



A Unified Framework for Solving Abel's and Linear Volterra Integral Equations and Their Neutrosophic Generalizations Using the GALM Transform

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Abstract

Integral equations, including Abel's integral equation and linear Volterra integral equations of both the first and second kinds and neutrosophic Abel's integral equation and linear Volterra integral equations of both the first and second kinds, regularly appear in advanced problems across biology, chemistry, physics, and engineering, often modeling systems with memory effects or time-dependent interactions. This study explores the GALM transform as a powerful and unified method for solving these equations. The exact solution of Abel's integral equation and its neutrosophic version is derived, demonstrating the transform's simplicity and efficiency through practical applications. Additionally, the GALM transform is employed to solve linear Volterra integral equations of the first and second kinds with their neutrosophic generalizations, with illustrative examples provided to validate its effectiveness. By addressing a wide range of problems, this research establishes the GALM transform as an accurate, reliable, and versatile tool, offering significant advantages over traditional methods in solving complex scientific and engineering equations.

Keywords: Linear Volterra integral equations of the first and second kinds; Abel's integral equation; Neutrosophic Abel's equation; Neutrosophic integral equation; GALM transform; Convolution theorem; Inverse GALM transform

1. Introduction

Integral equations play a role extensively in mathematical analysis and the modeling of various systems in applied sciences, such as biology, chemistry, physics, and engineering [1-3]. Among these, the linear Volterra integral equations of the first and second kinds, along with Abel's integral equation, frequently arise in problems involving memory effects or time-dependent interactions.

The linear Volterra integral equation of the first kind is expressed as:

$$f(x) + \lambda \int_{x_0}^x k(x,t) u(t)dt = 0 \quad (1)$$

Where $u(t)$ is the unknown function, $k(x,t)$ is the kernel, λ is a numerical parameter, and $f(x)$ is a given real-valued function, similarly, the linear Volterra integral equation of the second kind is given by:

$$u(x) = f(x) + \lambda \int_{x_0}^x k(x,t) u(t)dt, \quad (2)$$

With the same notations as above, Abel's integral equation, another cornerstone in this field is formulated as:

$$f(x) + \lambda \int_0^x \left(\frac{1}{\sqrt{k(x-t)}} \right) u(t) dt = 0 \quad (3)$$

To solve each these integral equations efficiently, we utilize the GALM transform, a novel integral transform [5] represented by the operator $G\{f(t)\}$ or simply $T(s)$, defined as:

$$T(s) = s^n \int_0^\infty a^{-t} f(st) dt$$

where $a > 1$, $t \in [0, \infty)$, and s is a complex parameter with $Re(s) > 0$. Importantly, the parameter s is independent of time t .

Neutrosophic generalizations of previous equations can be derived by putting neutrosophic variables and constants $a + bI$ or $x + yI$ instead of a or x . [9]. For more details about neutrosophic generalization of differential and integral equations with many different methods for solving these equations, see [10].

The present work aims to rigorously derive the exact solutions for these integral equations using the GALM transform. The study emphasizes reducing computational complexity and ensuring analytical precision [4]. By demonstrating the GALM transform's effectiveness and versatility across a range of applications [5-8], we establish it as a reliable and efficient tool for solving complex integral equations. To better understand its potential, it is important to explore the key properties that make the GALM transform a powerful method in solving such equations.

2. Properties of GALM integral transform

In this section, we explore the fundamental properties of the GALM integral transform, which play a crucial role in solving integral equations and simplifying complex expressions. These properties form the basis for applying the GALM transform effectively, allowing for an efficient approach to solving the first-kind linear Volterra integral equation. Key attributes, such as linearity, scaling, and convolution, will be discussed, along with any specific conditions required for their application. Understanding these properties is essential for effectively utilizing the GALM transform in our analytical approach.

2.1 Linearity property

GALM transform is a linear operator; Thus, for constants α and β :

$$G\{\alpha f(t) \pm \beta g(t)\} = \alpha F(s) \pm \beta G(s).$$

2.2 Scaling Property:

If the GALM transform of the function $f(t)$ is $T(s)$, then the GALM transform of the function $f(bt)$ is given by:

$$G\{f(bt)\} = \frac{T(bs)}{b^n}$$

2.3 Shifting Property of GALM Transform:

If $G\{f(t)\} = T(s)$ where $G\{f(t)\} = s^n \int_{t=0}^\infty a^{-t} f(st) dt$, then:

$$G\{e^{bt} f(t)\} = \frac{\ln(a)}{\ln(a) - bs} T\left(s \frac{\ln(a)}{\ln(a) - bs}\right)$$

2.4 Convolution theorem for GALM transform:

If the GALM integral transform of the function $f(t)$ and $g(t)$ are $F(s)$ and $G(s)$, respectively, then the GALM transform of their convolution $f(t) * g(t)$ is expressed as:

$$G\{f(t) * g(t)\} = S^{1-n} [F(s) \cdot G(s)]$$

This property highlights the convolution theorem for the GALM transform, allowing the convolution of two functions in the time domain to be simplified into a product of their transforms in the frequency domain.

2.5 The GALM transform of some elementary functions:

Table 1 presents the GALM transforms of several elementary functions. These transforms provide a powerful tool for analyzing and solving various types of integral equations. The table below summarizes the relationships between the functions and their corresponding GALM transforms:

Table 1: Summarizes the relationships between the functions and their corresponding GALM transforms

