Stability Solution of Fractional Randomly System

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Abstract

In this paper, we study stability Solution of Fractional Randomly System. Two methods are provided to check the stability of such system in mean sense. The first method is based on integral inequalities. The second method is based on Lyapunov function. Stable in mean sense, asymptotically stable in mean sense are shown by using generalized Gromwell inequality. Stable in mean sense, asymptotically stable in mean sense, Mittag-Leffler stable in mean sense are shown by using generalized Lyapunov method.

Keywords: Randomly Parameters; Norm, expected value; Lyapunov function; Integral inequalities; Stable; Mean sense; Laplace transform; Fractional derivative

1. Introduction

Recently, the topic of fractional differential equations has attracted much research which play a great a main role in varies applications. Many areas such as physics, biophysics, aerodynamics, control theory, viscoelasticity, capacitor theory, electrical circuit, description of memory and hereditary properties [1] used the fractional models instead of classical models.

Recently, stability of fractional differential systems has attracted many authors interest [2]. In 1996, Matignon [3] firstly studied the stability of linear fractional differential system with the Caputo derivative. Since then, many researchers have done further studies on the stability of linear fractional differential systems [4].

Analysis of the stability of stochastic systems becomes necessary both in theoretical and applied aspects. The mathematical theory of stability of solution of stochastic differential equations consists of two main directions. These are the direct (second) Lyapunov method [5] and Burton's fixed-point method [6]. The questions of existence, uniqueness and stability of solutions of stochastic partial differential equations were the subject of analysis in [7]. New results were also obtained for stochastic integro-differential equations [8].

This paper is devoted to the development of integral inequalities for Fractional non-homogenous Randomly System of the form

$$ {}^{C}D_{0^{+}}^{\alpha}\mathrm{X}(t)=\mathrm{A}\mathrm{X}(t)+f\left(t,\omega(t),\mathrm{X}(t)\right) \qquad (1)$$

Where $0<\alpha\leq 1$, $\omega(t)$ denotes the randomly parameters, $f:[0,\infty)\times\mathbb{R}^{n}\times\mathbb{R}^{n}\rightarrow\mathbb{R}^{n}$ is piecewise continuous in t and locally Lipchitz with respect to X in mean sense on $[0,\infty)\times\mathbb{R}^{n}\times\mathbb{R}^{n}$, and assume that $f\left(\omega(t),t,0\right)=0$, $\mathrm{A}\in\mathbb{R}^{n\times n}$ is a constant matrix.

This paper also is devoted to the development of Lyapunov function for Fractional Randomly System of the form

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$$ {}^{C}D_{0^+}^{\alpha}\mathrm{X}(t)=\mathrm{A}(\omega)\mathrm{X}(t)+f\left(t,\omega(t),\mathrm{X}(t)\right) \qquad (2)$$

Where $0<\alpha\leq 1$, $\omega(t)$ denotes the randomly parameters, $f:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ is piecewise continuous in t and locally Lipchitz with respect to X in mean sense on $[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, and assume that $f\left(\omega(t),t,0\right)=0$, $\mathrm{A}\in\mathbb{R}^{n\times n}$ is a constant matrix.

The paper is structured as follows. In section 2 firstly we present some definitions and properties about fractional calculus and Mittag-Leffler function. Then we give definition and some properties about Laplace transform. We give main results in section 3 as follows. In part 3-1 we obtain the solution of system (1) and prove the existence of this solution by using Laplace transform. In part 3-2, we study the different kinds of stability of solution of system (1) by using integral inequalities method. In part 3-3, we study the different kinds of stability of solution of system (2) by using Lyapunov function method.

2. Material and Methods

We embark on this section by briefly presenting some necessary facts and definitions from fractional calculus and special functions which are used throughout the paper.

Definition 1: [9] Let $T>0$, $f\in L^1\left([t_0,T],\mathbb{R}\right)$ be a differentiable function. The Riemann-Liouville integral of fractional order $\alpha>0$ is defined by

$$\mathrm{I}_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}f(\tau)\left(t-\tau\right)^{\alpha-1}d\tau \;;\quad t>0$$

Where $\Gamma(\alpha)=\int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau$ is Gamma function.

Definition 2: [9] Let $\alpha\in(0,1]$, $T>0$, $f\in L^1\left([0,T],\mathbb{R}\right)$ be a differentiable function. The Caputo fractional derivative of $f(t)$ is defined as

$${}^{c}D_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}f'(\tau)d\tau \qquad ;t\in[0,T]$$

Property 1: [9] Let $n<\alpha<n-1\,;n\in\mathbb{N}$, $T>0$, $f\in L^1\left([0,T],\mathbb{R}\right)$ be a differentiable function. The following formulas hold

$${}^{c}D_{0^+}^{\alpha}\mathrm{I}_{0^+}^{\alpha}f(t)=f(t)$$

$$\mathrm{I}_{0^+}^{\alpha}\,{}^{c}D_{0^+}^{\alpha}f(t)=f(t)-\sum_{\kappa=0}^{n-1}\frac{t^{\kappa}f^{(\kappa)}(0)}{\Gamma(\kappa+1)}$$

