The Degree of Best Approximation of Functions via Some Linear Operators

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Abstract

The concentration of linear operators is unpretentious to prove in measurable space $L_p[0,1]$ but there is few works in weighted space, here we will include characteristics of approximate of unrestrained functions in measured space by lined operators via direct and converse approximation theorems. In addition, the relationship between modulus of softness and K- functional where, we proven are together tools equivalence.

Keywords: Approximation of functions; Linear operators; Weighted space; Modulus of smoothness; K-functional

1. Introduction

Weighted spaces are pleasant in many branches of mathematics and specifically the approximation theory see [1,2,3], the authors introduce some concept which closed related with weighted space.
Measurable spaces with one variable were presented by Sharapll [3] provide us with important tolls for explaining various application in mechanic like modelling of electro rhea logical fluids. As a problem of this fact, introduced on the fundamental results of these space have been studied recently in the integral operators. Corresponding problems of approximation functions can be found in the studies.

In 2022 Auad and Klaeif established some theorems of Best co-proximal and Co-chebyshev in weighted space and their found the relationship among better approximate. In addition, best simultaneous approximate. of unbounded functions [4]. In 2021 Ahmet, introduce some approximation properties of functions in terms positive linear operators of the trigonometric series in weighted space with one variable [7]. To learn more about the concept of approximation of functions in more detail, see the following references [8 , 9].

One of the most important notes observed in investigations of approximation of functions is definition off modulus of smoothness and K-functional, which will provide us the better tools for dealing with the degree best approximation of unbounded functions also direct and inverse theorems.

In this work, we want to present the equivalent relationship between the two approximation tools, the modulus smoothness function and K-functional; we can use this relationship to prove the direct and opposite formulae of unrestricted functions in measured spaces.

Let $L_{\ell,p}[-1,1]$ denoted the space of all unrestricted functions with criterion given by

$$\|\varphi\|_{\ell,p} = \left(\int_{-1}^{1} |\varphi(x)\ell(x)|^p \, dx\right)^{\frac{1}{p}} < \infty \tag{1}$$

, $1 \le p < \infty$, $\ell : [-1,1] \to \mathbb{R}^+$, $x \in [-1,1]$ and $\varphi \in L_{\ell,p}[-1,1]$ weight function belong to the space $\mathcal{G}$ which contain of all weight functions. The degree of best approximation is given by

DOI: https://doi.org/10.54216/IJNS.260416

$$E_k(\varphi)_{\ell,p} = \underbrace{inf}_{\mathcal{p}_k\in\mathbb{P}_k} \|\varphi - \mathcal{p}_k\|_{\ell,p} \qquad (2)$$

$k \in \mathbb{N}$, $\mathcal{p}_k$ is polynomial of degree less than or equal k and $\mathbb{P}_k$ the subspace of weighted space $L_{\ell,p}[-1,1]$ contain the polynomial $\mathcal{p}_k$.

For $k \in \mathbb{N}$ and $\delta > 0$, the modulus of smoothness of unbounded functions is define by

$$\omega_k(\varphi,\delta)_{\ell,p} = \underbrace{sup}_{k>\delta} \left\|\Delta_\delta^k \, \varphi(.)\right\|_{\ell,p} . \qquad (3)$$

Where, $\Delta_\delta^k \, \varphi(x) = \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i} \varphi(\delta + x)$, (4)

as well as the K-functional of unrestricted functions in weighted space equipped by

$$K_k(\varphi,\delta)_{\ell,p} = \underbrace{inf}_{\mathcal{q}\in\Psi}\left\{\| \varphi - \mathcal{q}\|_{\ell,p} + \delta^k\left\|\mathcal{q}^{(k)}\right\|_{\ell,p}\right\}, \qquad (5)$$

where Ψ subspace of weighted space $L_{\ell,p}[-1,1]$.

