A Numerical Study of Neutrosophic Finite Difference Method and Some Applications

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Abstract

In this paper, we present some results about the neutrosophic-generalized version of finite-difference method, where we prove its essential properties, and we apply it to many different examples to clarify the validity of our work. In addition, some numerical tables related to the results will be clarified and presented.

Keywords: Neutrosophic equation; Neutrosophic FDM; Numerical table; Numerical application

1. Introduction

Numerical analysis is one of the most prominent branches of applied mathematics, as it is primarily concerned with studying the numerical solutions of many differential and algebraic equations alike [1]. Where we see in [2] an application of numerical methods with the aim of finding approximations for exact solutions to differential equations, where numerical tables and graphical diagrams are widely used in extracting results [3].

Numerical analysis has many applications, especially in other sciences, where numerical algorithms are widely used in physics, mathematics, computer science, and economics [4]. On the other hand, we find an application of numerical methods in finding numerical approximations to solutions to algebraic equations that branch out from many physical and economic problems [5].

One of the most important numerical methods used in the study of numerical analysis is the finite difference method, which has been used by many researchers in numerical studies related to the aforementioned equations see [6-8]. Neutrosophic generalizations of previous equations can be derived by putting neutrosophic variables and constants $a + bI$ or $x + yI$ instead of a or x.[9]. For more details about neutrosophic generalization of differential and integral equations with many different methods for solving these equations, see [10].

This has motivated us to present some results about the neutrosophic-generalized version of finite-difference method, where we prove its essential properties, and we apply it to many different examples to clarify the validity of our work. In addition, some numerical tables related to the results will be clarified and presented.

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2. Main Results

- **Neutrosophic Finite-Difference Method (NFDM)**

Taking a two-point boundary-value problem with a second order differential equation, which takes the form:

$$f" = g(x+yI, f, f'), \qquad a+cI \le x+yI \le b+dI,$$

$$f(a+cI) = \alpha + sI, f(b) = \beta + kI$$

where g is a function, $a+cI$ and $b+dI$ are the end points, and $f(a+cI) = \alpha + sI, f(b) = \beta + kI$

are the boundary conditions.

Example 2.1

$$f" = \frac{1}{8}(32 + 2(x+yI)^3 - ff'), \quad 1+I \le x+yI \le 3+I$$

$$f(1+I) = 17 + I, f(3+I) = \frac{43}{3} + I$$

Theorem 2.1

Suppose the function g in the boundary-value problem

$$f" = g(x+yI, f, f'), \qquad a+cI \le x+yI \le b+dI, \qquad f(a+cI) = \alpha + sI, f(b+dI) = \beta + kI$$

is continuous on

$$D(I) = ((x+yI, f, f') | a+cI \le x+yI \le b+dI, -\infty < f < \infty, -\infty < f' < \infty)$$

and that the partial derivatives g_f and $g_{f'}$ are also continuous in D(I)

If:

1) $g_f(x+yI, f, f') > 0$ for all $(x+yI, f, f') \in D(I)$, and

2) A constant M(I) exists with $|g_{f'}(x+yI, f, f')| \le M(I)$ for all $(x+yI, f, f') \in D(I)$, then the boundary-value problem has a unique solution.

A nonlinear boundary-value problem takes on the form of:

$$f" = g(x+yI, f, f'), a+cI \le x+yI \le b+dI, \qquad f(a+cI) = \alpha + sI, f(b+dI) = \beta + kI$$

(1) g and the partial derivatives g_f and $g_{f'}$ are continuous on:

$$D(I) = ((x+yI, f, fy') | a+cI \le x+yI \le b+dI, -\infty < f < \infty, -\infty < f' < \infty)$$

(2) $g_f(x+yI, f, f') \ge \delta$+uI on D(I), for some $\delta + uI > 0$.