Definition 3: [10] We consider the matrix Mittag-Leffler functions with one and two parameters which are defined by

$$\mathrm{E}_{\alpha}\left(t^{\alpha}\mathrm{A}\right)=\sum_{\kappa=0}^{\infty}\frac{\mathrm{A}^{\kappa}t^{\kappa\alpha}}{\Gamma(\alpha\kappa+1)}$$

$$\mathrm{E}_{\alpha,\beta}\left(t^{\alpha}\mathrm{A}\right)=\sum_{\kappa=0}^{\infty}\frac{\mathrm{A}^{\kappa}t^{\kappa\alpha}}{\Gamma(\alpha\kappa+\beta)} \qquad ;\left(\alpha,\beta>0,\mathrm{A}\in\mathbb{R}^{n\times n}\right)$$

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Lemma 1: [9] Let $\alpha, \lambda, \kappa > 0$ be arbitrary real numbers. The function $\mathrm{E}_{\alpha}(\cdot)$ has the following property

$$\left\| \mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right) \right\| \leq \kappa \mathrm{E}_{\alpha}\left(-\lambda t^{\alpha}\right)$$

Lemma 2: [9] Let $\alpha, \lambda, \kappa > 0$ be arbitrary real numbers. The function $\mathrm{E}_{\alpha,\alpha}(\cdot)$ has the following property

$$\left\| \mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}t^{\alpha}\right) \right\| \leq \kappa \mathrm{E}_{\alpha,\alpha}\left(-\lambda t^{\alpha}\right)$$

Lemma 3: [11] Let $Z \in \mathbb{C}$ such that

$$\frac{\alpha\pi}{2} < \arg(Z) < 2\pi - \frac{\alpha\pi}{2} \qquad ; \quad \left|\arg(Z)\right| > \frac{\alpha\pi}{2}$$

Then

$$\mathrm{E}_{\alpha}\left(-\lambda t^{\alpha}\right) \cong \begin{cases} 1 - \lambda \dfrac{t^{\alpha}}{\Gamma(1+\alpha)} \cong \exp\left\{-\dfrac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}\right\} & ; \; t \to 0^{+} \\ \dfrac{t^{-\alpha}}{\lambda \Gamma(1-\alpha)} & ; \; t \to \infty \end{cases}$$

Where $0 < \alpha \leq 1$, $\lambda > 0$.

Lemma 4: [11] Let $Z \in \mathbb{C}$ such that

$$\frac{\alpha\pi}{2} < \arg(Z) < 2\pi - \frac{\alpha\pi}{2} \qquad ; \quad \left|\arg(Z)\right| > \frac{\alpha\pi}{2}$$

Then

$$\mathrm{E}_{\alpha,\beta}\left(-\lambda t^{\alpha}\right) \cong \begin{cases} \dfrac{1}{\Gamma(\beta)} - \lambda \dfrac{t^{\alpha}}{\Gamma(\beta+\alpha)} & ; \; t \to 0^{+} \\ \dfrac{t^{-\alpha+\beta-1}}{\lambda \Gamma(\beta-\alpha)} & ; \; t \to \infty \end{cases}$$

Where $0 < \alpha \leq 1$, $\alpha \leq \beta$, $\lambda > 0$.

Definition 4:[12] Laplace transform of a real function $f(t)$ is defined as

$$F(s) = \mathbf{L}\left\{f(t)\right\} = \int_{0}^{\infty} e^{-st} f(t) dt$$

Properties: [12] 1-The Laplace transform of Riemann-Liouville integration is given as

$$\mathbf{L}\left\{\mathrm{I}_{0^{+}}^{\alpha} f(t)\right\} = s^{-\alpha} F(s)$$

2- The Laplace transform of Caputo derivative is given as

$$\mathbf{L}\left\{{}^{c}D_{0^{+}}^{\alpha} f(t)\right\} = s^{\alpha} F(s) - \sum_{\kappa=0}^{n-1} s^{\alpha-\kappa-1} f^{(\kappa)}(0) \qquad ; \; n-1 < \alpha < n$$

3- The Laplace transform of Mittag-Lefler is given as

$$\mathbf{L}\left\{t^{\beta-1} \mathrm{E}_{\alpha,\beta}\left(\mp \lambda t^{\alpha}\right)\right\} = s^{\alpha-\beta}\left(s^{\alpha} \pm \lambda\right)^{-1}$$

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Definition 5:[12] The convolution of the functions $f(t)$ and $g(t)$ is defined as

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau=\int_0^t f(\tau)g(t-\tau)d\tau$$

Property 2: [12] The inverse Laplace transform of the convolution is defined as

$$\mathbf{L}^{-1}\{F(s)\cdot G(s)\}=f(t)*g(t)$$

3. Results and Discussion

-We consider system of Fractional non-homogenous Randomly System has the following form

$$^{C}D_{0^+}^{\alpha}\mathrm{X}(t)=\mathrm{AX}(t)+f(t,\omega(t),\mathrm{X}(t)) \qquad (1)$$

Where $0<\alpha\le 1$, $\omega(t)$ denotes the randomly parameters, $f:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is piecewise continuous in t and locally Lipchitz with respect to X in mean sense on $[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, and assume that $f(\omega(t),t,0)=0$, $\mathrm{A}\in\mathbb{R}^{n\times n}$ is a constant matrix.