Let $\mathfrak{F} \in L_{\ell,p}[-1,1]^d$, denoted the multivariable function with d variable, $d \in \mathbb{N}$ and multi smoothness of modulus with d variables by

$$\omega_k{}^d(\mathfrak{F},\delta)_{\ell,p,d} = \underbrace{sup}_{k>\delta} \left\|\Delta_{\delta,d}^k \, \mathfrak{F}(.)\right\|_{\ell,p,d} \qquad (6)$$

Also, multi K-functional of unrestricted functions in measured space equipped via

$$K_k{}^d(\mathfrak{F},\delta)_{\ell,p} = \underbrace{inf}_{\mathfrak{Y}\in\Psi^d} \left\{\| \mathfrak{F} - \mathfrak{Y}\|_{\ell,p,d} + \delta^k\left\|\mathfrak{Y}_d{}^{(k)}\right\|_{\ell,p,d}\right\} \qquad (7)$$

2. Auxiliary lemmas

In this section, we introduce some lemma,s which essential it's in our major outcomes and through these lemmas, we will study the basic properties of modulus of smoothness.

Lemma 2.1[4] : Let $k \in \mathbb{N}$, $\delta > 0$ and $g \in L_p[-1,1]$, $1 \le p < \infty$. Then

i. For any $\alpha > 0$, we have $\left\|\Delta_\delta^k \, \rho\right\|_p \le (\alpha + 1)^k\left\|\Delta_\delta^k \, \rho\right\|_p$.

ii. $\left\|\Delta_\delta^k \, \rho\right\|_p \le 2^k\| \rho\|_p$.

iii. $\left\|\Delta_\delta^k \, \rho\right\|_p \cong \delta^{-k}\left\| D^{(k)}\rho\right\|_p$.

Lemma 2.2 [4] The modulus of smoothness $\omega_k(\rho,\delta)_p$ satisfies the following

i. For $i < j$, $\omega_j(\rho,\delta)_p \le 2^{j-i}\omega_i(\rho,\delta)_p$.

ii. For $\alpha > 0$, $\omega_j(\rho,\alpha\delta)_p \le (\alpha + 1)^j\omega_j(\rho,\delta)_p$.

Now, we define the following operator

$\Gamma_k(\varphi,t) = \int_{-1}^{1}\varphi(u)G_k\,(< t,u >)du; k \in \mathbb{N}$, (6)

where, $G_k(x) = \sum_{i=0}^{2k}\xi(\frac{i}{k})\,\frac{i+\alpha}{\alpha}P_k^\alpha(x)$, $x \in [-1,1]$, (7)

P_k^α is Gegenbour polynomial with index α and ξ continuous function on positive real numbers with zero such that $\xi(u) = 1$, $u \in [0,1]$.

Lemma 2.3[3] Let $\varphi \in L_{\ell,p}[-1,1]$, $\mathbb{P}_k$ the subspace of weighted space $L_{\ell,p}[-1,1]$ and $1 \le p < \infty$. Then

i. $\Gamma_k(\varphi,t) \in \mathbb{P}_k$ and $\Gamma_k(\varphi,t) = \varphi(t)$, for $t \in [-1,1]$.

ii. For $k \in \mathbb{N}$; $\|\Gamma_k(\varphi,.)\|_{\ell,p} \le C\|\varphi\|_{\ell,p}$, C positive constant

iii. For $k \in \mathbb{N}$; $\|\varphi - \Gamma_k(\varphi,.)\|_{\ell,p} \le CE_k(\varphi)_{\ell,p}$, C positive constant.

Lemma 2.4 [4] For $\varphi \in L_{\ell,p}[-1,1]$ and $1 \le p < \infty$. Then

$\Delta_\delta^k\Gamma_k(\varphi,t) = \Gamma_k\left(\Delta_\delta^k(\varphi,t)\right)$.

In particular for $\delta > 0$,

$\omega_k(\varphi - \Gamma_k(\varphi),\delta)_{\ell,p} \le C\omega_k(\varphi,\delta)_{\ell,p}$.

Proof: We define the operator with following property on the closed interval $[-1,1]$,

$\mathcal{M}_\beta(\psi,u) = \psi(\beta u)$, for $\beta \in [-1,1]$.