(3) there exists:

$$k = \max_{(x+yI, f, f') \in D(I)} |g_f\,(x+yI, f, f')| \text{ , and } L = \max_{(x+yI, f, f') \in D} |g_{f'}\,(x+yI, f, f')|$$

For:

$$f" = p(x+yI)f' + q(x+yI)f + r(x+yI)$$

is expanded using f in a third Taylor polynomial about $x_i + y_i I$ evaluated at $x_{i+1} + y_{i+1}I$ and $x_{i-1} + y_{i-1}I$,

$$f"(x_i + y_iI) = \frac{1}{h^2}[f(x_{i+1} + y_{i+1}I) - 2f(x_i + y_iI) + f(x_{i-1} + y_{i-1}I)] - \frac{h^2}{12} f^{(4)}\,(\xi_i + v_iI) \qquad (5.1)$$

for some $\xi_i + v_iI$ in $(x_{i-1} + y_{i-1}I, x_{i+1} + y_{i+1}I)$, and:

$$f'(x_i + y_iI) = \frac{1}{2h}[y(x_{i+1} + y_{i+1}I) - y(x_{i-1} + y_{i-1}I)] - \frac{h^2}{6} f'''\,(\eta_i + u_iI) \qquad (5.2)$$

for some $\eta_i + u_iI$ in $(x_{i-1} + y_{i-1}I, x_{i+1} + y_{i+1}I)$.

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Step 1:

We have to divide the interval $[a+cI, b+dI]$ into $(N+I)$ equal subintervals which gives us:

$$h=\frac{(b+dI-a-cI)}{(N+I)}$$

with end points at $x_i+y_iI = a+cI+ih$ for .

Step 2:

$$f"(x_i+y_iI)= g(x_i+y_iI, f(x_i+y_iI), f'(x_i+y_iI))$$

Hence

$$\frac{f(x_{i+1}+y_{i+1}I)-2f(x_i+y_iI)+f(x_{i+1}+y_{i+1}I)}{h^2} = g\left(x_i+y_iI, f(x_i+y_iI), \frac{y(x_{i+1}+y_{i+1}I)-y(x_{i-1}+y_{i-1}I)}{2h}-\frac{h^2}{6}f'''(\eta_i)\right) + \frac{h^2}{12}f^{(4)}(\xi_i) \qquad (5.3)$$

for some ξ_i+v_iI and η_i+u_iI in the interval $(x_{i-1}+y_{i-1}I, x_{i+1}+y_{i+1}I)$.

Step 3:

We have:

$$d_0=\alpha+sI, \qquad d_{N+1}=\beta+kI$$

And:

$$-\frac{d_{i+1}-2d_i+d_{i-1}}{h^2+I}+f\left(x_i,d_i,\frac{d_{i+1}-d_{i-1}}{2h+I}\right)=0$$

for $i = 1,2,\dots,N$.

Step 4:

$$2d_1-d_2+h^2f\left(x_1,w_1,\frac{d_2-\alpha}{2h+I}+I\right)-\alpha=0$$
$$-d_1+2d_2-d_3+h^2f\left(x_2,d_2,\frac{d_3-d_1}{2h+I}+I\right)=0$$
$$\vdots \qquad (5.4)$$
$$-d_{N-2}+2d_{N-1}-d_N+h^2f\left(x_{N-1},d_{N-1},\frac{d_N-d_{N-2}}{2h+I}\right)=0$$
$$-d_{N-1}+2d_N+h^2f\left(x_N,d_N,\frac{\beta-d_{N-1}}{2h+I}\right)-\beta-kI=0$$

Step 5:

$d^{(0)} = (d_1^{(0)}, d_2^{(0)}, \dots, d_N^{(0)})^t$, $F(\boldsymbol{w}^{(0)})$ and defining the Jacobian matrix as follows:

$$J(d_1,d_2,\dots,d_N)_{ij}=-1+\frac{h+I}{2}g_{f'}\left(x_i,d_i,\frac{d_{i+1}-d_{i-1}}{2h+I}\right), \text{for } i=j-1 \text{ and } j=2,\dots,N$$

$$J(d_1,d_2,\dots,d_N)_{ij}=2+h^2g_f\left(x_i,d_i,\frac{d_{i+1}-d_{i-1}}{2h}\right), \text{for } i=j \text{ and } j=1,\dots,N$$

$$J(d_1,d_2,\dots,d_N)_{ij}=-1-\frac{h+I}{2}g_{f'}\left(x_i,d_i,\frac{d_{i+1}-d_{i-1}}{2h+I}\right), \text{for } i=j+1 \text{ and } j=1,\dots,N-1 \qquad (5.5)$$

where $d_0=\alpha+sI$ and $d_{N+1}=\beta+kI$.