3.1. Solution the system of Fractional non-homogenous Randomly System using Laplace transform

In this part, to find solution the system of Fractional non-homogenous Randomly System we state and prove the following theorem.

Theorem (1): for $0<\alpha<1$, the solution of (1) is

$$\mathrm{X}(t)=\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\mathrm{X}_0+\int_0^t(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}(t-\tau)^{\alpha}\right)f(\tau,\omega(\tau),\mathrm{X}(\tau))d\tau \qquad (3)$$

Proof: By applying Laplace transform for (1), we have

$$\mathbf{L}\left\{^{C}D_{0^+}^{\alpha}\mathrm{X}(t)\right\}=\mathbf{L}\left\{\mathrm{A\,X}(t)+f(t,\omega(t),\mathrm{X}(t))\right\}$$
$$\mathbf{L}\left\{^{C}D_{0^+}^{\alpha}\mathrm{X}(t)\right\}=\mathbf{L}\left\{\mathrm{A\,X}(t)\right\}+\mathbf{L}\left\{f(t,\omega(t),\mathrm{X}(t))\right\}$$

Set $\mathbf{L}\left\{f(t,\omega(t),\mathrm{X}(t))\right\}=F(s,\omega(s),\mathrm{X}(s))$, then we get

$$s^{\alpha}X(s)-s^{\alpha-1}\mathrm{X}_0=\mathrm{A}X(s)+F(s,\omega(s),\mathrm{X}(s))$$
$$(s^{\alpha}I-\mathrm{A})X(s)=s^{\alpha-1}\mathrm{X}_0+F(s,\omega(s),\mathrm{X}(s))$$

Which implies

$$X(s)=s^{\alpha-1}(s^{\alpha}I-\mathrm{A})^{-1}\mathrm{X}_0+(s^{\alpha}I-\mathrm{A})^{-1}F(s,\omega(s),\mathrm{X}(s))$$

Now, taking Laplace inverse to both side , we obtain

$$\mathbf{L}^{-1}\{X(s)\}=\mathbf{L}^{-1}\left\{s^{\alpha-1}(s^{\alpha}I-\mathrm{A})^{-1}\mathrm{X}_0+(s^{\alpha}I-\mathrm{A})^{-1}F(s,\omega(s),\mathrm{X}(s))\right\}$$

$$\mathbf{L}^{-1}\{X(s)\}=\mathbf{L}^{-1}\left\{s^{\alpha-1}(s^{\alpha}I-\mathrm{A})^{-1}\mathrm{X}_0\right\}+\mathbf{L}^{-1}\left\{(s^{\alpha}I-\mathrm{A})^{-1}F(s,\omega(s),\mathrm{X}(s))\right\}$$

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By convolution theorem of Laplace transform on the second term of last equation, we have

$$\mathbf{L}^{-1}\left\{X\left(s\right)\right\}=\mathbf{L}^{-1}\left\{s^{\alpha-1}\left(s^{\alpha}I-\mathrm{A}\right)^{-1}\right\}\mathrm{X}_0+\mathbf{L}^{-1}\left\{\left(s^{\alpha}I-\mathrm{A}\right)^{-1}\right\}*\mathbf{L}^{-1}\left\{F\left(s,\omega\left(s\right),\mathrm{X}\left(s\right)\right)\right\}$$

$$\mathrm{X}\left(t\right)=\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\mathrm{X}_0+t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}t^{\alpha}\right)*f\left(t,\omega\left(t\right),\mathrm{X}(t)\right)$$

Whish implies

$$\mathrm{X}\left(t\right)=\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\mathrm{X}_0+\int_0^t\left(t-\tau\right)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}(t-\tau)^{\alpha}\right)f\left(\tau,\mathrm{X}(\tau)\right)d\tau$$

3.2. Stability of Solution the Fractional non-homogenous Randomly System

In this part, we introduce some concepts of stability in mean sense in the case of Fractional non-homogenous Randomly System

Definition 6: [9] The zero solution of system $\left(1\right)$ is said to be stable in the mean sense if for any $\varepsilon>0$, there exists $\delta\left(\varepsilon\right)>0$ such that for any initial condition satisfying $\left\|\mathrm{X}_0\right\|<\delta\left(\varepsilon\right)$, the expected value of the norm of the solution satisfies $E\left(\left\|\mathrm{X}\left(t\right)\right\|\right)<\varepsilon$ for any $t\geq 0$.