$f(t)$	$T(s) = G\{f(t)\}$
1	$\frac{s^n}{\ln(a)}$
t	$\frac{s^{n+1}}{(\ln(a))^2}$
t^m $m=0,1,2,$	$\frac{m! s^{n+m}}{(\ln(a))^{m+1}}$
t^m $m > -1$	$\frac{\Gamma(m+1)s^{n+m}}{(\ln(a))^{m+1}}$ Where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \text{Re}(z) > 0$
e^{bt}	$\frac{s^n}{\ln(a) - bs}$
$\sin(bt)$	$\frac{b s^{n+1}}{(\ln(a))^2 + b^2s^2}$
$\cos(bt)$	$\frac{s^n \ln(a)}{(\ln(a))^2 + b^2s^2}$
$\sinh(bt)$	$\frac{b s^{n+1}}{(\ln(a))^2 - b^2s^2}$
$\cosh(bt)$	$\frac{s^n \ln(a)}{(\ln(a))^2 - b^2s^2}$
$t e^{bt}$	$\frac{s^{n+1}}{(\ln(a) - bs)^2}$
$t^m e^{bt}$	$\frac{m! s^{n+m}}{(\ln(a) - bs)^{m+1}}$
$e^{ct} \sin(bt)$	$\frac{b s^{n+1}}{(\ln(a) - cs)^2 + b^2s^2}$
$e^{ct} \cos(bt)$	$\frac{s^n (\ln(a) - cs)}{(\ln(a) - cs)^2 + b^2s^2}$
$e^{ct} \sinh(bt)$	$\frac{b s^{n+1}}{(\ln(a) - cs)^2 - b^2s^2}$
$e^{ct} \cosh(bt)$	$\frac{s^n (\ln(a) - cs)}{(\ln(a) - cs)^2 - b^2s^2}$

2.6 Inverse of GALM integral transform:

The inverse of the GALM integral transform is given by:

$$G^{-1}\{T(s)\} = f(t) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{T(s)}{S^{n-1}} a^{\frac{t}{s}} ds$$

Where ϵ is a real number greater than the real part of the poles of $T(s)$, and the integration is performed along a vertical contour. This inversion formula allows the recovery of the original time function $f(t)$ from its transformed counterpart $T(s)$.

3. Solving Linear Volterra Integral Equations of the First Kind Using the Galm Transform

In this section, we explore an efficient method for solving such equations by applying the GALM transform. The GALM transform offers a powerful tool for transforming integral equations into simpler algebraic equations in the frequency domain. By utilizing the properties of the GALM transform, we can reduce the computational complexity and provide exact solutions for a wide range of linear Volterra integral equations.

In the present work, we assume that the kernel $k(x, t)$ of the linear Volterra integral equation of the first kind as defined in equation (1) is a difference kernel, which can be represented as a function of the difference $(x - t)$. Consequently, equation (1) can be rewritten as:

$$f(x) = \int_0^x k(x-t)u(t)dt \quad (4)$$

By applying the GALM transform to both sides of equation (4), we obtain:

$$G\{f(x)\} = G\left\{\int_0^x k(x-t)u(t)dt\right\} \quad (5)$$

Using the convolution theorem of the GALM transform, this becomes:

$$G\{f(x)\} = S^{1-n}G\{k(x)\}G\{u(x)\}$$

Which implies that

$$G\{u(x)\} = \left[\frac{S^{n-1}G\{f(x)\}}{G\{k(x)\}} \right] \quad (6)$$

Now, applying the inverse GALM transform to both sides of equation (6), we get:

$$u(x) = G^{-1}\left\{\left[\frac{S^{n-1}G\{f(x)\}}{G\{k(x)\}}\right]\right\} \quad (7)$$

This expression gives the required solution to equation (4).

3.1 APPLICATIONS

To illustrate the efficacy of the GALM transform in solving linear Volterra integral equations of the first kind, several applications are presented. These applications involve linear Volterra integral equations of the first kind with kernels that include exponential functions, hyperbolic functions, trigonometric functions, and others [5].

Application 3.1.1: Consider a linear Volterra integral equation of the first kind with a kernel that includes an exponential function:

$$x = \int_0^x e^{2(x-t)}u(t)dt \quad (8)$$

Solution:

Applying the GALM transform to both sides of equation (8), we obtain:

$$G\{x\} = G\left\{\int_0^x e^{2(x-t)}u(t)dt\right\} \quad (9)$$

Using the convolution theorem of the GALM transform on equation (9), we obtain:

$$G\{x\} = s^{1-n}G\{e^{2x}\}G\{u(x)\},$$

Which leads to:

$$\begin{aligned}\frac{s^{n+1}}{(\ln(a))^2} &= s^{1-n} \left(\frac{s^n}{\ln(a) - 2s} \right) U(x), \\ \frac{s^{n+1}}{(\ln(a))^2} &= \left(\frac{s}{\ln(a) - 2s} \right) U(x), \\ \frac{s^{n+1}}{(\ln(a))^2} \left(\frac{\ln(a) - 2s}{s} \right) &= U(x), \\ \frac{s^n \ln(a)}{(\ln(a))^2} - \frac{2s}{(\ln(a))^2} &= U(x). \\ \frac{s^n}{\ln(a)} - \frac{2s}{(\ln(a))^2} &= U(x)\end{aligned}\quad (10)$$

Taking the inverse GALM transform of both sides of equation (10), we get:

$$G^{-1} \left\{ \frac{s^n}{\ln(a)} \right\} - G^{-1} \left\{ \frac{2s}{(\ln(a))^2} \right\} = G^{-1} \{U(x)\}.$$

Thus, we obtain the exact solution to equation (8):

$$u(x) = 1 - 2x \quad (11)$$

Application 3.2.2: Consider linear Volterra integral equation of first kind whose kernel containing exponential function

$$\sin x = \int_0^x e^{3(x-t)} u(t) dt \quad (12)$$

Solution:

Applying the GALM transform to both sides of equation (12), we obtain:

$$G\{\sin(x)\} = G \left\{ \int_0^x e^{3(x-t)} u(t) dt \right\} \quad (13)$$

Using the convolution theorem of the GALM transform (property 2.4) on equation (13), we obtain:

$$G\{\sin(x)\} = s^{1-n} G\{e^{3x}\} U(x),$$

Which leads to:

$$\begin{aligned}\frac{s^{n+1}}{(\ln(a))^2 + s^2} &= s^{1-n} \left(\frac{s^n}{\ln(a) - 3s} \right) U(x), \\ \frac{s^{n+1}}{(\ln(a))^2 + s^2} &= \left(\frac{s}{\ln(a) - 3s} \right) U(x), \\ \frac{s^{n+1}}{(\ln(a))^2 + s^2} \left(\frac{\ln(a) - 3s}{s} \right) &= U(x). \\ \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} - \frac{3s^{n+1}}{(\ln(a))^2 + s^2} &= U(x)\end{aligned}\quad (14)$$