Remark 2.1

We can find the initial approximation $\boldsymbol{d}^{(0)}$by using the following equation equation:

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$$\boldsymbol{d}^{(0)} = \alpha + sI + \frac{\beta + kI - \alpha - sI}{b + dI - a - cI}(x_i - a - cI)$$

$$J(d_1, \dots, d_N)(e_1, \dots, e_N)^t = -F(d_1, d_2, \dots, d_N)$$

where $d_i^{(k)} = d_i^{(k-1)} + l_i$, for each $i = 1, 2, \dots, N$.

Crout LU Factorization

Since $J(\boldsymbol{d})$ is tridiagonal, it takes on the form:

$$J(\boldsymbol{d}) = \begin{bmatrix} j_{11} & j_{12} & 0 & \dots & \dots & \dots & \dots & 0 \\ j_{21} & j_{22} & j_{23} & 0 & \dots & \dots & \dots & 0 \\ 0 & j_{32} & j_{33} & j_{34} & 0 & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & j_{i-1,j} \\ 0 & 0 & 0 & 0 & 0 & 0 & j_{i,j-1} & j_{ij} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_{11} & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ q_{21} & q_{22} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & q_{32} & q_{33} & 0 & \dots & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{i,j-1} & q_{ij} \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & h_{12} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & h_{23} & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & h_{34} & 0 & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & h_{i-1,j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(1) Computing the first column of Q, where $q_{i1} = a_{i1}$

(2) Computing the first row of H, where $h_{1j} = \frac{a_{1j}}{l_{11}}$

(3) Alternately computing the columns of Q and the rows of H, where:

$$q_{ij} = a_{ij} - \sum_{k=1}^{j-1} q_{ik} h_{kj} \text{, for } j \le i, i = 1,2, \dots, N$$

$$h_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} q_{ik} h_{kj}}{q_{ii}} \text{, for } i \le j, j = 2,3, \dots, N$$

$J(d_1, \dots, d_N)(e_1, \dots, e_N)^t = -F(d_1, d_2, \dots, d_N)$

- **MATLAB Application**

In this section, we will solve the boundary value problem of nonlinear ordinary differential equation

$f'' = \frac{1}{8}(32 + 2(x + yI)^3 - ff'), \quad 1 + I \le x + yI \le 3 + I \; f(1 + I) = 17 + I, f(3 + I) = \frac{43}{3} + I$

with $h = 0.2 + I$

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Step 1:

Since we know that $h = 0.2 + I$, this means that our interval [1+I,3+I] is divided into $N + 1 + I = 20 + I$ equal subintervals.

Step 2:

Next, we will define the boundary conditions such that $d_0 = 17 + I$ and $d_{20} = 14.23415 + I$.

Step 3:

Using the equation define initial approximation $\boldsymbol{d}^{(0)}$.

$$d(0) = \begin{pmatrix} 16.23167 + I, 16.732234 + I, 16.6 + I, 16.433687 + I, 16.33333333 + I, 16.2 + I, 16.2897 + I, 15.988653 + I, \\ 15.8 + I, 15.667 + I, 15.523 + I, 15.4 + I, 15.223 + I, 15.13498 + I, 15 + I, 14.866218 + I, 14.7332673 + I, 14.47161 + I \end{pmatrix}^t$$

Step 4:

We know that $N = 19 + I$, which implies that $F(\boldsymbol{d})$ is

19×19 nonlinear system. $F(\boldsymbol{d})$ is:

$$2d_1 - d_2 + 0.01\left(3 + I + 0.3121 + I + \frac{d_1(d_2 - 17)}{1.6 + I}\right) - 17 = 0$$

$$-d_1 + 2d_2 - d_3 + 0.01\left(3 + I + 0.4776 + \frac{d_2(d_3 - d_1)}{1.6 + I}\right) = 0$$

$$-d_2 + 2d_3 - d_4 + 0.01\left(3 + I + 0.51314 + \frac{d_3(d_4 - d_2)}{1.6 + I}\right) = 0$$