Definition 7: [9] The zero solution of system $\left(1\right)$ is said to be asymptotically stable in the mean sense if, in addition, to being stable in the mean sense, it is true that for each X_0, there exists a $\delta>0$ such that such that $\lim_{t\to\infty}E\left(\left\|\mathrm{X}\left(t\right)\right\|\right)=0$ whenever $\left\|\mathrm{X}_0\right\|<\delta$

Definition 8: [9] The zero solution of system $\left(1\right)$ is said to be Mittag-Leffler stable in the mean sense if for each X_0, the expected value of norm of the solution satisfies

$$E\left(\left\|\mathrm{X}\left(t\right)\right\|\right)\leq\left\|\mathrm{X}_0\right\|E_{\alpha}\left(-\lambda t^{\alpha}\right)$$

Where $0<\alpha\leq 1$, $\lambda>0$

Lemma 5: [9] Let $0<\alpha\leq 1$, $b\left(t\right)$ and $l\left(t\right)$ be continuous, nonnegative functions on $\left[0,T\right]$ and $u\left(t\right)$ be a continuous, nonnegative function on $\left[0,T\right]$ with

$$u\left(t\right)\leq b\left(t\right)+\frac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}l\left(s\right)u\left(s\right)ds\quad;\ t\in\left[0,T\right]$$

Then it has

$$u\left(t\right)\leq\left[B\left(t\right)+\int_0^t L\left(s\right)B\left(s\right)\exp\left(\int_s^t L\left(\tau\right)d\tau\right)ds\right]^{\beta}\qquad;t\in\left[0,T\right]$$

If $b\left(t\right)$ is nondecreasing on $\left[0,T\right]$, then the inequality is reduced to

$$u\left(t\right)\leq\left[B\left(t\right)+\int_0^t L\left(s\right)B\left(s\right)\exp\left(\int_s^t L\left(\tau\right)d\tau\right)ds\right]^{\beta}\qquad;t\in\left[0,T\right]$$

If $b\left(t\right)\equiv 0$ on $\left[0,T\right]$, then $u\left(t\right)\equiv 0$, where $0<\beta<\alpha\leq 1$, and

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$$B(t)=2^{\frac{1}{\beta-1}}b^{\frac{1}{\beta}}(t)$$

$$L(t)=\frac{2^{\frac{1}{\beta-1}}}{\left(\Gamma(\alpha)\right)^{\frac{1}{\beta}}}\left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{\frac{1-\beta}{\beta}}t^{\frac{\alpha-\beta}{\beta}}l^{\frac{1}{\beta}}(t)$$

Next, we prove the boundness of the solution of system (1) by using some generalized Gronwall inequalities and then provide another alternative approach to verify the stability of system (1) in mean sense.

Lemma 6: Let the function $f(t,\omega(t),\mathrm{X})$ in system (1) satisfies the relation

$$E\left\|f(t,\omega(t),\mathrm{X})-f(t,\omega(t),\mathrm{Y})\right\|\le L\left(t,E\left\|\mathrm{X}-Y\right\|\right) \qquad (4)$$

And let L verifies the condition

$$0\le L(t,u)-L(t,\mathrm{v})\le M(t,v)(u-v) \qquad ;\mathrm{u}\ge\mathrm{v}\ge 0 \qquad (5)$$

Where $M:[0,\infty)\times\mathbb{R}\rightarrow\mathbb{R}$ is a nonnegative continuous function. Then we have the estimate

$$E\left\|\mathrm{X}(t)-\mathrm{X}_0\right\|\le\left[B(t)+\int_0^t\phi(s)B(s)\exp\left(\int_s^t\phi(\tau)d\tau\right)ds\right]^{\beta} \qquad ;t\ge 0 \qquad (6)$$

Where $t\ge 0$, $0<\beta<\alpha<1$, and

$$B(t)=2^{\frac{1}{\beta-1}}\left(\frac{\left\|\mathrm{X}_0\right\|}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left(\left\|\mathrm{A}\right\|+M(\tau,0)\right)d\tau\right)^{\frac{1}{\beta}}$$

$$\phi(t)=\frac{2^{\frac{1}{\beta-1}}}{\left(\Gamma(\alpha)\right)^{\frac{1}{\beta}}}\left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{\frac{1-\beta}{\beta}}t^{\frac{\alpha-\beta}{\beta}}\left(\left\|\mathrm{A}\right\|+M(t,0)\right)^{\frac{1}{\beta}}$$

Proof: Let $\mathrm{X}(t)$ be the solution of system (1). Then we have

$$\mathrm{X}(t)=\mathrm{X}_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\mathrm{A}\,\mathrm{X}(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)d\tau$$

Let us define the mapping $Y(t):[0,\infty)\rightarrow\mathbb{R}^n$ as

$$Y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\mathrm{A}\,\mathrm{X}(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)d\tau$$