Taking the inverse GALM transform of both sides of equation (14), we obtain:

$$G^{-1} \left\{ \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} \right\} - G^{-1} \left\{ \frac{3s^{n+1}}{(\ln(a))^2 + s^2} \right\} = G^{-1} \{U(x)\}.$$

Thus, we obtain the exact solution to equation (12):

$$U(x) = \cos x - 3 \sin x \quad (15)$$

Application 3.2.3: Consider linear Volterra integral equation of first kind whose kernel containing cosine function:

$$x = \int_0^x \cos(x-t) u(t) dt \quad (16)$$

Solution:

Applying the GALM transform to both sides of equation (16), we have:

$$G\{x\} = G\left\{\int_0^x \cos(x-t)u(t)dt\right\} \quad (17)$$

Using convolution theorem of GALM transform on (17), we have:

$$G\{x\} = S^{1-n}G\{\cos x\}U(x),$$

Which leads to:

$$\frac{s^{n+1}}{(ln(a))^2} = S^{1-n} \left(\frac{s^n ln(a)}{(ln(a))^2 + s^2} \right) U(x),$$

$$\frac{s^{n+1}}{(ln(a))^2} = \left(\frac{s ln(a)}{(ln(a))^2 + s^2} \right) U(x),$$

$$\frac{s^{n+1}}{(ln(a))^2} \left(\frac{(ln(a))^2 + s^2}{s ln(a)} \right) = U(x),$$

$$\frac{s^n}{ln(a)} + \frac{s^{n+2}}{(ln(a))^3} = U(x) \quad (18)$$

Taking the inverse of the both sides of (16), we obtain:

$$G^{-1}\left\{\frac{s^n}{ln(a)}\right\} + G^{-1}\left\{\frac{s^{n+2}}{(ln(a))^3}\right\} = G^{-1}\{U(x)\}.$$

Finally, We get the exact solution of equation (14):

$$u(x) = 1 + \frac{x^2}{2} \quad (19)$$

Application 3.2.4: Consider linear Volterra integral equation of first kind whose kernel linear in the argument x and t :

$$x^2 = \frac{1}{3} \int_0^x (x-t) u(t) dt \quad (20)$$

Solution: Take the GALM transform to both side of equation (20), we have:

$$G\{x^2\} = G\left\{\frac{1}{3} \int_0^x (x-t)u(t)dt\right\} \quad (21)$$

Using convolution theorem of GALM transform on (21), we have:

$$G\{x^2\} = \frac{1}{3} S^{1-n} G\{x\}U(x),$$

Which leads to:

$$\frac{2!S^{n+2}}{(ln(a))^3} = \frac{1}{3} S^{1-n} \left(\frac{s^{n+1}}{(ln(a))^2} \right) U(x),$$

$$\frac{6S^{n+2}}{(ln(a))^3} \left(\frac{(ln(a))^2}{s^2} \right) = U(x),$$

$$U(x) = \frac{6S^n}{ln(a)} \quad (22)$$

Taking the inverse of the both sides of (22), we obtain:

$$G^{-1}\left\{\frac{6S^n}{ln(a)}\right\} = G^{-1}\{U(x)\}.$$

So that, the exact solution of equation is (20):

$$u(x) = 6 \quad (23)$$

Application 3.2.5: Consider linear Volterra integral equation of first kind whose kernel containing sine function

$$x = \int_0^x \sin(x-t) u(t) dt \quad (24)$$

Solution: Applying the GALM transform to both sides of equation (24), we have:

$$G\{x\} = G \left\{ \int_0^x \sin(x-t) u(t) dt \right\} \quad (25)$$

Using the convolution theorem of GALM transform on (25), we have:

$$G \{x\} = S^{1-n} G\{ \sin(x) \} U(x) ,$$

Which leads to:

$$\frac{s^{n+1}}{(\ln(a))^2} = S^{(1-n)} \left(\frac{s^{n+1}}{(\ln(a))^2 + s^2} \right) U(x),$$

$$\frac{s^{n+1}}{(\ln(a))^2} \left(\frac{(\ln(a))^2 + s^2}{2s^2} \right) = U(x) ,$$

$$\frac{s^{n-1} (\ln(a))^2}{2(\ln(a))^2} + \frac{s^{n+1}}{2(\ln(a))^2} = U(x),$$

$$\frac{s^{n-1}}{2} + \frac{s^{n+1}}{2(\ln(a))^2} = U(x) \quad (26)$$

Taking the inverse of the both sides of (24), we obtain:

$$G^{-1} \left\{ \frac{s^{n-1}}{2} \right\} + G^{-1} \left\{ \frac{s^{n+1}}{2(\ln(a))^2} \right\} = G^{-1}\{ U(x) \}.$$

Finally we get the exact solution of equation (24):

$$u(x) = \frac{-1}{2x} + \frac{x}{2} \quad (27)$$

4. Solving Linear Volterra Integral Equations of the Second Kind Using the Galm Transform

To highlight the efficacy of the GALM transform in solving linear Volterra integral equations of the second kind, we present applications with diverse kernels [13-17], including exponential, hyperbolic, and trigonometric functions, emphasizing the method's robustness in addressing the inherent complexities of these equations.