From definition the operator $\Gamma_k(\varphi,t)$, $\Delta_\delta^k\,\varphi(x)$ and lemma 2.1, we obtain

$$\begin{aligned}\mathcal{M}_\beta(\Gamma_k(\varphi),u) &= \int_{-1}^{1}\varphi(v)G_k\,(<\beta u,v>)dv\\ &= \int_{-1}^{1}\varphi(v)G_k\,(<u,\beta^{-1}v>)dv\\ &= \int_{-1}^{1}\varphi(\beta v)G_k\,(|<u,v>|)dv\\ &= \Gamma_k\big(\mathcal{M}_\beta(\varphi),u\big)\,.\end{aligned}$$

Now, $\left\|\Delta_\delta^k(\varphi-\Gamma_k(\varphi,.))\right\|_{\ell,p} = \left\|\Delta_\delta^k(\varphi)-\Delta_\delta^k(\Gamma_k(\varphi,.))\right\|_{\ell,p}$

$$\begin{aligned}&= \left\|\Delta_\delta^k(\varphi)-\Gamma_k(\Delta_\delta^k(\varphi,.))\right\|_{\ell,p}\\ &\le (1+C)\left\|\Delta_\delta^k(\varphi,.)\right\|_{\ell,p}\end{aligned}$$

$$sup\left\|\Delta_\delta^k(\varphi-\Gamma_k(\varphi,.))\right\|_{\ell,p} \le (1+C)\;sup\left\|\Delta_\delta^k(\varphi,.)\right\|_{\ell,p}$$

$$\omega_k(\varphi-\Gamma_k(\varphi,.),\delta)_{\ell,p}\;\le (1+C)\;\omega_k(\varphi,\delta)_{\ell,p}\,.$$

3. Main theorems

We begin with frontal and reverse theorems for best approximation of unbounded functions in $L_{\ell,p}[-1,1]$ via modulis of softness through some lined operators which belong subspace of $L_{\ell,p}[-1,1]$.

Theorem 3.1 Let $\varphi \in L_{\ell,p}[-1,1]$, $1 \le p < \infty$ and C positive constant dependent on p. Then

$E_k(\varphi)_{\ell,p}\;\le\;C(p)\omega_k(\varphi,\delta)_{\ell,p}$, where $\delta = k^{-1}$.

Proof: For$k = 1$, $1 \le p < \infty$, using lemma 2.1, (2) and (3), we have

$$\begin{aligned}E_k(\varphi)_{\ell,p} &\le \|\varphi(.)-\Gamma_k(\varphi,.)\|_{\ell,p}\;\le\;\int_{-1}^{1}\omega_k(\varphi,\delta)_{\ell,p}\,du\\ &\le\;C(p)\omega_k(\varphi,\delta)_{\ell,p}\,.\end{aligned}$$

Now, for $k > 1$, we follows the induction mathematical on k, by lemma 2.1 and lemma 2.3, we have the following:

Suppose that the following inequality hold

$E_k(\varphi)_{\ell,p}\;\le\;C(p)\omega_k(\varphi,\delta)_{\ell,p}$, for some $k \ge 1$.

The definition of $\Gamma_k(\varphi,t)$ implies $\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,t) = 0$.

So, $\left\|\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.)\right\|_{\ell,p} \le C\;E_k(\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.))_{\ell,p}$

$$\le\;C(p)\omega_k\left(\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.)\right)_{\ell,p}\,.$$

On other hand, using lemma 2.3and lemma 2.4, we obtain.

For $\in \mathbb{N}$, $\omega_k\left(\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,\delta)\right)_{\ell,p} = C(k)j^k\int_j^{2i}\frac{\omega_{k+1}\left(\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.),x\right)_{\ell,p}}{x^{k+1}}\,dx + C(k)j^k 2^{k+1}\left\|\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.)\right\|_{\ell,p}\int_{2^i}^{1}\frac{dx}{x^{k+1}}$

$$\le C_1(i,k)\omega_{k+1}\left(\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.),x\right)_{\ell,p} + C_2(x)2^{-i}\left\|\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.)\right\|_{\ell,p}$$

