}\right)+\left(\frac{(1+\overline{T}_1)^{\lambda_2}-(1-\overline{T}_1)^{\lambda_2}}{(1+\overline{T}_1)^{\lambda_2}+(1-\overline{T}_1)^{\lambda_2}}\right)\right)},\right.$$

$$\left.\frac{\left(\frac{2(\overline{I}_1)^{\lambda_1}}{(2-\overline{I}_1)^{\lambda_1}+(\overline{I}_1)^{\lambda_1}}\right)+\left(\frac{2(\overline{I}_1)^{\lambda_2}}{(2-\overline{I}_1)^{\lambda_2}+(\overline{I}_1)^{\lambda_2}}\right)}{1+\left(\left(1-\frac{2(\overline{I}_1)^{\lambda_1}}{(2-\overline{I}_1)^{\lambda_1}+(\overline{I}_1)^{\lambda_1}}\right)+\left(1-\frac{2(\overline{I}_1)^{\lambda_2}}{(2-\overline{I}_1)^{\lambda_2}+(\overline{I}_1)^{\lambda_2}}\right)\right)},\frac{\left(\frac{2(\overline{F}_1)^{\lambda_1}}{(2-\overline{F}_1)^{\lambda_1}+(\overline{F}_1)^{\lambda_1}}\right)+\left(\frac{2(\overline{F}_1)^{\lambda_2}}{(2-\overline{F}_1)^{\lambda_2}+(\overline{F}_1)^{\lambda_2}}\right)}{1+\left(\left(1-\frac{2(\overline{F}_1)^{\lambda_1}}{(2-\overline{F}_1)^{\lambda_1}+(\overline{F}_1)^{\lambda_1}}\right)+\left(1-\frac{2(\overline{F}_1)^{\lambda_2}}{(2-\overline{F}_1)^{\lambda_2}+(\overline{F}_1)^{\lambda_2}}\right)\right)}\right)\right\rangle$$

$$=\left\langle\left(\frac{(1+\underline{T}_1)^{\lambda_1}(1+\underline{T}_1)^{\lambda_2}-(1-\underline{T}_1)^{\lambda_1}(1-\underline{T}_1)^{\lambda_2}}{(1+\underline{T}_1)^{\lambda_1}(1+\underline{T}_1)^{\lambda_2}+(1-\underline{T}_1)^{\lambda_1}(1-\underline{T}_1)^{\lambda_2}},\frac{2(\underline{I}_1)^{\lambda_1}(\underline{I}_1)^{\lambda_2}}{(2-\underline{I}_1)^{\lambda_1}(2-\underline{I}_1)^{\lambda_2}+(\underline{I}_1)^{\lambda_1}(\underline{I}_1)^{\lambda_2}},\frac{2(\underline{F}_1)^{\lambda_1}(\underline{F}_1)^{\lambda_2}}{(2-\underline{F}_1)^{\lambda_1}(2-\underline{F}_1)^{\lambda_2}+(\underline{F}_1)^{\lambda_1}(\underline{F}_1)^{\lambda_2}}\right),\right.$$

$$\left.\left(\frac{(1+\overline{T}_1)^{\lambda_1}(1+\overline{T}_1)^{\lambda_2}-(1-\overline{T}_1)^{\lambda_1}(1-\overline{T}_1)^{\lambda_2}}{(1+\overline{T}_1)^{\lambda_1}(1+\overline{T}_1)^{\lambda_2}+(1-\overline{T}_1)^{\lambda_1}(1-\overline{T}_1)^{\lambda_2}},\frac{2(\overline{I}_1)^{\lambda_1}(\overline{I}_1)^{\lambda_2}}{(2-\overline{I}_1)^{\lambda_1}(2-\overline{I}_1)^{\lambda_2}+(\overline{I}_1)^{\lambda_1}(\overline{I}_1)^{\lambda_2}},\frac{2(\overline{F}_1)^{\lambda_1}(\overline{F}_1)^{\lambda_2}}{(2-\overline{F}_1)^{\lambda_1}(2-\overline{F}_1)^{\lambda_2}+(\overline{F}_1)^{\lambda_1}(\overline{F}_1)^{\lambda_2}}\right)\right\rangle$$

$$=\left\langle\left(\frac{(1+\underline{T}_1)^{\lambda_1+\lambda_2}-(1-\underline{T}_1)^{\lambda_1+\lambda_2}}{(1+\underline{T}_1)^{\lambda_1+\lambda_2}+(1-\underline{T}_1)^{\lambda_1+\lambda_2}},\frac{2(\underline{I}_1)^{\lambda_1+\lambda_2}}{(2-\underline{I}_1)^{\lambda_1+\lambda_2}+(\underline{I}_1)^{\lambda_1+\lambda_2}},\frac{2(\underline{F}_1)^{\lambda_1+\lambda_2}}{(2-\underline{F}_1)^{\lambda_1+\lambda_2}+(\underline{F}_1)^{\lambda_1+\lambda_2}}\right),\right.$$

$$\left.\left(\frac{(1+\overline{T}_1)^{\lambda_1+\lambda_2}-(1-\overline{T}_1)^{\lambda_1+\lambda_2}}{(1+\overline{T}_1)^{\lambda_1+\lambda_2}+(1-\overline{T}_1)^{\lambda_1+\lambda_2}},\frac{2(\overline{I}_1)^{\lambda_1+\lambda_2}}{(2-\overline{I}_1)^{\lambda_1+\lambda_2}+(\overline{I}_1)^{\lambda_1+\lambda_2}},\frac{2(\overline{F}_1)^{\lambda_1+\lambda_2}}{(2-\overline{F}_1)^{\lambda_1+\lambda_2}+(\overline{F}_1)^{\lambda_1+\lambda_2}}\right)\right\rangle$$