$$-d_3 + 2d_4 - d_5 + 0.01\left(3 + I + 0.5567 + \frac{d_4(d_5 - d_3)}{1.6 + I}\right) = 0$$

$$-d_4 + 2d_5 - d_6 + 0.01\left(3 + I + 0.87791 + \frac{d_5(d_6 - d_4)}{1.6 + I}\right) = 0$$

$$-d_5 + 2d_6 - d_7 + 0.01\left(3 + I + 1.02445 + \frac{d_6(d_7 - d_5)}{1.6 + I}\right) = 0$$

$$-d_6 + 2d_7 - d_8 + 0.01\left(3 + I + 1.2431 + \frac{d_7(d_8 - d_6)}{1.6 + I}\right) = 0$$

$$-d_7 + 2d_8 - d_9 + 0.01\left(3 + I + 1.458 + \frac{d_8(d_9 - d_7)}{1.6 + I}\right) = 0$$

$$-d_8 + 2d_9 - d_{10} + 0.01\left(3 + I + 1.72215 + \frac{d_9(d_{10} - d_8)}{1.6 + I}\right) = 0$$

$$-d_9 + 2d_{10} - d_{11} + 0.01\left(3 + I + 2.67 + \frac{d_{10}(d_{11} - d_9)}{1.6 + I}\right) = 0$$

$$-d_{10} + 2d_{11} - d_{12} + 0.01\left(3 + I + 2.3778 + \frac{d_{11}(d_{12} - d_{10})}{1.6 + I}\right) = 0$$

$$-d_{11} + 2d_{12} - d_{13} + 0.01\left(3 + I + 2.6221 + \frac{d_{12}(d_{13} - d_{11})}{1.6 + I}\right) = 0$$

$$-d_{12} + 2d_{13} - d_{14} + 0.01\left(3 + I + 3.3215 + \frac{d_{13}(d_{14} - d_{12})}{1.6 + I}\right) = 0$$

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$$-d_{13}+2d_{14}-d_{15}+0.01\left(3+I+3.4117+\frac{d_{14}(d_{15}-d_{13})}{1.6+I}\right)=0$$

$$-d_{14}+2d_{15}-d_{16}+0.01\left(3+I+3.90625+\frac{d_{15}(d_{16}-d_{14})}{1.6+I}\right)=0$$

$$-d_{15}+2d_{16}-d_{17}+0.01\left(3+I+4.3213+\frac{d_{16}(d_{17}-d_{15})}{1.6+I}\right)=0$$

$$-d_{16}+2d_{17}-d_{18}+0.01\left(3+I+4.95565+\frac{d_{17}(d_{18}-d_{16})}{1.6+I}\right)=0$$

$$-d_{17}+2d_{18}-d_{19}+0.01\left(3+I+5.4567+\frac{d_{18}(d_{19}-d_{17})}{1.6+I}\right)=0$$

$$-d_{18}+2d_{19}+0.01\left(3+I+6.09725+\frac{d_{19}(14.22163-d_{18})}{1.6+I}\right)-14.333333=0$$