Choosing i such that

$$\frac{1}{4} \le C_1(i)C_1(k)2^{-ik} \le \frac{1}{2}$$

DOI: https://doi.org/10.54216/IJNS.260416

Thus, $\left\|\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.)\right\|_{\ell,p} \leq C\,\omega_{k+1}\left(\varphi(.)-\Gamma_{\left\lfloor\frac{k}{4}\right\rfloor}(\varphi,.),x\right)_{\ell,p}$

$$\leq C\,\omega_{k+1}(\varphi,\delta)_{\ell,p}.$$

Theorem 3.2 For $\varphi \in L_{\ell,p}[-1,1]$, $1 \leq p < \infty$, C positive constant and k natural numbers. Then

$\omega_k(\varphi,\delta^{-1})_{\ell,p} \leq C\delta^{-k}\sum_{i=1}^{k} i^{k-1}\,\delta^{-1}E_{i-1}(\varphi)_{\ell,p}$.

Proof: From lemma 2.1 and 2.3 , we have

$$\omega_k(\varphi,\delta^{-1})_{\ell,p} = sup\left\|\Delta_\delta^k(\varphi,.)\right\|_{\ell,p} \leq 2^k\|\varphi\|_{\ell,p}$$

$$\leq 2^k\left(\|\varphi(.)-\Gamma_k(\varphi,.)\|_{\ell,p}+\|\Gamma_k(\varphi,.)\|_{\ell,p}\right)$$

$$\leq 2^k\left(C\,E_k(\varphi)_{\ell,p}+\|\Gamma_k(\varphi,.)\|_{\ell,p}\right)$$

$$\leq 2^k\left(C\,E_k(\varphi)_{\ell,p}+(C+1)\,E_k(\varphi)_{\ell,p}\right)$$

$$\leq C\delta^{-k}\sum_{i=1}^{k} i^{k-1}\,E_{i-1}(\varphi)_{\ell,p}\ .$$

4. Approximation via a new pair of approximate tools

In this part, we present additional two tools of modulus of smoothness and K-functional on the weighted and apply them to find the better approximation on the weighted space. In addition, both the frontal and the reverse approximate. Theorems are considered in terms gadget K-functional function on weighted space.

Now we begin to present the proof of the equivalence of the two gadgets of degree k, which are K-functional and modulus of smoothness. Moreover, we introduced anther operator to approximate unbounded functions in weighted space is called the differential operator it is one of the important operator that have been used in our current work and is symbolized by $\Lambda(g)$ which as interpolate with the function φ.

The differential operator $\Lambda_{r,s}(\varphi)$ are connected to usual partial derivative when the independent variable has more than one in terms the following form

$$\frac{\partial}{\partial t_s}\left[\varphi\left(\frac{t}{\|t\|}\right)\right]_{\|t\|=1} = \partial_s\varphi - t_s\sum_{r=1}^{n} t_r\partial_r\varphi = -\sum_{r=1}^{n} t_r\Lambda_{r,s}(\varphi) \qquad (8)$$

On the other hand, the direct approximation theorem of the function φ in weighted space and the converse theorem of the same function are both considered in this section.

Now, we start to prove that modulus of smoothness and K-functional are equivalence.

Theorem 4.1 Let $\varphi \in L_{\ell,p}[-1,1]$, $1 \leq p < \infty$ and k natural number. Then

$$\omega_k(\varphi,\delta)_{\ell,p} \cong K_k(\varphi,\delta)_{\ell,p}\ ,\ \text{for}\ \ \delta > 0. \qquad (9)$$

Proof: From lemma 2.1 (ii) and (iii), we obtain

$\left\|\Delta_\delta^k(\varphi)\right\|_{\ell,p} \leq \|\varphi\|_{\ell,p}$ and $\left\|\Delta_\delta^k(\varphi)\right\|_{\ell,p} \leq \left\|\Delta_\delta^k(\varphi-\vartheta)\right\|_{\ell,p}+\left\|\Delta_\delta^k(\vartheta)\right\|_{\ell,p}$

$$\leq \|\varphi-\vartheta\|_{\ell,p}+\delta^k\|\vartheta\|_{\ell,p},$$

$\Rightarrow \omega_k(\varphi,\delta)_{\ell,p} \leq C_\circ\,K_k(\varphi,\delta)_{\ell,p}$ (10)

We need to prove that the converse of (10), its sufficient that proof of (6) complete.