And when for $(\lambda_1 \oplus_E \lambda_2) A$ as follows

$$=\left\langle\left(\frac{(1+\underline{T}_1)^{\lambda_1+\lambda_2}-(1-\underline{T}_1)^{\lambda_1+\lambda_2}}{(1+\underline{T}_1)^{\lambda_1+\lambda_2}+(1-\underline{T}_1)^{\lambda_1+\lambda_2}},\frac{2(\underline{I}_1)^{\lambda_1+\lambda_2}}{(2-\underline{I}_1)^{\lambda_1+\lambda_2}+(\underline{I}_1)^{\lambda_1+\lambda_2}},\frac{2(\underline{F}_1)^{\lambda_1+\lambda_2}}{(2-\underline{F}_1)^{\lambda_1+\lambda_2}+(\underline{F}_1)^{\lambda_1+\lambda_2}}\right),\right.$$

$$\left.\left(\frac{(1+\overline{T}_1)^{\lambda_1+\lambda_2}-(1-\overline{T}_1)^{\lambda_1+\lambda_2}}{(1+\overline{T}_1)^{\lambda_1+\lambda_2}+(1-\overline{T}_1)^{\lambda_1+\lambda_2}},\frac{2(\overline{I}_1)^{\lambda_1+\lambda_2}}{(2-\overline{I}_1)^{\lambda_1+\lambda_2}+(\overline{I}_1)^{\lambda_1+\lambda_2}},\frac{2(\overline{F}_1)^{\lambda_1+\lambda_2}}{(2-\overline{F}_1)^{\lambda_1+\lambda_2}+(\overline{F}_1)^{\lambda_1+\lambda_2}}\right)\right\rangle$$

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Then, it proves that $\lambda_1 A \oplus_E \lambda_2 A = (\lambda_1 \oplus_E \lambda_2) A$.

(f) $A^{\lambda_1} \otimes_E A^{\lambda_2} = A^{\lambda_1+\lambda_2}$

Since $A^{\lambda_1} \otimes_E A^{\lambda_2}$ we obtain

$$
=\left\langle\left(\frac{\left(\frac{2(\underline{T}_1)^{\lambda_1}}{(2-\underline{T}_1)^{\lambda_1}+(\underline{T}_1)^{\lambda_1}}\right)+\left(\frac{2(\underline{T}_1)^{\lambda_2}}{(2-\underline{T}_1)^{\lambda_2}+(\underline{T}_1)^{\lambda_2}}\right)}{1+\left(\left(1-\frac{2(\underline{T}_1)^{\lambda_1}}{(2-\underline{T}_1)^{\lambda_1}+(\underline{T}_1)^{\lambda_1}}\right)+\left(1-\frac{2(\underline{T}_1)^{\lambda_2}}{(2-\underline{T}_1)^{\lambda_2}+(\underline{T}_1)^{\lambda_2}}\right)\right)},\frac{\left(\frac{(1+\underline{I}_1)^{\lambda_1}-(1-\underline{I}_1)^{\lambda_1}}{(1+\underline{I}_1)^{\lambda_1}+(1-\underline{I}_1)^{\lambda_1}}\right)+\left(\frac{(1+\underline{I}_1)^{\lambda_2}-(1-\underline{I}_1)^{\lambda_2}}{(1+\underline{I}_1)^{\lambda_2}+(1-\underline{I}_1)^{\lambda_2}}\right)}{1+\left(\left(\frac{(1+\underline{I}_1)^{\lambda_1}-(1-\underline{I}_1)^{\lambda_1}}{(1+\underline{I}_1)^{\lambda_1}+(1-\underline{I}_1)^{\lambda_1}}\right)+\left(\frac{(1+\underline{I}_1)^{\lambda_2}-(1-\underline{I}_1)^{\lambda_2}}{(1+\underline{I}_1)^{\lambda_2}+(1-\underline{I}_1)^{\lambda_2}}\right)\right)},\right.\right.
$$

$$
\left.\frac{\left(\frac{(1+\underline{F}_1)^{\lambda_1}-(1-\underline{F}_1)^{\lambda_1}}{(1+\underline{F}_1)^{\lambda_1}+(1-\underline{F}_1)^{\lambda_1}}\right)+\left(\frac{(1+\underline{F}_1)^{\lambda_2}-(1-\underline{F}_1)^{\lambda_2}}{(1+\underline{F}_1)^{\lambda_2}+(1-\underline{F}_1)^{\lambda_2}}\right)}{1+\left(\left(\frac{(1+\underline{F}_1)^{\lambda_1}-(1-\underline{F}_1)^{\lambda_1}}{(1+\underline{F}_1)^{\lambda_1}+(1-\underline{F}_1)^{\lambda_1}}\right)+\left(\frac{(1+\underline{F}_1)^{\lambda_2}-(1-\underline{F}_1)^{\lambda_2}}{(1+\underline{F}_1)^{\lambda_2}+(1-\underline{F}_1)^{\lambda_2}}\right)\right)}\right),\left(\frac{\left(\frac{2(\overline{T}_1)^{\lambda_1}}{(2-\overline{T}_1)^{\lambda_1}+(\overline{T}_1)^{\lambda_1}}\right)+\left(\frac{2(\overline{T}_1)^{\lambda_2}}{(2-\overline{T}_1)^{\lambda_2}+(\overline{T}_1)^{\lambda_2}}\right)}{1+\left(\left(1-\frac{2(\overline{T}_1)^{\lambda_1}}{(2-\overline{T}_1)^{\lambda_1}+(\overline{T}_1)^{\lambda_1}}\right)+\left(1-\frac{2(\overline{T}_1)^{\lambda_2}}{(2-\overline{T}_1)^{\lambda_2}+(\overline{T}_1)^{\lambda_2}}\right)\right)},\right.
$$