Application 4.1: Consider linear Volterra integral equation of second kind whose kernel containing sin function, then equation (2) reads;

$$u(t) = \cos(x) + \int_0^x \sin(x-t)u(t)dt \dots\dots\dots (28)$$

Solution: Take the GALM transform to both side of equation (26) we have:

$$G\{ u(t) \} = G\{ \cos(x) \} + G\{ \int_0^x \sin(x-t)u(t)dt \} \dots\dots\dots(29)$$

Using convolution theorem of GALM transform on (29), we have :

$$G\{u(t)\} = G\{\cos(x)\} + s^{1-n}G\{\sin(x)\}G\{u(t)\},$$

We get:

$$U(x) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} + s^{1-n} \left(\frac{s^{n+1}}{(\ln(a))^2 + s^2} \right) U(x)$$

$$U(x) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} + \left(\frac{s^2}{(\ln(a))^2 + s^2} \right) U(x),$$

$$U(x) - \left(\frac{s^2}{(\ln(a))^2 + s^2} \right) U(x) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2},$$

$$U(x) \left(1 - \frac{s^2}{(\ln(a))^2 + s^2} \right) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2}$$

$$U(x) \left(\frac{(\ln(a))^2}{(\ln(a))^2 + s^2} \right) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2},$$

$$\begin{aligned}
 U(x) \left(1 - \frac{s^2}{(\ln(a))^2 + s^2} \right) &= \frac{s^{n+1}}{(\ln(a))^2}, \\
 U(x) &= \frac{s^{n+1}}{(\ln(a))^2} \left(\frac{(\ln(a))^2 + s^2}{(\ln(a))^2} \right), \\
 U(x) &= \frac{s^{n+1}(\ln(a))^2}{(\ln(a))^4} + \frac{s^{n+3}}{(\ln(a))^4}, \\
 U(x) &= \frac{s^{n+1}}{(\ln(a))^2} + \frac{s^{n+3}}{(\ln(a))^4} \dots \dots \dots \quad \dots(38)
 \end{aligned}$$

Take the inverse of the both sides of (38), we obtain:

$$G^{-1}\{U(x)\} = G^{-1}\left\{\frac{s^{n+1}}{(\ln(a))^2}\right\} + G^{-1}\left\{\frac{s^{n+3}}{(\ln(a))^4}\right\},$$

We get the exact solution of equation (34):

$$u(x) = x + \frac{1}{6}x^3 \dots \dots \dots \quad \dots(39)$$

Application 4.4: Consider linear Volterra integral equation of second kind whose kernel containing exponential function

$$u(t) = e^x - \cos x - 2 \int_0^x e^{(x-t)}u(t)dt \dots \dots \dots \quad \dots (40)$$

Solution: Take the GALM transform to both side of equation (40)

$$G\{u(t)\} = G\{e^x\} - G\{\cos x\} - 2G\left\{\int_0^x e^{(x-t)}u(t)dt\right\} \dots \dots \dots (41)$$

Using convolution theorem of GALM transform on (41),we have :

$$G\{u(x)\} = G\{e^x\} - G\{\cos x\} - 2S^{1-n}G\{e^x\}G\{u(x)\}, \text{ we get:}$$

$$\begin{aligned}
 U(x) &= \frac{s^n}{\ln(a)-s} - \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} - 2S^{1-n} \left(\frac{s^n}{\ln(a)-s} \right) U(x) \\
 U(t) &= \frac{s^n}{\ln(a)-s} - \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} - 2 \left(\frac{s}{\ln(a)-s} \right) U(x), \\
 U(x) + 2 \left(\frac{s}{\ln(a)-s} \right) U(t) &= \frac{s^n}{\ln(a)-s} - \frac{s^n \ln(a)}{(\ln(a))^2 + s^2}, \\
 U(x) \left(1 + \frac{2s}{\ln(a)-s} \right) &= \frac{s^n}{\ln(a)-s} - \frac{s^n \ln(a)}{(\ln(a))^2 + s^2}, \\
 U(x) &= \left(\frac{s^n}{\ln(a)-s} - \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} \right) \left(\frac{\ln(a)-s}{\ln(a)+s} \right), \\
 U(x) &= \frac{s^{n+2} + s^{n+1} \ln(a)}{((\ln(a))^2 + s^2)(\ln(a)+s)}. \\
 U(x) &= \frac{s^{n+1} (\ln(a)+s)}{(\ln(a))^2 + s^2 (\ln(a)+s)}. \\
 U(x) &= \frac{s^{n+1}}{(\ln(a))^2 + s^2} \dots \dots \dots \quad \dots(42)
 \end{aligned}$$

Take the inverse of the both sides of (42), we obtain:

$$G^{-1}\{U(x)\} = G^{-1}\left\{\frac{s^{n+1}}{((\ln(a))^2 + s^2)}\right\},$$

we get the exact solution of equation (42):

$$u(x) = \sin(x) \dots \dots \dots \quad \dots(43)$$

Application 4.5: Consider linear Volterra integral equation of second kind whose kernel containing exponential function

$$u(t) = x + \int_0^x e^{-(x-t)}u(t)dt \dots \dots \dots \quad \dots (44)$$

Solution: Take the GALM transform to both side of equation (44):

$$G\{u(t)\} = G\{x\} + G\left\{\int_0^x e^{-(x-t)}u(t)dt\right\} \dots \dots \dots (45)$$

Using convolution theorem of GALM transform on (45), we have:

$G\{u(x)\} = G\{x\} + S^{1-n}G\{e^{-x}\}G\{u(x)\}$, we get:

$$\begin{aligned}
 U(x) &= \frac{s^{n+1}}{(\ln(a))^2} + S^{1-n} \left(\frac{s^n}{\ln(a)+s} \right) U(x), \\
 U(x) - \left(\frac{s}{\ln(a)+s} \right) U(x) &= \frac{s^{n+1}}{(\ln(a))^2}, \\
 U(x) \left(1 - \frac{s}{\ln(a)+s} \right) &= \frac{s^{n+1}}{(\ln(a))^2}, \\
 U(x) &= \frac{s^{n+1}}{(\ln(a))^2} \left(\frac{\ln(a)+s}{\ln(a)} \right), \\
 U(x) &= \frac{s^{n+1} \ln(a) + s^{n+2}}{(\ln(a))^3}, \\
 U(x) &= \frac{s^{n+1}}{(\ln(a))^2} + \frac{s^{n+2}}{(\ln(a))^3} \dots \dots \dots \quad .(46)
 \end{aligned}$$