From (5) , lemma 2.1 and lemma 2.3, we obtain

$$K_k(\varphi,\delta)_{\ell,p} \leq \|\varphi(.)-\Gamma_k(\varphi,.)\|_{\ell,p}+\delta^k\left\|D^{(k)}\Gamma_k(\varphi,.)\right\|_{\ell,p}$$

$$\leq \omega_k(\varphi,\delta)_{\ell,p}+\delta^k\delta^{-k}\left\|\Delta_\delta^k\Gamma_k(\varphi,.)\right\|_{\ell,p}$$

$$\leq C_1\omega_k(\varphi,\delta)_{\ell,p}\ .$$

We shall, study the degree of better approximate. Of unrestricted functions in measured space via the differential operator. $\Lambda_{r,s}(\varphi)$

Theorem 4.2 Let $\varphi \in L_{\ell,p}[-1,1]$, $1 \leq p < \infty$, and k natural number. Then

$E_{2k}(\varphi)_{\ell,p} \leq \frac{C}{k}\max E_k(D_j\varphi)_{\ell,p}$.

Proof: From lemmas 2.3 and 2.4 , we have

DOI: https://doi.org/10.54216/IJNS.260416

$D^{(k)}\Gamma_k(\varphi,t) = D^{(k)}\int_{-1}^{1}\varphi(u)G_k\,(<t,u>)du = \int_{-1}^{1}D^{(k)}\varphi(u)G_k\,(<t,u>)du$ and $D^{(k)}\Gamma_k(\varphi,t) = \Gamma_k D^{(k)}(\varphi,t)$.

Thus, $E_{2k}(\varphi)_{\ell,p} \le E_{2k}(\varphi - \Gamma_k(\varphi))_{\ell,p} + E_{2k}(\Gamma_k(\varphi))_{\ell,p}$

$$\le C\,K_k(\varphi - \Gamma_k(\varphi))_{\ell,p} + C\,K_k(\Gamma_k(\varphi))_{\ell,p}$$

$$\le \frac{C}{k}\max\left\{\left\|D^{(k)}(\varphi - \Gamma_k(\varphi))\right\|_{\ell,p} + \left\|D^{(k)}(\Gamma_k(\varphi))\right\|_{\ell,p}\right\}$$

$$\le \frac{C}{k}\max E_k(D_j\varphi)_{\ell,p}.$$

5. Approximation of multi- functions

In this part, we present new types of multivariable functions to approximate them using new approximation tools. We will start by studding the elementary features of multi modulus of smoothness. Moreover, the frontal and reverse formulas of multi functions in terms multi smoothness modulus are considered as well as, the connection inter alia multi K-functional and multi smoothness of modulus has been given.

New, we introduce some propositions which need its in our main results.

proposition 5.1: If $\mathfrak{F} \in L_{\ell,p}[-1,1]^d$, $1 \le p < \infty$ and$k \in \mathbb{N}$, then
$\left\|\Delta_{\delta,d}^{k}\,\mathfrak{F}(.)\right\|_{\ell,p,d} \le |u|^k\left\|D_d^k\mathfrak{F}\right\|_{\ell,p,d}$.

Proof: Let $\Lambda_d(u,v) = \mathfrak{F}(\mathrm{P}_{d,u},v)$. For each $u_1,u_2 > 0$ and $v = (v_1,v_2,\dots,v_d)$
$\Lambda_d(u_1+u_2,v) = \mathfrak{F}(\mathrm{P}_{d,u_1+u_2,},v) = \Lambda_d\big(u_1,\mathrm{P}_{d,u_2,},v\big)$.