$$
\left.\left.\frac{\left(\frac{(1+\overline{I}_1)^{\lambda_1}-(1-\overline{I}_1)^{\lambda_1}}{(1+\overline{I}_1)^{\lambda_1}+(1-\overline{I}_1)^{\lambda_1}}\right)+\left(\frac{(1+\overline{I}_1)^{\lambda_2}-(1-\overline{I}_1)^{\lambda_2}}{(1+\overline{I}_1)^{\lambda_2}+(1-\overline{I}_1)^{\lambda_2}}\right)}{1+\left(\left(\frac{(1+\overline{I}_1)^{\lambda_1}-(1-\overline{I}_1)^{\lambda_1}}{(1+\overline{I}_1)^{\lambda_1}+(1-\overline{I}_1)^{\lambda_1}}\right)+\left(\frac{(1+\overline{I}_1)^{\lambda_2}-(1-\overline{I}_1)^{\lambda_2}}{(1+\overline{I}_1)^{\lambda_2}+(1-\overline{I}_1)^{\lambda_2}}\right)\right)},\frac{\left(\frac{(1+\overline{F}_1)^{\lambda_1}-(1-\overline{F}_1)^{\lambda_1}}{(1+\overline{F}_1)^{\lambda_1}+(1-\overline{F}_1)^{\lambda_1}}\right)+\left(\frac{(1+\overline{F}_1)^{\lambda_2}-(1-\overline{F}_1)^{\lambda_2}}{(1+\overline{F}_1)^{\lambda_2}+(1-\overline{F}_1)^{\lambda_2}}\right)}{1+\left(\left(\frac{(1+\overline{F}_1)^{\lambda_1}-(1-\overline{F}_1)^{\lambda_1}}{(1+\overline{F}_1)^{\lambda_1}+(1-\overline{F}_1)^{\lambda_1}}\right)+\left(\frac{(1+\overline{F}_1)^{\lambda_2}-(1-\overline{F}_1)^{\lambda_2}}{(1+\overline{F}_1)^{\lambda_2}+(1-\overline{F}_1)^{\lambda_2}}\right)\right)}\right)\right\rangle
$$

$$
=\left\langle\left(\frac{2(\underline{T}_1)^{\lambda_1}(\underline{T}_1)^{\lambda_2}}{(2-\underline{T}_1)^{\lambda_1}(2-\underline{T}_1)^{\lambda_2}+(\underline{T}_1)^{\lambda_1}(\underline{T}_1)^{\lambda_2}},\frac{(1+\underline{I}_1)^{\lambda_1}(1+\underline{I}_1)^{\lambda_2}-(1-\underline{I}_1)^{\lambda_1}(1-\underline{I}_1)^{\lambda_2}}{(1+\underline{I}_1)^{\lambda_1}(1+\underline{I}_1)^{\lambda_2}+(1-\underline{I}_1)^{\lambda_1}(1-\underline{I}_1)^{\lambda_2}},\frac{(1+\underline{F}_1)^{\lambda_1}(1+\underline{F}_1)^{\lambda_2}-(1-\underline{F}_1)^{\lambda_1}(1-\underline{F}_1)^{\lambda_2}}{(1+\underline{F}_1)^{\lambda_1}(1+\underline{F}_1)^{\lambda_2}+(1-\underline{F}_1)^{\lambda_1}(1-\underline{F}_1)^{\lambda_2}}\right),\right.
$$

$$
\left.\left(\frac{2(\overline{T}_1)^{\lambda_1}(\overline{T}_1)^{\lambda_2}}{(2-\overline{T}_1)^{\lambda_1}(2-\overline{T}_1)^{\lambda_2}+(\overline{T}_1)^{\lambda_1}(\overline{T}_1)^{\lambda_2}},\frac{(1+\overline{I}_1)^{\lambda_1}(1+\overline{I}_1)^{\lambda_2}-(1-\overline{I}_1)^{\lambda_1}(1-\overline{I}_1)^{\lambda_2}}{(1+\overline{I}_1)^{\lambda_1}(1+\overline{I}_1)^{\lambda_2}+(1-\overline{I}_1)^{\lambda_1}(1-\overline{I}_1)^{\lambda_2}},\frac{(1+\overline{F}_1)^{\lambda_1}(1+\overline{F}_1)^{\lambda_2}-(1-\overline{F}_1)^{\lambda_1}(1-\overline{F}_1)^{\lambda_2}}{(1+\overline{F}_1)^{\lambda_1}(1+\overline{F}_1)^{\lambda_2}+(1-\overline{F}_1)^{\lambda_1}(1-\overline{F}_1)^{\lambda_2}}\right)\right\rangle
$$

$$
=\left\langle\left(\frac{2(\underline{T}_1)^{\lambda_1+\lambda_2}}{(2-\underline{T}_1)^{\lambda_1+\lambda_2}+(\underline{T}_1)^{\lambda_1+\lambda_2}},\frac{(1+\underline{I}_1)^{\lambda_1+\lambda_2}-(1-\underline{I}_1)^{\lambda_1+\lambda_2}}{(1+\underline{I}_1)^{\lambda_1+\lambda_2}+(1-\underline{I}_1)^{\lambda_1+\lambda_2}},\frac{(1+\underline{F}_1)^{\lambda_1+\lambda_2}-(1-\underline{F}_1)^{\lambda_1+\lambda_2}}{(1+\underline{F}_1)^{\lambda_1+\lambda_2}+(1-\underline{F}_1)^{\lambda_1+\lambda_2}}\right),\right.
$$