Take the inverse of the both sides of (44), we obtain:

$$G^{-1}\{U(x)\} = G^{-1}\left\{\frac{s^{n+1}}{(\ln(a))^2}\right\} + G^{-1}\left\{\frac{s^{n+2}}{(\ln(a))^3}\right\}.$$

So that the exact solution of equation (42) reads:

$$u(x) = x + \frac{x^2}{2} \dots \dots \dots \quad .(47)$$

5. Solving Abel’s Integral Equation Using The Galm Transform

Application 5.1: Consider the Abel’s integral equation:

$$x = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \dots \dots \dots \quad (48)$$

Solution: Take the GALM transform to both side of equation (46),we have :

$$G\{x\} = G\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t) dt\right\} \dots \dots \dots \quad (49)$$

Using convolution theorem of GALM transform on (47), we have $G\{x\} = S^{1-n}G\left\{x^{-\frac{1}{2}}\right\}G\{u(x)\}$, we get:

$$\begin{aligned}
 \frac{s^{n+1}}{(\ln(a))^2} &= S^{1-n} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} \right) U(x) \\
 \frac{s^{n+1}}{(\ln(a))^2} &= \frac{s^{\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} U(x) \\
 \frac{s^{n+1}}{(\ln(a))^2} \left(\frac{(\ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) &= U(x) \\
 \frac{s^{n+\frac{1}{2}}}{\sqrt{\pi} (\ln(a))^{\frac{1}{2}}} &= U(x) \dots \dots \dots \quad (50)
 \end{aligned}$$

Take the inverse of the both sides of (50), we obtain:

$$\frac{1}{\sqrt{\pi}} G^{-1}\left\{\frac{s^{n+\frac{1}{2}}}{(\ln(a))^{\frac{1}{2}}}\right\} = G^{-1}\{U(x)\}.$$

We get the exact solution of equation (48):

$$u(x) = \frac{2\sqrt{x}}{\pi} \dots \dots \dots \quad (51)$$

Application 5.2 : Consider the Abel’s integral equation:

$$3x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \dots \dots \dots \quad (52)$$

Solution: Take the GALM transform to both side of equation (52) we have:

$$3G\{x^2\} = G\left\{\int_0^x \frac{1}{\sqrt{(x-t)}} u(t) dt\right\} \dots\dots\dots (53)$$

Using convolution theorem of GALM transform on (53), we have

$$3G\{x^2\} = S^{1-n}G\{x^{-\frac{1}{2}}\}G\{u(x)\}, \text{we get:}$$

$$\frac{6 s^{n+2}}{(ln(a))^3} = S^{(1-n)} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(ln(a))^{\frac{1}{2}}} \right) U(x)$$

$$\frac{6 s^{n+2}}{(ln(a))^3} = \left(\frac{s^{\frac{1}{2}} \sqrt{\pi}}{(ln(a))^{\frac{1}{2}}} \right) U(x)$$

$$\frac{6 s^{n+2}}{(ln(a))^3} \left(\frac{(ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) = U(x)$$

$$\frac{6}{\sqrt{\pi}} \left(\frac{s^{n+\frac{3}{2}}}{(ln(a))^{\frac{5}{2}}} \right) = U(x) \tag{54}$$

Take the inverse of the both sides of (54), we obtain:

$$\frac{6}{\sqrt{\pi}} G^{-1} \left\{ \frac{s^{n+\frac{3}{2}}}{(ln(a))^{\frac{5}{2}}} \right\} = G^{-1}\{U(x)\}.$$

We get the exact solution of equation (52):

$$\frac{8}{\pi} \sqrt{x^3} = u(x) \tag{55}$$

Application 5.3: Consider the Abel’s integral equation:

$$\frac{4}{3} \sqrt{x^3} = \int_0^x \frac{1}{\sqrt{(x-t)}} u(t) dt \tag{56}$$

Solution: Take the GALM transform to both side of equation (56) we have:

$$\frac{4}{3} G\{\sqrt{x^3}\} = G \int_0^x \frac{1}{\sqrt{(x-t)}} u(t) dt \tag{57}$$

Using convolution theorem of GALM transform on (57), we have

$$\frac{4}{3} G\{\sqrt{x^3}\} = S^{1-n}G\{x^{-\frac{1}{2}}\}G\{u(x)\}, \text{we get:}$$

$$\frac{4}{3} \left(\frac{s^{n+\frac{3}{2}}}{(ln(a))^{\frac{5}{2}}} \right) = S^{(1-n)} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(ln(a))^{\frac{1}{2}}} \right) U(x)$$

$$\frac{4}{3} \left(\frac{3 \sqrt{\pi} s^{n+\frac{3}{2}}}{4 (ln(a))^{\frac{5}{2}}} \right) = \left(\frac{s^{\frac{1}{2}} \sqrt{\pi}}{(ln(a))^{\frac{1}{2}}} \right) U(x)$$

$$\left(\frac{\sqrt{\pi} s^{n+\frac{3}{2}}}{(ln(a))^{\frac{5}{2}}} \right) \left(\frac{(ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) = U(x)$$

$$\frac{s^{n+1}}{(ln(a))^2} = U(x) \tag{58}$$

Take the inverse of the both sides of (58), we obtain:

$$G^{-1} \left\{ \frac{s^{n+1}}{(\ln(a))^2} \right\} = G^{-1}\{U(x)\},$$

We get the exact solution of equation (52):

$$u(x) = x \tag{59}$$

Application 5.4 : Consider the Abel’s integral equation:

$$1 + x + x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \dots\dots\dots \tag{60}$$

Solution: Take the GALM transform to both side of equation (60) , we have :

$$G\{1 + x + x^2\} = G\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t) dt\right\} \dots\dots\dots \tag{61}$$