From definition difference symmetric of$\Delta_{\delta,d}^{k}$, that

$$\Delta_{\delta,d}^{k}\mathfrak{F}(v) = \int_0^u \dots \int_0^u D_{u^k}^{k}\,\Lambda_d(u_1+u_{21}+\dots+u_d,v)du_1\dots du_1$$

$$= \int_0^u \dots \int_0^u D_{u^k}^{k}\,\Lambda_d\big(0,\ \mathrm{P}_{d,u_1+u_2,\dots,,u_d,},v\big)du_1\dots du_1\ .$$

So, $\left\|\Delta_{\delta,d}^{k}\,\mathfrak{F}(.)\right\|_{\ell,p,d} \le \int_0^u \dots \int_0^u\left\|D_{u^k}^{k}\Lambda_d\big(0,\ \mathrm{P}_{d,u_1+u_2,\dots,,u_d,},v\big)\right\|_{\ell,p}du_1\dots du_1$.

Also, $\left\|\Delta_{\delta,d}^{k}\,\mathfrak{F}(.)\right\|_{\ell,p} \le \int_0^u \dots \int_0^u\left\|D_{u^k}^{k}\mathfrak{F}(.)\right\|_{\ell,p,d}du_1\dots du_1$

$$\le\ u^k\left\|D_{u^k}^{k}\mathfrak{F}(.)\right\|_{\ell,p,d}.$$

Proposition 5.2: If $\mathfrak{F} \in L_{\ell,p}[-1,1]^d$, $1 \le p < \infty$ and $k \in \mathbb{N}$, then

i. $\omega_k{}^d(\mathfrak{F},u)_{\ell,p,d} \le C(.p)\|\mathfrak{F}(.)\|_{\ell,p,d}$
ii. For $\delta \ge 1$ and $u \in \mathbb{R}^+$, $\omega_k{}^d(\mathfrak{F},\delta u)_{\ell,p,d} \le C(k,p)(1+k\delta u)^{k/p}\delta^k\omega_k{}^d(\mathfrak{F},u)_{\ell,p,d}$.
iii. Let $i,j \in \mathbb{N}\cup\{0\}$ and $u \in (0,\frac{1}{2^i j})$
, $\omega_k{}^d(\mathfrak{F},u)_{\ell,p,d} \le C(i,k)\ \omega_{k+1}{}^d(\mathfrak{F},u)_{\ell,p,d} + C(k)\|\mathfrak{F}(.)\|_{\ell,p,d}$.

Thus, using 5.2, we have established the upper and lower limits of the inequality estimates

$$\omega_k{}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d} \le C(i,k)\left\|D_{u^k}^{k}\mathfrak{F}(.)\right\|_{\ell,p,d} \qquad (8)$$

$$\left\|D_{u^k}^{k}\mathfrak{F}(.)\right\|_{\ell,p,d} \le\ C(k)\frac{1}{2^i j}\ \omega_k{}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d} \qquad (9)$$

Theorem 5.3: Let $\mathfrak{F} \in L_{\ell,p}[-1,1]^d$, $1 \le p < \infty$, $k \in \mathbb{N}$ and $\mathfrak{Y}$ be operator in $\Omega^d$is subspace of $L_{\ell,p}[-1,1]^d$. Then, there is constant C dependent on k and $p$such that

$E_k(\mathfrak{F})_{\ell,p,d} \le C(k,p)\omega_k{}^d(\mathfrak{F},\gamma)_{\ell,p,d}$, $\gamma > 0$.

Proof: We take the difference between the function $\mathfrak{F}$ and the operator $\mathfrak{Y}$, then

$$|\mathfrak{F}(u) - \mathfrak{Y}(\mathrm{u})| = \frac{2}{\gamma}\left|\int_{\frac{2}{\gamma}}^{\gamma}\left\{\frac{1}{j^k}\int_0^j \dots \int_0^j \Delta_{\gamma,d}^{k}\,\mathfrak{F}(\mathrm{u})\,dv_1\dots dv_d\right\}dj\right|.$$