Then it has $\mathrm{X}(t)=\mathrm{X}_0+Y(t)$. Noting the fact $f(\tau,\omega(\tau),0)=0$, we get

$$E\left\|f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)\right\|\le L\left(\tau,E\left\|\mathrm{X}\right\|\right)$$

On the other hand, from (5) we have $L(t,0)=0$ and $L(t,u)\le M(t,0)u$. Then by (5) and (4) we have

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$$E\left\|Y(t)\right\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|A\right\| E\left\|X(\tau)\right\| d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} E\left\|f\left(\tau,\omega(\tau),X(\tau)\right)\right\| d\tau$$

$$E\left\|Y(t)\right\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|A\right\| E\left\{\left\|X_0\right\|+\left\|Y(\tau)\right\|\right\} d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} L\left(\tau, E\left\{\left\|X_0\right\|+\left\|Y(\tau)\right\|\right\}\right) d\tau$$

$$E\left\|Y(t)\right\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|A\right\|\left\|X_0\right\| d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|A\right\| E\left\|Y(\tau)\right\| d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} L\left(\tau,\left\|X_0\right\|\right) d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} L\left(\tau, E\left\|Y(\tau)\right\|\right) d\tau$$

$$E\left\|Y(t)\right\| \le \frac{\left\|X_0\right\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|A\right\| d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|A\right\| E\left\|Y(\tau)\right\| d\tau$$

$$+\frac{\left\|X_0\right\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} M(\tau,0) d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} M(\tau,0) E\left\|Y(\tau)\right\| d\tau$$

$$E\left\|Y(t)\right\| \le \frac{\left\|X_0\right\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\left\|A\right\|+M(\tau,0)\right] d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\left\|A\right\|+M(\tau,0)\right] E\left\|Y(\tau)\right\| d\tau$$

According to lemma 5 we can obtain estimate (6). The proof is completed.

Lemma 7: If the function $f\left(t,\omega(t),X\right)$ in system (1) satisfies the condition in the lemma 6 and also there exist two positive constants δ_0 and $\bar{M}$ such that

$$\int_0^\infty s^{\frac{\alpha-\beta}{\beta}}\left(\left\|A\right\|+M(s,\delta)\right)^{\frac{1}{\beta}} ds \le \bar{M}$$

For all $0 \le \delta \le \delta_0$, then there exists $0 < \tilde{M}$ such that for all $t \ge 0$,

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$$E\left\|X(t)-X_0\right\| \le \tilde{M}$$

Proof: With the help of Holder's inequality, we have

$$\int_0^t (t-\tau)^{\alpha-1}\left(\|A\|+M(\tau,0)\right)d\tau = \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-\alpha}\tau^{\alpha-\beta}\left(\|A\|+M(\tau,0)\right)d\tau$$

$$\le \left[\int_0^t \left((t-\tau)^{\alpha-1}\tau^{\beta-\alpha}\right)^{\frac{1}{1-\beta}}d\tau\right]^{1-\beta}\left[\int_0^t \left(\tau^{\frac{\alpha-\beta}{\beta}}\left(\|A\|+M(\tau,0)\right)\right)^{\frac{1}{\beta}}d\tau\right]^{\beta}$$

$$=\left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{1-\beta}\left[\int_0^t \left(\tau^{\frac{\alpha-\beta}{\beta}}\left(\|A\|+M(\tau,0)\right)\right)^{\frac{1}{\beta}}d\tau\right]^{\beta}$$

This implies that

$$\lim_{t\to\infty}\int_0^t (t-\tau)^{\alpha-1}\left(\|A\|+M(\tau,0)\right)d\tau \le \bar{M}^{\beta}\left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{1-\beta}$$

It follows form lemma 6 that the solution $X(t)$ is bounded for all $t \ge 0$, and the boundedness is denoted as $\tilde{M}$. The proof is completed.

Theorem 2: If the function $f(t,\omega(t),X)$ in system (1) satisfies the condition in the lemma 6 and also there exist two positive constants δ_0 and $\bar{M}$ such that

$$\int_0^\infty s^{\frac{\alpha-\beta}{\beta}}\left(\|A\|+M(s,\delta)\right)^{\frac{1}{\beta}}ds \le \bar{M}$$

For all $0 \le \delta \le \delta_0$, then the zero solution of system (1) is stable in mean sense.

Proof: Let $\varepsilon > 0$ and $X(t)$ be the solution of system (1). We choose δ such that

$$\delta(\varepsilon)=\min\left\{\frac{\varepsilon}{2},\delta_0,\frac{\varepsilon}{2\tilde{M}}\right\}$$

Where $\tilde{M}$ is the constant boundedness in lemma 7.

Then, for $\|X_0\| < \delta(\varepsilon)$, we have

$$E\left\|X(t)\right\| \le E\left\{\|X_0\|+\|X-X_0\|\right\} \le \frac{\varepsilon}{2}+\|X_0\|\tilde{M} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

This implies that the zero solution of system (1) is stable in mean sense.