Using convolution theorem of GALM transform on (61), we have

$$G\{1 + x + x^2\} = S^{1-n} G\{x^{-\frac{1}{2}}\} G\{u(x)\},$$

We get :

$$\left(\frac{s^n}{\ln(a)} + \frac{s^{n+1}}{(\ln(a))^2} + \frac{2s^{n+2}}{(\ln(a))^3}\right) = S^{1-n} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}}\right) U(x)$$

$$\left(\frac{s^n}{\ln(a)} + \frac{s^{n+1}}{(\ln(a))^2} + \frac{2s^{n+2}}{(\ln(a))^3}\right) = \left(\frac{s^{\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}}\right) U(x)$$

$$\left(\frac{s^n}{\ln(a)} + \frac{s^{n+1}}{(\ln(a))^2} + \frac{2s^{n+2}}{(\ln(a))^3}\right) \left(\frac{(\ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}}\right) = U(x),$$

$$\frac{1}{\sqrt{\pi}} \left(\frac{s^{n-\frac{1}{2}}}{(\ln(a))^{\frac{1}{2}}} + \frac{s^{n+\frac{1}{2}}}{(\ln(a))^{\frac{3}{2}}} + \frac{2s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}}\right) = U(x) \dots\dots\dots \tag{62}$$

Take the inverse of the both sides of (62), we obtain:

$$\frac{1}{\sqrt{\pi}} G^{-1} \left\{ \frac{s^{n-\frac{1}{2}}}{(\ln(a))^{\frac{1}{2}}} + \frac{s^{n+\frac{1}{2}}}{(\ln(a))^{\frac{3}{2}}} + \frac{2s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} \right\} = G^{-1}\{U(x)\}$$

We get the exact solution of equation (60):

$$\frac{1}{\pi} \left(x^{-\frac{1}{2}} + 2x^{\frac{1}{2}} + \frac{8}{3}x^{\frac{3}{2}}\right) = u(x) \dots\dots\dots \tag{63}$$

Application 5.5: Consider the Abel’s integral equation:

$$2x^{\frac{1}{2}} + \frac{8}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \dots\dots\dots \tag{64}$$

Solution: Take the GALM transform to both side of equation (64) we have:

$$2G\{x^{\frac{1}{2}}\} + \frac{8}{3}G\{x^{\frac{3}{2}}\} = G\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t) dt\right\} \dots\dots\dots \tag{65}$$

Using convolution theorem of GALM transform on (65), we have

$$2G\{x^{\frac{1}{2}}\} + \frac{8}{3}G\{x^{\frac{3}{2}}\} = S^{1-n} G\{x^{-\frac{1}{2}}\} G\{u(x)\}$$

Then, we get :

$$2 \left(\frac{\sqrt{\pi} s^{n+\frac{1}{2}}}{2(\ln(a))^{\frac{3}{2}}}\right) + \frac{8}{3} \left(\frac{3\sqrt{\pi} s^{n+\frac{3}{2}}}{4(\ln(a))^{\frac{5}{2}}}\right) = S^{1-n} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}}\right) U(x)$$

$$\frac{\sqrt{\pi} s^{n+\frac{1}{2}}}{(\ln(a))^{\frac{3}{2}}} + \frac{2\sqrt{\pi} s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} = \left(\frac{s^{\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}}\right) U(x),$$

$$\left(\frac{\sqrt{\pi} s^{n+\frac{1}{2}}}{(\ln(a))^{\frac{3}{2}}} + \frac{2\sqrt{\pi} s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} \right) \left(\frac{(\ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) = U(x),$$

$$\frac{s^n}{\ln(a)} + \frac{2s^{n+1}}{(\ln(a))^2} = U(x) \dots \dots \dots (66)$$

Take the inverse of the both sides of (64), we obtain: $G^{-1} \left\{ \frac{s^n}{\ln(a)} + \frac{2s^{n+1}}{(\ln(a))^2} \right\} = G^{-1} \{U(x)\}$,

We get the exact solution of equation (18):

$$u(x) = 1 + 2x = \dots \dots \dots (67)$$

6. Solving Neutrosophic Linear Volterra Integral Equations of the First Kind Using the Galm Transform

In the present work, we assume that the kernel $k(x + yI, t)$ of the linear Volterra integral equation of the first kind as defined in equation (1) is a difference kernel, which can be represented as a function of the difference $(x - t)$. Consequently, equation (1) can be rewritten as:

$$f(x + yI) = \int_0^{x+yI} k(x + yI - t)u(t)dt \quad (68)$$

$$G\{f(x + yI)\} = G \left\{ \int_0^x k(x + yI - t)u(t)dt \right\} \quad (69)$$

$$G\{f(x + yI)\} = S^{1-n}G\{k(x + yI)\}G\{u(x + yI)\}$$

Which implies that

$$G\{u(x + yI)\} = \left[\frac{S^{n-1}G\{f(x + yI)\}}{G\{k(x + yI)\}} \right] \quad (70)$$

$$u(x + yI) = G^{-1} \left\{ \left[\frac{S^{n-1}G\{f(x + yI)\}}{G\{k(x + yI)\}} \right] \right\} \quad (71)$$