$$
\left.\left(\frac{2(\overline{T}_1)^{\lambda_1+\lambda_2}}{(2-\overline{T}_1)^{\lambda_1+\lambda_2}+(\overline{T}_1)^{\lambda_1+\lambda_2}},\frac{(1+\overline{I}_1)^{\lambda_1+\lambda_2}-(1-\overline{I}_1)^{\lambda_1+\lambda_2}}{(1+\overline{I}_1)^{\lambda_1+\lambda_2}+(1-\overline{I}_1)^{\lambda_1+\lambda_2}},\frac{(1+\overline{F}_1)^{\lambda_1+\lambda_2}-(1-\overline{F}_1)^{\lambda_1+\lambda_2}}{(1+\overline{F}_1)^{\lambda_1+\lambda_2}+(1-\overline{F}_1)^{\lambda_1+\lambda_2}}\right)\right\rangle
$$

Then, for $A^{\lambda_1+\lambda_2}$ as follows

$$
=\left\langle\left(\frac{2(\underline{T}_1)^{\lambda_1+\lambda_2}}{(2-\underline{T}_1)^{\lambda_1+\lambda_2}+(\underline{T}_1)^{\lambda_1+\lambda_2}},\frac{(1+\underline{I}_1)^{\lambda_1+\lambda_2}-(1-\underline{I}_1)^{\lambda_1+\lambda_2}}{(1+\underline{I}_1)^{\lambda_1+\lambda_2}+(1-\underline{I}_1)^{\lambda_1+\lambda_2}},\frac{(1+\underline{F}_1)^{\lambda_1+\lambda_2}-(1-\underline{F}_1)^{\lambda_1+\lambda_2}}{(1+\underline{F}_1)^{\lambda_1+\lambda_2}+(1-\underline{F}_1)^{\lambda_1+\lambda_2}}\right),\right.
$$

$$
\left.\left(\frac{2(\overline{T}_1)^{\lambda_1+\lambda_2}}{(2-\overline{T}_1)^{\lambda_1+\lambda_2}+(\overline{T}_1)^{\lambda_1+\lambda_2}},\frac{(1+\overline{I}_1)^{\lambda_1+\lambda_2}-(1-\overline{I}_1)^{\lambda_1+\lambda_2}}{(1+\overline{I}_1)^{\lambda_1+\lambda_2}+(1-\overline{I}_1)^{\lambda_1+\lambda_2}},\frac{(1+\overline{F}_1)^{\lambda_1+\lambda_2}-(1-\overline{F}_1)^{\lambda_1+\lambda_2}}{(1+\overline{F}_1)^{\lambda_1+\lambda_2}+(1-\overline{F}_1)^{\lambda_1+\lambda_2}}\right)\right\rangle
$$

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Hence, it has proven that $A^{\lambda_1} \otimes_E A^{\lambda_2} = A^{\lambda_1+\lambda_2}$.

(g) $\left(A \oplus_E B\right) \oplus_E C = A \oplus_E \left(B \oplus_E C\right)$

$$= \left\langle \left(\frac{\left(\frac{\underline{T}_1+\underline{T}_2}{1+\left(\left(\underline{T}_1\right)\left(\underline{T}_2\right)\right)}\right)+\left(\underline{T}_3\right)}{1+\left(\frac{\underline{T}_1+\underline{T}_2}{1+\left(\left(\underline{T}_1\right)\left(\underline{T}_2\right)\right)}\right)+\left(\underline{T}_3\right)}, \frac{\left(\frac{\left(\underline{I}_1\right)\left(\underline{I}_2\right)}{1+\left(1-\underline{I}_1\right)\left(1-\underline{I}_2\right)}\right)\left(\underline{I}_3\right)}{1+\left(1-\frac{\left(\underline{I}_1\right)\left(\underline{I}_2\right)}{1+\left(1-\underline{I}_1\right)\left(1-\underline{I}_2\right)}\right)\left(1-\underline{I}_3\right)}, \frac{\left(\frac{\left(\underline{F}_1\right)\left(\underline{F}_2\right)}{1+\left(1-\underline{F}_1\right)\left(1-\underline{F}_2\right)}\right)\left(\underline{F}_3\right)}{1+\left(1-\frac{\left(\underline{F}_1\right)\left(\underline{F}_2\right)}{1+\left(1-\underline{F}_1\right)\left(1-\underline{F}_2\right)}\right)\left(1-\underline{F}_3\right)} \right),\right.$$

$$\left. \left(\frac{\left(\frac{\overline{T}_1+\overline{T}_2}{1+\left(\left(\overline{T}_1\right)\left(\overline{T}_2\right)\right)}\right)+\left(\overline{T}_3\right)}{1+\left(\frac{\overline{T}_1+\overline{T}_2}{1+\left(\left(\overline{T}_1\right)\left(\overline{T}_2\right)\right)}\right)+\left(\overline{T}_3\right)}, \frac{\left(\frac{\left(\overline{I}_1\right)\left(\overline{I}_2\right)}{1+\left(1-\overline{I}_1\right)\left(1-\overline{I}_2\right)}\right)\left(\overline{I}_3\right)}{1+\left(1-\frac{\left(\overline{I}_1\right)\left(\overline{I}_2\right)}{1+\left(1-\overline{I}_1\right)\left(1-\overline{I}_2\right)}\right)\left(1-\overline{I}_3\right)}, \frac{\left(\frac{\left(\overline{F}_1\right)\left(\overline{F}_2\right)}{1+\left(1-\overline{F}_1\right)\left(1-\overline{F}_2\right)}\right)\left(\overline{F}_3\right)}{1+\left(1-\frac{\left(\overline{I}_1\right)\left(\overline{I}_2\right)}{1+\left(1-\overline{I}_1\right)\left(1-\overline{I}_2\right)}\right)\left(1-\overline{F}_3\right)} \right) \right\rangle$$