Theorem 3: Assum that, for any $t \ge 0$, it has $f(t,\omega(t),0)=0$, and there exists a constant $L > 0$ such that for $X, Y \in \mathbb{R}^n$, it has

$$E\left\|f(t,\omega(t),X)-f(t,\omega(t),Y)\right\| \le LE\left\|X-Y\right\| \tag{7}$$

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Then the zero solution of system (1) is asymptotically stable in mean sense.

Proof: Combining the fact $f\left(t,\omega(t),0\right)=0$ and inequality (7), we have

$$E\left\|f\left(t,\omega(t),\mathrm{X}\right)-f\left(t,\omega(t),0\right)\right\|\leq LE\left\|\mathrm{X}\right\|$$

Let $\mathrm{X}(t)$ be the solution of system (1). Then by the theorem 1, we have

$$\mathrm{X}(t)=\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\mathrm{X}_0+\int_0^t(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}(t-\tau)^{\alpha}\right)f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)d\tau \qquad (3)$$

According to lemma 1, we have

$$\left\|\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\right\|\leq\kappa\mathrm{E}_{\alpha}\left(-\lambda t^{\alpha}\right)$$

According to lemma 2, we have

$$\left\|\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}t^{\alpha}\right)\right\|\leq\kappa\mathrm{E}_{\alpha,\alpha}\left(-\lambda t^{\alpha}\right)$$

Now, taking norm to both side of equation (3), we obtain

$$\left\|\mathrm{X}(t)\right\|=\left\|\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\mathrm{X}_0+\int_0^t(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}(t-\tau)^{\alpha}\right)f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)d\tau\right\|$$

$$\left\|\mathrm{X}(t)\right\|\leq\left\|\mathrm{E}_{\alpha}\left(\mathrm{A}t^{\alpha}\right)\right\|\left\|\mathrm{X}_0\right\|+\int_0^t(t-\tau)^{\alpha-1}\left\|\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}(t-\tau)^{\alpha}\right)\right\|\left\|f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)\right\|d\tau$$

$$\left\|\mathrm{X}(t)\right\|\leq\kappa\mathrm{E}_{\alpha}\left(-\lambda t^{\alpha}\right)\left\|\mathrm{X}_0\right\|+\kappa\int_0^t(t-\tau)^{\alpha-1}\left\|\mathrm{E}_{\alpha,\alpha}\left(-\lambda(t-\tau)^{\alpha}\right)\right\|\left\|f\left(\tau,\omega(\tau),\mathrm{X}(\tau)\right)\right\|d\tau \qquad (8)$$

Then, taking the expectation on both sides of inequality (8), we have

$$E\left\|\mathrm{X}(t)\right\|\leq\kappa\mathrm{E}_{\alpha}\left(-\lambda t^{\alpha}\right)\left\|\mathrm{X}_0\right\|+L\kappa\int_0^t(t-\tau)^{\alpha-1}\left\|\mathrm{E}_{\alpha,\alpha}\left(-\lambda(t-\tau)^{\alpha}\right)\right\|E\left\|\mathrm{X}(\tau)\right\|d\tau$$

Bu using lemma 3 and lemma 4, we have

$$\lim_{t\to\infty}\mathrm{E}_{\alpha}\left(-\lambda t^{\alpha}\right)=0 \qquad \& \qquad \lim_{t\to\infty}\mathrm{E}_{\alpha,\alpha}\left(-\lambda t^{\alpha}\right)=0$$

It follows that $\lim\limits_{t\to\infty}E\left\|\mathrm{X}(t)\right\|=0$. The proof is completed.

We consider system of fractional differential equations with randomly parameters has the following form

$$^{C}D_{0^+}^{\alpha}\mathrm{X}(t)=\mathrm{A}(\omega)\mathrm{X}(t)+f\left(t,\omega(t),\mathrm{X}(t)\right) \qquad (2)$$

Where $0<\alpha\leq 1$, $\omega(t)$ denotes the randomly parameters, $f:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is piecewise continuous in t and locally Lipchitz with respect to X in mean sense on $[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, and assume that $f\left(\omega(t),t,0\right)=0$, $\mathrm{A}\in\mathbb{R}^{n\times n}$ is a constant matrix.

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3.3. Stability of Solution of Fractional Randomly System

In this part, we will use the generalized Lyapunov function method to prove the stability of system (2). The method is to construct a scalar function $V\left(t,\mathrm{X}(t)\right)$, which is continuous in both t and X, has first partial derivatives in these variables and equals zero only at $\mathrm{X}(t)=0$. Such a function is called a Lyapunov function.