Neutrosophic Application 5.1: Consider a linear neutrosophic Volterra integral equation of the first kind with a kernel that includes an exponential function:

$$x + yI = \int_0^{x+yI} e^{2(x+yI-t)}u(t)dt \quad (72)$$

Solution:

$$G\{x + yI\} = G \left\{ \int_0^{x+yI} e^{2(x+yI-t)}u(t)dt \right\} \quad (73)$$

$$G\{x + yI\} = S^{1-n}G\{e^{2(x+yI)}\}G\{u(x + yI)\} ,$$

Which leads to:

$$\frac{s^{n+1}}{(\ln(a))^2} = s^{1-n} \left(\frac{s^n}{\ln(a) - 2S} \right) U(x + yI),$$

$$\frac{s^{n+1}}{(\ln(a))^2} = \left(\frac{s}{\ln(a) - 2S} \right) U(x + yI),$$

$$\frac{s^{n+1}}{(\ln(a))^2} \left(\frac{\ln(a) - 2s}{s} \right) = U(x + yI),$$

$$\frac{s^n \ln(a)}{(\ln(a))^2} - \frac{2s}{(\ln(a))^2} = U(x + yI).$$

$$\frac{s^n}{\ln(a)} - \frac{2s}{(\ln(a))^2} = U(x + yI) \tag{74}$$

$$G^{-1}\left\{\frac{s^n}{\ln(a)}\right\} - G^{-1}\left\{\frac{2s}{(\ln(a))^2}\right\} = G^{-1}\{U(x + yI)\}.$$

$$u(x + yI) = 1 - 2(x + yI) \tag{75}$$

7. Solving Neutrosophic Linear Volterra Integral Equations of the Second Kind Using the Galm Transform

Neutrosophic Application 6.1: $u(t) = \cos(x + yI) + \int_0^{x+yI} \sin(x + yI - t)u(t)dt \dots\dots\dots \tag{28}$

Solution:

$$G\{u(t)\} = G\{\cos(x + yI)\} + G\left\{\int_0^x \sin(x + yI - t)u(t)dt\right\} \dots\dots\dots \tag{76}$$

$$G\{u(t)\} = G\{\cos(x + yI)\} + s^{1-n}G\{\sin(x + yI)\}G\{u(t)\},$$

We get:

$$U(x + yI) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} + s^{1-n} \left(\frac{s^{n+1}}{(\ln(a))^2 + s^2}\right) U(x + yI)$$

$$U(x + yI) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} + \left(\frac{s^2}{(\ln(a))^2 + s^2}\right) U(x + yI),$$

$$U(x + yI) - \left(\frac{s^2}{(\ln(a))^2 + s^2}\right) U(x + yI) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2},$$

$$U(x + yI) \left(1 - \frac{s^2}{(\ln(a))^2 + s^2}\right) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2}$$

$$U(x + yI) \left(\frac{(\ln(a))^2}{(\ln(a))^2 + s^2}\right) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2},$$

$$U(x + yI) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} \left(\frac{(\ln(a))^2 + s^2}{(\ln(a))^2}\right).$$

$$U(x + yI) = \frac{s^n}{\ln(a)} \dots\dots\dots \tag{77}$$

$$G^{-1}\{U(x + yI)\} = G^{-1}\left\{\frac{s^n}{\ln(a)}\right\}$$

$$u(x + yI) = 1 \dots\dots\dots \tag{78}$$

Neutrosophic Application 6.2: Consider linear neutrosophic Volterra integral equation of second kind whose kernel linear in the argument x and t

$$u(t) = 1 - \int_0^{x+yI} (x + yI - t)u(t)dt \dots\dots\dots \tag{79}$$

Solution:

$$G\{u(t)\} = G\{1\} - G\left\{\int_0^{x+yI} (x + yI - t)u(t)dt\right\} \dots\dots\dots \tag{80}$$

$G\{u(t)\} = G\{1\} - S^{1-n}G\{x + yI\}G\{u(t)\}$, we get:

$$U(x + yI) = \frac{s^n}{\ln(a)} - S^{1-n} \left(\frac{s^{n+1}}{(\ln(a))^2}\right) U(x + yI)$$

$$U(x + yI) + \left(\frac{s^2}{(\ln(a))^2}\right) U(x + yI) = \frac{s^n}{\ln(a)},$$

$$U(x + yI) \left(1 + \frac{s^2}{(\ln(a))^2}\right) = \frac{s^n}{\ln(a)},$$

$$U(x + yI) = \frac{s^n}{\ln(a)} \left(\frac{(\ln(a))^2}{(\ln(a))^2 + s^2} \right),$$

$$U(x + yI) = \frac{s^n \ln(a)}{(\ln(a))^2 + s^2} \dots \dots \dots (81)$$

$$G^{-1}\{U(x + yI)\} = G^{-1}\left\{\frac{s^n \ln(a)}{(\ln(a))^2 + s^2}\right\},$$

$$u(x) = \cos(x + yI) \dots \dots \dots (82)$$

Neutrosophic Application 6.3: Consider linear neutrosophic Volterra integral equation of second kind whose kernel containing sin function

$$u(t) = x + yI + \int_0^{x+yI} \sin(x + yI - t)u(t)dt \dots \dots \dots (83)$$

$$G\{u(t)\} = G\{x + yI\} + G\left\{\int_0^{x+yI} \sin(x + yI - t)u(t)dt\right\} \dots \dots \dots (84)$$

$$G\{u(t)\} = G\{x + yI\} + S^{1-n}G\{\sin(x + yI)\}G\{u(t)\},$$

we get:

$$U(x + yI) = \frac{s^{n+1}}{(\ln(a))^2} + S^{1-n}\left(\frac{s^{n+1}}{(\ln(a))^2 + s^2}\right)U(x + yI),$$

$$U(x + yI) - \left(\frac{s^2}{(\ln(a))^2 + s^2}\right)U(x + yI) = \frac{s^{n+1}}{(\ln(a))^2},$$

$$U(x + yI) \left(1 - \frac{s^2}{(\ln(a))^2 + s^2}\right) = \frac{s^{n+1}}{(\ln(a))^2},$$

$$U(x + yI) = \frac{s^{n+1}}{(\ln(a))^2} \left(\frac{(\ln(a))^2 + s^2}{(\ln(a))^2}\right),$$

$$U(x + yI) = \frac{s^{n+1}(\ln(a))^2}{(\ln(a))^4} + \frac{s^{n+3}}{(\ln(a))^4},$$

$$U(x + yI) = \frac{s^{n+1}}{(\ln(a))^2} + \frac{s^{n+3}}{(\ln(a))^4} \dots \dots \dots (85)$$

$$G^{-1}\{U(x + yI)\} = G^{-1}\left\{\frac{s^{n+1}}{(\ln(a))^2}\right\} + G^{-1}\left\{\frac{s^{n+3}}{(\ln(a))^4}\right\},$$

$$u(x + yI) = x + yI + \frac{1}{6}(x + yI)^3 \dots \dots \dots (86)$$

8. Solving Neutrosophic Abel’s Integral Equation Using the Galm Transform

Neutrosophic Application 7.1: Consider the neutrosophic Abel’s integral equation:

$$x + yI = \int_0^{x+yI} \frac{1}{\sqrt{(x+yI-t)}} u(t)dt \dots \dots \dots (87)$$