x_i+yI	w_i+I	$\boldsymbol{w}^{(0)}$+I	$\boldsymbol{w}^{(1)}$+I	$\boldsymbol{w}^{(2)}$+I	$\boldsymbol{w}^{(3)}$+I	$\boldsymbol{w}^{(4)}$+I	$\boldsymbol{w}^{(5)}$+I
1.0+I	w_0 + I	19+I	19+I	19.+I	19.+I	19.+I	19.+I
1.2+I	w_1+I	18.86671+I	18.7641+I	18.75606+I	18.7606+I	18.7606+I	18.7606+I
1.4+I	w_2+I	18.73331+I	18.5212+I	18.51735+I	18.51635+I	18.5135+I	18.5135+I
1.6+I	w_3+I	18.6000+I	18.2714+I	18.28579+I	18.2859+I	18.2859+I	18.2859+I
1.8+I	w_4 + I	18.46671+I	18.0152+I	17.9974+I	17.99674+I	17.9974+I	17.9974+I
2+I	w_5 + I	18.33331+I	17.7532+I	17.72979+I	17.7299+I	17.7299+I	17.7299+I
2.2+I	w_6+I	18.2000+I	17.4867+I	17.45478+I	17.4578+I	17.4578+I	17.4578+I
2.4+I	w_7+I	18.06617+I	17.2175+I	17.13831+I	17.18631+I	17.1831+I	17.1831+I
2.6+I	w_8 + I	17.93133+I	16.9477+I	16.904485+I	16.90856+I	16.9085+I	16.9085+I
2.8+I	w_9+I	17.8000+I	16.6808+I	16.63377+I	16.63767+I	16.6377+I	16.6377+I
3+I	w_{10} + I	17.66167+I	16.4208+I	16.373552+I	16.3752+I	16.3752+I	16.3752+I
3.2+I	w_{11} + I	1'7.53133+I	16.1733+I	16.124369+I	16.12691+I	16.1269+I	16.1269+I
3.4+I	w_{12} + I	17.41+I	15.9449+I	15.897597+I	15.89917+I	15.8997+I	15.8997+I
3.6+I	w_{13}+I	17.26167+I	15.7443+I	15.70522+I	15.7022+I	15.7022+I	15.7022+I
3.8+I	w_{14} + I	17.13313+I	15.5820+I	15.54548+I	15.54248+I	15.5448+I	15.5448+I
4+I	w_{15}+I	17.0000+I	15.4710+I	15.43597+I	15.42397+I	15.4397+I	15.4397+I
4.2+I	w_{16} + I	16.86167+I	15.4271+I	15.45017+I	15.42017+I	15.4017+I	15.4017+I
4.4+I	w_{17}+I	16.73233+I	15.4694+I	15.44583+I	15.44283+I	15.4483+I	15.4483+I
4.6+I	w_{18} + I	16.62059+I	15.6008+I	15.55999+I	15.59929+I	15.5999+I	15.5999+I
4.8+I	w_{19}+I	16.90589+I	15.8854+I	15.85844+I	15.88434+I	15.8844+I	15.8844+I
5+I	w_{20} + I	16.33533+I	15.3333+I	15.35333+I	15.333333+I	15.3333+I	15.3333+I

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3. Conclusion

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References

[1] J. R. Cash, "Block Runge-Kutta methods for numerical integration of initial value problems in ordinary diff. eq.s Part II: the stiff case," *Math. Of Computation*, vol. 40, no. 161, pp. 193-206, 1983.

[2] J. C. Butcher, "Numerical methods for diff. eq.s and applications," *The Arabian Journal for Science and Engineering*, Dharan, Saudi Arabia, vol. 22, no. 2, pp. 17-29, 1997.

[3] S. H. Lee and J. Kim, "Numerical Solutions for High-Order Differential Equations Using Adaptive Block Methods," *Journal of Computational and Applied Mathematics*, vol. 387, pp. 112-125, 2021.

[4] A. M. Amin, I. Shah, M. Asif, K. M. Abualnaja, E. E. Mahmoud, and A. Abdel–Aty, "A powerful numerical technique for treating twelfth-order boundary value problems," *Open Physics*, vol. 18, pp. 1048-1062, 2020.

[5] D. G. Zill, *A First Course in Differential Equations With Modeling Application*, Brooks/Cole, Cengage Learning, 2008.

[6] P. B. Worland, "Parallel methods for the numerical solutions of ordinary differential equations," *IEEE Transactions on Computers*, vol. 25, pp. 1045-1048, 1976.

[7] P. C. Chakravarti and P. B. Worland, "A class of self-starting methods for the numerical solution of y f (x, y)," *BIT*, vol. 11, pp. 368-383, 1971.

[8] R. Al-Deiakeh et al., "Lie symmetry analysis, explicit solutions, and conservation laws of the time-fractional Fisher equation in two-dimensional space," *Journal of Ocean Engineering and Science*, vol. 7, no. 4, pp. 345-352, 2022.

[9] M. A. Khan and R. S. Gupta, "A Comparative Study of Numerical Techniques for Solving Stiff Differential Equations," *Applied Numerical Mathematics*, vol. 184, pp. 56-70, 2023.

[10] T. S. Nguyen and P. L. Tran, "Efficient Algorithms for Boundary Value Problems in Differential Equations," *Mathematics and Computers in Simulation*, vol. 198, pp. 310-325, 2022.

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