$$= \left\langle \left(\frac{\left(\frac{\underline{T}_2+\underline{T}_3}{1+\left(\left(\underline{T}_2\right)\left(\underline{T}_3\right)\right)}\right)+\left(\underline{T}_1\right)}{1+\left(\frac{\underline{T}_2+\underline{T}_3}{1+\left(\left(\underline{T}_2\right)\left(\underline{T}_3\right)\right)}\right)+\left(\underline{T}_1\right)}, \frac{\left(\frac{\left(\underline{I}_2\right)\left(\underline{I}_3\right)}{1+\left(1-\underline{I}_2\right)\left(1-\underline{I}_3\right)}\right)\left(\underline{I}_1\right)}{1+\left(1-\frac{\left(\underline{I}_2\right)\left(\underline{I}_3\right)}{1+\left(1-\underline{I}_2\right)\left(1-\underline{I}_3\right)}\right)\left(1-\underline{I}_1\right)}, \frac{\left(\frac{\left(\underline{F}_2\right)\left(\underline{F}_3\right)}{1+\left(1-\underline{F}_2\right)\left(1-\underline{F}_3\right)}\right)\left(\underline{F}_1\right)}{1+\left(1-\frac{\left(\underline{F}_2\right)\left(\underline{F}_3\right)}{1+\left(1-\underline{F}_2\right)\left(1-\underline{F}_1\right)}\right)\left(1-\underline{F}_1\right)} \right),\right.$$

$$\left. \left(\frac{\left(\frac{\overline{T}_2+\overline{T}_3}{1+\left(\left(\overline{T}_2\right)\left(\overline{T}_3\right)\right)}\right)+\left(\overline{T}_1\right)}{1+\left(\frac{\overline{T}_2+\overline{T}_3}{1+\left(\left(\overline{T}_2\right)\left(\overline{T}_3\right)\right)}\right)+\left(\overline{T}_1\right)}, \frac{\left(\frac{\left(\overline{I}_2\right)\left(\overline{I}_3\right)}{1+\left(1-\overline{I}_2\right)\left(1-\overline{I}_3\right)}\right)\left(\overline{I}_1\right)}{1+\left(1-\frac{\left(\overline{I}_2\right)\left(\overline{I}_3\right)}{1+\left(1-\overline{I}_2\right)\left(1-\overline{I}_3\right)}\right)\left(1-\overline{I}_1\right)}, \frac{\left(\frac{\left(\overline{F}_2\right)\left(\overline{F}_3\right)}{1+\left(1-\overline{F}_2\right)\left(1-\overline{F}_3\right)}\right)\left(\overline{F}_1\right)}{1+\left(1-\frac{\left(\overline{I}_2\right)\left(\overline{I}_3\right)}{1+\left(1-\overline{I}_2\right)\left(1-\overline{I}_3\right)}\right)\left(1-\overline{F}_1\right)} \right) \right\rangle$$

$= A \oplus_E \left(B \oplus_E C\right)$

□

(h) $\left(A \otimes_E B\right) \otimes_E C = A \otimes_E \left(B \otimes_E C\right)$

$$= \left\langle \left(\frac{\left(\frac{\left(\underline{T}_1\right)\left(\underline{T}_2\right)}{1+\left(1-\underline{T}_1\right)\left(1-\underline{T}_2\right)}\right)\left(\underline{T}_3\right)}{1+\left(1-\frac{\left(\underline{T}_1\right)\left(\underline{T}_2\right)}{1+\left(1-\underline{T}_1\right)\left(1-\underline{T}_2\right)}\right)\left(1-\underline{T}_3\right)}, \frac{\left(\frac{\underline{I}_1+\underline{I}_2}{1+\left(\left(\underline{I}_1\right)\left(\underline{I}_2\right)\right)}\right)+\left(\underline{I}_3\right)}{1+\left(\frac{\underline{I}_1+\underline{I}_2}{1+\left(\left(\underline{I}_1\right)\left(\underline{I}_2\right)\right)}\right)+\left(\underline{I}_3\right)}, \frac{\left(\frac{\underline{F}_1+\underline{F}_2}{1+\left(\left(\underline{F}_1\right)\left(\underline{F}_2\right)\right)}\right)+\left(\underline{F}_3\right)}{1+\left(\frac{\underline{F}_1+\underline{F}_2}{1+\left(\left(\underline{F}_1\right)\left(\underline{F}_2\right)\right)}\right)+\left(\underline{F}_3\right)} \right),\right.$$

$$\left. \left(\frac{\left(\frac{\left(\overline{T}_1\right)\left(\overline{T}_2\right)}{1+\left(1-\overline{T}_1\right)\left(1-\overline{T}_2\right)}\right)\left(\overline{T}_3\right)}{1+\left(1-\frac{\left(\overline{T}_1\right)\left(\overline{T}_2\right)}{1+\left(1-\overline{T}_1\right)\left(1-\overline{T}_2\right)}\right)\left(1-\overline{T}_3\right)}, \frac{\left(\frac{\overline{I}_1+\overline{I}_2}{1+\left(\left(\overline{I}_1\right)\left(\overline{I}_2\right)\right)}\right)+\left(\overline{I}_3\right)}{1+\left(\frac{\overline{I}_1+\overline{I}_2}{1+\left(\left(\overline{I}_1\right)\left(\overline{I}_2\right)\right)}\right)+\left(\overline{I}_3\right)}, \frac{\left(\frac{\overline{F}_1+\overline{F}_2}{1+\left(\left(\overline{F}_1\right)\left(\overline{F}_2\right)\right)}\right)+\left(\overline{F}_3\right)}{1+\left(\frac{\overline{F}_1+\overline{F}_2}{1+\left(\left(\overline{F}_1\right)\left(\overline{F}_2\right)\right)}\right)+\left(\overline{F}_3\right)} \right) \right\rangle$$