Note 1: For any solution X, the $\alpha-$order Caputo derivative of the scaler function $V\left(t,\mathrm{X}(t)\right)$ is calculated as

$$^{C}D_{0^+}^{\alpha}V\left(t,\mathrm{X}(t)\right)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{V}\left(\tau,\mathrm{X}(\tau)\right)d\tau$$

Where

$$\dot{V}\left(t,\mathrm{X}(t)\right)=\frac{dV}{dt}=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial \mathrm{X}}\frac{d\mathrm{X}}{dt}$$

Theorem 4: Let $0<\alpha\leq 1$, and let $V\left(t,\mathrm{X}(t)\right):[0,\infty)\times\mathbb{R}^n\rightarrow\mathbb{R}$ be continuously differential function and locally Lipschitz continuous with respect to X and satisfy the following conditions:

$(i)\ V(t,0)=0$

$(ii)\ V(t,\mathrm{X})\geq a\|\mathrm{X}\|$

$(iii)\ E\left(^{C}D_{0^+}^{\alpha}V(t,\mathrm{X})\right)\leq 0$

Then the zero solution of system (2) is stable in mean sense.

Proof: Note that $V\left(t,\mathrm{X}(t)\right)$ can be written as

$$V\left(t,\mathrm{X}(t)\right)=V_0+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{^{C}D_{0^+}^{\alpha}V\left(\tau,\mathrm{X}(\tau)\right)}{(t-\tau)^{1-\alpha}}d\tau \qquad (9)$$

Where $V_0=V\left(0,\mathrm{X}_0\right)$ only depends on the initial state X_0.

Taking the expected value on the both sides of equality (9)

$$E\left(V\left(t,\mathrm{X}(t)\right)\right)=V_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}E\left(^{C}D_{0^+}^{\alpha}V\left(\tau,\mathrm{X}(\tau)\right)\right)d\tau$$

Furthe more, combining condition (iii), we have

$$E\left(V\left(t,\mathrm{X}(t)\right)\right)\leq V_0$$

On the other hand, from condition (ii), we get

$$aE\|\mathrm{X}(t)\|\leq V_0$$

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Since $V\left(t,X(t)\right)$ is locally Lipschitz continuous with respect to X, there a constant L such that for some $\gamma>0$ and $0<\|X\|<\gamma$, it has $V\left(t,X\right)\leq L\|X\|$. Then, for any $\varepsilon>0$, if $\delta(\varepsilon)$ is chosen as $\delta(\varepsilon)=\min\left(\frac{a\varepsilon}{L},\gamma\right)$, then, for $\|X_0\|<\delta(\varepsilon)$, it has

$$aE\|X(t)\|\leq V_0\leq L\|X_0\|\leq a\varepsilon$$

This implies that the zero solution of system (2) is stable in mean sense. The proof is completed.

Result 1: Let $0<\alpha\leq 1$, and let $V\left(t,X(t)\right):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ be continuously differential function and locally Lipschitz continuous with respect to X and satisfy the following conditions:

$(i)\ V\left(t,0\right)=0$

$(ii)\ V\left(t,X\right)\geq a\|X\|$

$(iii)\ E\left(dV\left(t,X(t)\right)\right)\leq 0$

Then the zero solution of system (2) is stable in mean sense.

Lemma 8: [9] If $g(r)$ is an always increasing function defined for all $r\geq 0$ and $g(0)=0$, then if $E\{r\}\geq M>0$, there exists an $L(M)>0$ such that $E\{g(r)\}\geq L$.

Theorem 5: Under the hypothesis of Theorem 4, further assume that $V\left(t,X(t)\right)$ has the property that $E\left({}^{C}D_{0^+}^{\alpha}V\left(t,X(t)\right)\right)$ is negative definite, i.e.,

$$E\left({}^{C}D_{0^+}^{\alpha}V\left(t,X(t)\right)\right)\leq -h\left(\|X\|\right)\qquad(10)$$

Where $h(0)=0$, and $h\left(\|X\|\right)$ is an increasing function. Then the zero solution of system (2) is asymptotically stable in mean sense.

Proof: We will prove it by contradiction. Assum that $E\{\|X\|\}$ does not tend to zero as $t\to\infty$. Then it is possible to find a $\delta>0$ such that $E\{\|X\|\}>\delta$ for any $t\geq t_0$. It follows from Lemma 8 that there exists a $\kappa(\delta)>0$ such that

$$E\{h\|X\|\}\geq\kappa(\delta)\qquad(11)$$

On the other hand, from inequalities (10) and (11) we get

$$\begin{aligned}E\left\{V\left(t,X(t)\right)\right\}&\leq V_0-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}E\{h\|X\|\}d\tau\\&\leq V_0-\frac{\kappa(\delta)t^{\alpha}}{\Gamma(\alpha+1)}\end{aligned}$$

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This implies that if we choose t such that $t^{\alpha} > \frac{\Gamma(\alpha+1)V_0}{\kappa(\delta)}$, then it has $E\left(V\left(t,\mathrm{X}(t)\right)\right)<0$, which led to a contradiction with condition (ii). It follows that $\underset{t\to\infty}{E}\left\{\left\|\mathrm{X}(t)\right\|\right\}=0$. The proof is completed.