Solution: $G\{x + yI\} = G\left\{\int_0^{x+yI} \frac{1}{\sqrt{(x+yI-t)}} u(t)dt\right\} \dots \dots \dots (88)$

we have $G\{x + yI\} = S^{1-n}G\left\{x^{-\frac{1}{2}}\right\}G\{u(x + yI)\}$, we get:

$$\frac{s^{n+1}}{(\ln(a))^2} = S^{1-n} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} \right) U(x + yI)$$

$$\frac{s^{n+1}}{(\ln(a))^2} = \frac{s^{\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} U(x + yI)$$

$$\frac{s^{n+1}}{(\ln(a))^2} \left(\frac{(\ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) = U(x + yI)$$

$$\frac{s^{n+\frac{1}{2}}}{\sqrt{\pi} (\ln(a))^{\frac{3}{2}}} = U(x + yI) \dots \dots \dots (89)$$

$$\frac{1}{\sqrt{\pi}} G^{-1} \left\{ \frac{s^{n+\frac{1}{2}}}{(\ln(a))^{\frac{3}{2}}} \right\} = G^{-1}\{U(x + yI)\}.$$

$$u(x) = \frac{2\sqrt{x+yI}}{\pi} \dots\dots\dots (90)$$

Neutrosophic Application 7.2: Consider the neutrosophic Abel’s integral equation:

$$3(x + yI)^2 = \int_0^{x+yI} \frac{1}{\sqrt{(x+yI-t)}} u(t) dt \dots\dots\dots (91)$$

Solution: $3G\{(x + yI)^2\} = G\{\int_0^{x+yI} \frac{1}{\sqrt{(x+yI-t)}} u(t) dt\} \dots\dots\dots (92)$

$3G\{(x + yI)^2\} = S^{1-n}G\{x^{-\frac{1}{2}}\}G\{u(x + yI)\}$, we get:

$$\frac{6 s^{n+2}}{(\ln(a))^3} = S^{1-n} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} \right) U(x + yI)$$

$$\frac{6 s^{n+2}}{(\ln(a))^3} = \left(\frac{s^{\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} \right) U(x + yI)$$

$$\frac{6 s^{n+2}}{(\ln(a))^3} \left(\frac{(\ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) = U(x + yI)$$

$$\frac{6}{\sqrt{\pi}} \left(\frac{s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} \right) = U(x + yI) \quad (93)$$

$$\frac{6}{\sqrt{\pi}} G^{-1} \left\{ \frac{s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} \right\} = G^{-1}\{U(x + yI)\}.$$

$$\frac{8}{\pi} \sqrt{(x + yI)^3} = u(x + yI) \quad (94)$$

Neutrosophic Application 7.3: Consider the neutrosophic Abel’s integral equation:

$$\frac{4}{3} \sqrt{(x + yI)^3} = \int_0^{x+yI} \frac{1}{\sqrt{(x + yI - t)}} u(t) dt \quad (95)$$

Solution: $\frac{4}{3}G\{\sqrt{(x + yI)^3}\} = G\int_0^{x+yI} \frac{1}{\sqrt{(x+yI-t)}} u(t) dt \quad (96)$

$\frac{4}{3}G\{\sqrt{(x + yI)^3}\} = S^{1-n}G\{x^{-\frac{1}{2}}\}G\{u(x + yI)\}$, we get:

$$\frac{4}{3} \left(\frac{s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} \right) = S^{1-n} \left(\frac{s^{n-\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} \right) U(x + yI)$$

$$\frac{4}{3} \left(\frac{3 \sqrt{\pi} s^{n+\frac{3}{2}}}{4 (\ln(a))^{\frac{5}{2}}} \right) = \left(\frac{s^{\frac{1}{2}} \sqrt{\pi}}{(\ln(a))^{\frac{1}{2}}} \right) U(x + yI)$$

$$\left(\frac{\sqrt{\pi} s^{n+\frac{3}{2}}}{(\ln(a))^{\frac{5}{2}}} \right) \left(\frac{(\ln(a))^{\frac{1}{2}}}{s^{\frac{1}{2}} \sqrt{\pi}} \right) = U(x + yI)$$

$$\frac{s^{n+1}}{(\ln(a))^2} = U(x + yI) \quad (97)$$

$$G^{-1} \left\{ \frac{s^{n+1}}{(\ln(a))^2} \right\} = G^{-1}\{U(x + yI)\},$$

$$u(x + yI) = x + yI \quad (98)$$

8. Conclusion

In conclusion, the GALM transform has proven to be a powerful tool for solving linear Volterra integral equations of the first kind and neutrosophic linear Volterra integral equations of the first kind, as demonstrated in the examples presented. Its efficiency in reducing computational complexity and time makes it an attractive method for obtaining exact solutions [20]. Moreover, the potential application of the GALM transform to systems of classical and neutrosophic linear Volterra integral equations holds great promise for further broadening its scope and utility in various settings. Looking ahead, an extension of this approach to the second kind of Volterra integral equations and Abel's equation appears to be a natural progression. For the second kind of Volterra integral equations, where the unknown function appears both inside and outside the integral, the GALM transform could offer a similar reduction in complexity and improve solution accuracy. Likewise, its adaptation to solving Abel's integral equation, which frequently arises in nonlinear dynamical systems, could open up new avenues for research and practical applications, demonstrating its versatility in both theoretical and applied contexts.

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