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$$
= \left\langle \left(\frac{\left(\frac{(\underline{T}_2)(\underline{T}_3)}{1+(1-\underline{T}_2)(1-\underline{T}_3)}\right)(\underline{T}_1)}{1+\left(1-\frac{(\underline{T}_2)(\underline{T}_3)}{1+(1-\underline{T}_2)(1-\underline{T}_3)}\right)(1-\underline{T}_1)}, \frac{\left(\frac{\underline{I}_2+\underline{I}_3}{1+((\underline{I}_2)(\underline{I}_3))}\right)+(\underline{I}_1)}{1+\left(\frac{\underline{I}_2+\underline{I}_3}{1+((\underline{I}_2)(\underline{I}_3))}\right)+(\underline{I}_1)}, \frac{\left(\frac{\underline{F}_2+\underline{F}_3}{1+((\underline{F}_2)(\underline{F}_3))}\right)+(\underline{F}_1)}{1+\left(\frac{\underline{F}_2+\underline{F}_3}{1+((\underline{F}_2)(\underline{F}_3))}\right)+(\underline{F}_1)} \right), \right.
$$

$$
\left. \left(\frac{\left(\frac{(\overline{T}_2)(\overline{T}_3)}{1+(1-\overline{T}_2)(1-\overline{T}_3)}\right)(\overline{T}_1)}{1+\left(1-\frac{(\overline{T}_2)(\overline{T}_3)}{1+(1-\overline{T}_2)(1-\overline{T}_3)}\right)(1-\overline{T}_1)}, \frac{\left(\frac{\overline{I}_2+\overline{I}_3}{1+((\overline{I}_2)(\overline{I}_3))}\right)+(\overline{I}_1)}{1+\left(\frac{\overline{I}_2+\overline{I}_3}{1+((\overline{I}_2)(\overline{I}_3))}\right)+(\overline{I}_1)}, \frac{\left(\frac{\overline{F}_2+\overline{F}_3}{1+((\overline{F}_2)(\overline{F}_3))}\right)+(\overline{F}_1)}{1+\left(\frac{\overline{F}_2+\overline{F}_3}{1+((\overline{F}_2)(\overline{F}_3))}\right)+(\overline{F}_1)} \right) \right\rangle
$$

$$
= A \otimes_E (B \otimes_E C)
$$

□

5. Numerical Example

In this section, Mondal et al. [28] illustrate the applicability of the proposed RNS Einstein operations with the case study. A company wants to invest a specific amount of money in the most suitable investment fund. They have four options for deciding where to invest: car company (p_1) , food company (p_2), computer company (p_3), and arms company (p_4). They consider three factors, which are risk analysis (f_1), growth analysis (f_2), and environmental impact analysis (f_3). The four potential choices are to be chosen based on the criteria determined by the RN number provided by the three decision-makers.

Step 1: The decision maker constructs a decision matrix for four alternatives and three criteria using rough neutrosophic numbers as follows:

Table 1: Decision matrix with rough neutrosophic number

	f_1	f_2	f_3
p_1	$\langle (0.2,0.3,0.4), (0.4,0.3,0.4) \rangle$	$\langle (0.4,0.3,0.4), (0.6,0.1,0.4) \rangle$	$\langle (0.2,0.4,0.4), (0.4,0.4,0.4) \rangle$
p_2	$\langle (0.3,0.4,0.5), (0.4,0.3,0.6) \rangle$	$\langle (0.5,0.3,0.6), (0.7,0.1,0.4) \rangle$	$\langle (0.3,0.3,0.4), (0.4,0.3,0.3) \rangle$
p_3	$\langle (0.4,0.3,0.6), (0.5,0.2,0.3) \rangle$	$\langle (0.5,0.3,0.4), (0.6,0.3,0.3) \rangle$	$\langle (0.2,0.3,0.5), (0.4,0.1,0.3) \rangle$
p_4	$\langle (0.3,0.5,0.5), (0.4,0.3,0.1) \rangle$	$\langle (0.4,0.5,0.5), (0.5,0.3,0.3) \rangle$	$\langle (0.3,0.6,0.4), (0.4,0.5,0.4) \rangle$

Step 2: Determine the criteria weights by using Definition 2.6. The criteria weight calculated as follows:

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$$COS_j\left[N(R)\right]=\frac{1}{n}\sum_{i=1}^{n}\cos\frac{\pi}{2}\left(\frac{\left|4-2\left(\underline{T}_{ij}+\overline{T}_{ij}\right)+\underline{I}_{ij}+\overline{I}_{ij}+\underline{F}_{ij}+\overline{F}_{ij}\right|}{8}\right)$$

$$COS_1\left[N(R)\right]=\frac{1}{4}\left(\begin{array}{l}\cos\frac{\pi}{2}\left(\frac{\left|4-2(0.2+0.4)+0.3+0.3+0.4+0.4\right|}{8}\right)+\cos\frac{\pi}{2}\left(\frac{\left|4-2(0.3+0.4)+0.4+0.3+0.5+0.6\right|}{8}\right)+\\ \cos\frac{\pi}{2}\left(\frac{\left|4-2(0.4+0.5)+0.3+0.2+0.6+0.3\right|}{8}\right)+\cos\frac{\pi}{2}\left(\frac{\left|4-2(0.3+0.4)+0.5+0.3+0.5+0.1\right|}{8}\right)\end{array}\right)$$

$$=\frac{1}{4}\left(0.6788+0.6494+0.7604+0.7071\right)$$

$$COS_1\left[N(R)\right]=0.6989$$

Consequently,

$$COS_2\left[N(R)\right]=0.7988$$

$$COS_3\left[N(R)\right]=0.6779$$

Then,

$$w_j=\frac{COS_j\left[N(R)\right]}{\sum_j^n COS_j\left[N(R)\right]}$$

$$w_1=\frac{0.6989}{2.1756}=0.3212$$

$$w_2=\frac{0.7988}{2.1756}=0.3672$$

$$w_3=\frac{0.6779}{2.1756}=0.3116$$

Step 3: Determine the weighted value, refer to Proposition 4.1. (c) .

$$RNSE\left[p_1\middle|f_1,f_2,f_3\right]$$

$$=\left\langle\begin{array}{l}0.3212(0.2,0.3,0.4)\oplus_E 0.3672(0.4,0.3,0.4)\oplus_E 0.3116(0.2,0.4,0.4),\\ 0.3212(0.4,0.3,0.4)\oplus_E 0.3672(0.6,0.1,0.4)\oplus_E 0.3116(0.4,0.4,0.4)\end{array}\right\rangle$$

$$=\left\langle(0.2765,0.3287,0.4000),(0.4797,0.2240,0.4000)\right\rangle$$

$$RNSE\left[p_2\middle|f_1,f_2,f_3\right]$$

$$=\left\langle\begin{array}{l}0.3212(0.3,0.4,0.5)\oplus_E 0.3672(0.5,0.3,0.6)\oplus_E 0.3116(0.3,0.3,0.4),\\ 0.3212(0.4,0.3,0.2)\oplus_E 0.3672(0.7,0.1,0.4)\oplus_E 0.3116(0.4,0.3,0.3)\end{array}\right\rangle$$

$$=\left\langle(0.3779,0.3297,0.5010),(0.5274,0.2033,0.2948)\right\rangle$$

$$RNSE\left[p_3\middle|f_1,f_2,f_3\right]$$

$$=\left\langle\begin{array}{l}0.3212(0.3,0.5,0.5)\oplus_E 0.3672(0.4,0.5,0.5)\oplus_E 0.3116(0.3,0.6,0.4),\\ 0.3212(0.5,0.2,0.3)\oplus_E 0.3672(0.6,0.3,0.3)\oplus_E 0.3116(0.4,0.1,0.3)\end{array}\right\rangle$$

$$=\left\langle(0.3808,0.3000,0.4907),(0.5102,0.1889,0.3000)\right\rangle$$

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$RNSE\left[p_4 \middle| f_1, f_2, f_3\right]$

$$=\left\langle \begin{array}{c} 0.3212(0.3,0.5,0.5)\oplus_E 0.3672(0.4,0.5,0.5)\oplus_E 0.3116(0.3,0.6,0.4), \\ 0.3212(0.4,0.3,0.1)\oplus_E 0.3672(0.5,0.3,0.3)\oplus_E 0.3116(0.4,0.5,0.4) \end{array} \right\rangle$$

$$=\langle(0.3376,0.5299,0.4671),(0.4380,0.3541,0.2353)\rangle$$

Step 4: Determine the score values and accuracy value based on Definition 2.7, we compute the score values $S(p_i)(i=1,2,3,4)$.

$S(p_1)=0.5672, S(p_2)=0.5961, S(p_3)=0.6019, S(p_4)=0.5315$.

Because the score values vary, we do not calculate accuracy values.

Step 5: Order the priority. The alternatives are ranked as follows

$p_3 \succ p_2 \succ p_1 \succ p_4$. Therefore, the best alternative is a p_3, computer company.

6. Conclusion

In conclusion, this paper introduces the Einstein operator in rough neutrosophic set environments, overcoming the limitations of other aggregation methods. The proposed operator leverages the Einstein operation within a rough neutrosophic set, yielding desirable properties that are mathematically validated and providing a robust framework for effectively handling imperfect and incomplete information. This enhances decision-making processes and presents the properties of the new Einstein operational law based on rough neutrosophic sets (RNS). The research aims to apply this method to real-world multiple-criteria decision-making (MCDM) problems. The results from the numerical example adopted by Mondal et al. [28] demonstrate the applicability of the proposed rough neutrosophic Einstein operational laws. Additionally, future research should explore rough neutrosophic sets with other operations, such as the Frank and Dombi operations.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965.

[2] Z. Pawlak, "Rough sets and fuzzy sets," *Fuzzy Sets and Systems*, vol. 17, no. 1, pp. 99-102, 1985.

[3] F. Smarandache, *Neutrosophy: Neutrosophic Probability, Set, and Logic: Analytic Synthesis & Synthetic Analysis*, 1998.

[4] S. Broumi, F. Smarandache, and M. Dhar, "Rough neutrosophic sets," *Infinite Study*, vol. 32, pp. 493-502, 2014.

[5] S. Alias, D. Mohamad, and A. Shuib, "Rough neutrosophic multisets," *Neutrosophic Sets and Systems*, vol. 16, pp. 80-88, 2017.