Result 2: Under the hypothesis of Theorem 4, further assume that $V\left(t,\mathrm{X}(t)\right)$ has the property that $E\left(\frac{dV\left(t,\mathrm{X}\right)}{dt}\right)$ is negative definite, i.e.,

$$E\left(\frac{dV\left(t,\mathrm{X}\right)}{dt}\right)\leq -h\left(\left\|\mathrm{X}\right\|\right)$$

Where $h(0)=0$, and $h\left(\left\|\mathrm{X}\right\|\right)$ is an increasing function. Then the zero solution of system (2) is asymptotically stable in mean sense.

Theorem 6: Under the hypothesis of Theorem 5, further assume that there exists positive constant $a_i, p, q\ (i=1,2,3)$ and $\beta(0\leq\beta<1)$ such that $V\left(t,\mathrm{X}(t)\right)$ has the properties

$$(iv)\ \left\|\mathrm{X}\right\|^{q}\leq V\left(t,\mathrm{X}\right)\leq a_1\left\|\mathrm{X}\right\|^{p}+a_2 I_0^{\beta}\left\|\mathrm{X}\right\|^{q}$$

$$(v)\ E\left({}^{C}D_{0^+}^{\alpha}V\left(t,\mathrm{X}\right)\right)\leq -a_3 E\left\|\mathrm{X}\right\|^{p}$$

Then the zero solution of system (2) is Mittag-Leffler stable in mean sense.

Proof: From the second inequality in condition (iv) we have

$$-a_3\left\|\mathrm{X}\right\|^{p}\leq -\frac{a_3}{a_1}V\left(t,\mathrm{X}\right)+\frac{a_2a_3}{a_1}I_0^{\beta}\left\|\mathrm{X}\right\|^{q}$$

Also, from condition (v) we get

$$E\left({}^{C}D_{0^+}^{\alpha}V\left(t,\mathrm{X}\right)\right)\leq -\frac{a_3}{a_1}EV\left(t,\mathrm{X}\right)+\frac{a_2a_3}{a_1}E\left(I_{0^+}^{\beta}\left\|\mathrm{X}\right\|^{q}\right)\qquad(12)$$

On the other hand, from the first inequality in condition (iv) we have

$$\frac{a_2a_3}{a_1}I_{0^+}^{\beta}\left\|\mathrm{X}\right\|^{q}\leq\frac{a_2a_3}{a_1}I_{0^+}^{\beta}V\left(t,\mathrm{X}\right)\qquad(13)$$

Then, combining inequalities (12) and (13), we have

$$E\left({}^{C}D_{0^+}^{\alpha}V\left(t,\mathrm{X}\right)\right)\leq -\frac{a_3}{a_1}EV\left(t,\mathrm{X}\right)+\frac{a_2a_3}{a_1}I_{0^+}^{\beta}EV\left(t,\mathrm{X}\right)$$

According to Laplace transform and comparison principle in [], we have

$$EV\left(t,\mathrm{X}\right)\leq E_{\alpha}\left(-\frac{a_2a_3\Gamma(\beta)}{a_1}t^{\alpha+\beta}+\frac{a_3}{a_1}t^{\alpha}\right)$$

from condition (ii), we get

$$E\,\|X\| \le \frac{1}{a} E_{\alpha}\left(-\frac{a_2 a_3 \Gamma(\beta)}{a_1} t^{\alpha+\beta} + \frac{a_3}{a_1} t^{\alpha} \right)$$

The proof is completed.

Example:

Consider the system of fractional differential equations with randomly parameters

$$\frac{d^{\frac{1}{2}} x_1(t)}{dt^{\frac{1}{2}}} = \frac{-3}{1+t^2} x_2(t) + \frac{1}{1+t^2} x_2^2(t)$$

$$\frac{d^{\frac{1}{2}} x_2(t)}{dt^{\frac{1}{2}}} = \frac{-1}{1+t^2} x_1(t) - \frac{1}{1+t^2} x_2^2(t)$$

Introducing a Lyapunov function $V\,(t,x) = \frac{1}{2}\left(x_1^2 + x_2^2\right), V\,(t,0) = 0, V\,(t,x) > 0$, we have

$$\frac{d^{\frac{1}{2}} V\,(t,x)}{dt^{\frac{1}{2}}} = \frac{-3}{1+t^2} x_2 - \frac{1}{1+t^2} x_1$$

Then,

$$E\left(\frac{d^{\frac{1}{2}} V\,(t,x)}{dt^{\frac{1}{2}}} \right) = E\left(\frac{-3}{1+t^2} x_2 - \frac{1}{1+t^2} x_1 \right) < 0$$

Then, the zero solution of the given system is stable in the mean sense.

4. Conclusion

In this paper, we have studied the different kinds of stability of system (1) by using integral inequality method, and we have studied the different kinds of stability of system (2) by using Lyapunov function method. The importance of research into fractional differential equations with randomly parameters and their significance to future applications warrant continued. We propose some possible research topic in this active research which are studying the kinds of stability of these equations but with order $\alpha \in (n, n-1)$.

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