[6] Q. Jin, K. Hu, C. Bo, and L. Li, "A new single-valued neutrosophic rough sets and related topology," *Journal of Mathematics*, vol. 1, pp. 5522021, 2021.

[7] D. J. S. Martina and G. Deepa, "Application of multi-valued rough neutrosophic set and matrix in multi-criteria decision-making: Multi-valued neutrosophic rough set and matrix," *Mathematics in Applied Sciences and Engineering*, vol. 4, no. 3, pp. 227–248, 2023.

[8] C. Bo, X. Zhang, S. Shao, and F. Smarandache, "New multigranulation neutrosophic rough set with applications," *Symmetry*, vol. 10, no. 11, p. 578, 2018.

[9] H. Zhao and H. Y. Zhang, "On hesitant neutrosophic rough set over two universes and its application," *Artificial Intelligence Review*, vol. 53, pp. 4387-4406, 2020.

[10] K. Rogulj, J. Kilić Pamuković, and M. Ivić, "Hybrid MCDM based on VIKOR and cross entropy under rough neutrosophic set theory," *Mathematics*, vol. 9, no. 12, p. 1334, 2021.

[11] E. M. Kumari and M. Thirucheran, "Rough hesitant bipolar neutrosophic sets and its applications in game theory," *Advances in Applied Mathematical Sciences*, vol. 21, no. 1, pp. 493-517, 2021.

[12] S. Das, R. Das, and B. C. Tripathy, "Topology on rough pentapartitioned neutrosophic set," *Iraqi Journal of Science*, pp. 2630-2640, 2022.

DOI: https://doi.org/10.54216/IJNS.270105

[13] A. E. Samuel and R. Narmadhagnanam, "Pi-distance of rough neutrosophic sets for medical diagnosis," *Infinite Study*, 2019.

[14] J. S. M. Donbosco and D. Ganesan, "The energy of rough neutrosophic matrix and its application to MCDM problem for selecting the best building construction site," *Decision Making: Applications in Management and Engineering*, vol. 5, no. 2, pp. 30-45, 2022.

[15] B. Praba, S. Pooja, and N. Sivakumar, "Attribute based double bounded rough neutrosophic sets in facial expression detection," *Neutrosophic Sets and Systems*, vol. 49, no. 1, p. 21, 2022.

[16] M. Myvizh, "Choosing the best location in the hospital by MADM strategy using the energy of a rough neutrosophic matrix," *Mathematics in Engineering, Science & Aerospace (MESA)*, vol. 14, no. 3, 2023.

[17] M. Shazib Hameed, S. Ali, S. Shoaib, M. Mukhtar, M. Kashif Ishaq, and U. Mukhtiar, "On characterization of χ-single valued neutrosophic subgroups," *Journal of Mathematics and Computer Science*, vol. 24, no. 04, pp. 358–369, 2021.

[18] M. Abdel-Basset and M. Mohamed, "The role of single valued neutrosophic sets and rough sets in smart city: Imperfect and incomplete information systems," *Measurement*, vol. 124, 2018.

[19] M. Khan, M. Gulistan, N. Hassan, and A. M. Nasruddin, "Air pollution model using neutrosophic cubic Einstein averaging operators," *Neutrosophic Sets and Systems*, vol. 32, no. 1, p. 24, 2020.

[20] J. Ye, E. Türkarslan, M. Ünver, and M. Olgun, "Algebraic and Einstein weighted operators of neutrosophic enthalpy values for multi-criteria decision making in neutrosophic multi-valued set settings," *Granular Computing*, vol. 7, no. 3, pp. 479-487, 2022.

[21] H. M. A. Farid and M. Riaz, "Single-valued neutrosophic Einstein interactive aggregation operators with applications for material selection in engineering design: Case study of cryogenic storage tank," *Complex & Intelligent Systems*, vol. 8, no. 3, pp. 2131-2149, 2022.

[22] M. Jamil, F. Afzal, A. Akgül, S. Abdullah, A. Maqbool, A. Razzaque, and J. Awrejcewicz, "Einstein aggregation operators under bipolar neutrosophic environment with applications in multi-criteria decision-making," *Applied Sciences*, vol. 12, no. 19, p. 10045, 2022.

[23] M. Kamran, S. Ashraf, N. Salamat, M. Naeem, and T. Botmart, "Cyber security control selection based decision support algorithm under single valued neutrosophic hesitant fuzzy Einstein aggregation information," *AIMS Mathematics*, vol. 8, no. 3, pp. 5551-5573, 2023.

[24] D. Ajay, G. Selvachandran, J. Aldring, P. H. Thong, L. H. Son, and B. C. Cuong, "Einstein exponential operation laws of spherical fuzzy sets and aggregation operators in decision making," *Multimedia Tools and Applications*, vol. 82, no. 27, pp. 41767-41790, 2023.

[25] B. Li, J. Wang, L. Yang, and X. Li, "A novel generalized simplified neutrosophic number Einstein aggregation operator," *Infinite Study*, 2018.

[26] A. Fahmi, M. Aslam, and R. Ahmed, "Decision-making problem based on generalized interval-valued bipolar neutrosophic Einstein fuzzy aggregation operator," *Soft Computing*, vol. 27, no. 20, pp. 14533–14551, 2023.

[27] S. M. Zainal, A. T. Ab Ghani, and L. Abdullah, "On some logical relation properties for rough neutrosophic set," in *Journal of Physics: Conference Series*, vol. 1878, p. 012048, 2021.

[28] K. Mondal, S. Pramanik, and B. C. Giri, "Rough neutrosophic aggregation operators for multi-criteria decision-making," in *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, pp. 79-105, 2019.

DOI: https://doi.org/10.54216/IJNS.270105