From, Minkowski,s inequality, we have

DOI: https://doi.org/10.54216/IJNS.260416

$$\|\mathfrak{F}(.)-\ \mathfrak{Y}(.)\|_{\ell,p,d} \leq C(k,p)\frac{2}{\gamma}\int_{\frac{2}{\gamma}}^{\gamma}\left\{\frac{1}{j^{k-1}}\int_0^j \dots \left\|\frac{1}{j}\int_0^j \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv_1\right\|_{\ell,p,d} dv_2 \dots dv_d\right\}dj$$

$$=\ C(k,p)\frac{2}{\gamma}\int_{\frac{2}{\gamma}}^{\gamma}\left\{\frac{1}{j^{k-1}}\int_0^j \dots \left\|\frac{1}{j}\int_{v_2+\cdots v_d}^{jv_2+\cdots v_d} \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv_1\right\|_{\ell,p,d} dv_2 \dots dv_d\right\}dj \quad (10)$$

Since,

$$\left\|\frac{1}{j}\int_{v_2+\cdots v_d}^{j+v_2+\cdots v_d} \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv\right\|_{\ell,p,d} =\ \left\|\frac{1}{j}\left(\int_0^{j+v_2+\cdots v_d} \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv-\int_0^{v_2+\cdots v_d} \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv\right)\right\|_{\ell,p,d}$$

$$\leq \sup\left\|\frac{1}{(j+v_2+\cdots+v_d)/d}\int_0^{(j+v_2+\cdots+v_d)/d} \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv\right\|_{\ell,p,d}+$$

$$\sup\left\|\frac{1}{(j+v_2+\cdots+v_d)/d}\int_0^{(v_2+\cdots+v_d)/d} \Delta_{\gamma,d}^k\,\mathfrak{F}(\mathrm{u})\,dv\right\|_{\ell,p,d}.$$

$$=\ {\omega_k}^d(\mathfrak{F},\gamma)_{\ell,p,d}+{\omega_k}^d(\mathfrak{F},\gamma)_{\ell,p,d}=2\,{\omega_k}^d(\mathfrak{F},\gamma)_{\ell,p,d}.$$

From (10) and (11), we obtain

$$\|\mathfrak{F}(.)-\ \mathfrak{Y}(.)\|_{\ell,p,d} \leq C(k,p)\frac{2}{\gamma}\int_{\frac{2}{\gamma}}^{\gamma}\left\{\frac{1}{j^{k-1}}\int_0^j \dots \int_0^j {\omega_k}^d\,(\mathfrak{F},\gamma)dv_1\,dv_2 \dots dv_d\right\}dj$$

$$\leq\ C(k,p){\omega_k}^d\,(\mathfrak{F},\gamma)_{\ell,p,d}\frac{2}{\gamma}\int_{\gamma/2}^{\gamma}dv=C(k,p){\omega_k}^d\,(\mathfrak{F},\gamma)_{\ell,p,d}\ . \quad (11)$$

Theorem 5.4: Let $\mathfrak{F}\in L_{\ell,p}[-1,1]^d$, $1\leq p<\infty$, $k\in\mathbb{N}$ and $\mathfrak{Y}$ be operator in $\Omega^d$is subspace of $L_{\ell,p}[-1,1]^d$. Then, there is steady C count on k and $p$such that

$${\omega_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d}\approx\ {K_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d}.$$

Proof: From (9) and (11), we have

$${K_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d}\leq C(k,p)\ {\omega_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d} \quad (12)$$

From another direction, by propositions 5.1 and 5.2, we get

$${\omega_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d}\leq {\omega_k}^d(\mathfrak{F}-\mathfrak{N},\tfrac{1}{k})_{\ell,p,d}+{\omega_k}^d(\mathfrak{N},\tfrac{1}{k})_{\ell,p,d}$$

$$\leq C(k,p)\|\mathfrak{F}(.)-\ \mathfrak{N}(.)\|_{\ell,p,d}+C(k,p)\left\|D_{u^k}^k\mathfrak{N}(.)\right\|_{\ell,p,d} \quad (13)$$

By using (12) and (13), we attain the equivalent inter alia K-functional and modulis of softness in multi weighted spaces.

Theorem 5.5: Let $\mathfrak{F}\in L_{\ell,p}[-1,1]^d$, $1\leq p<\infty$, $k\in\mathbb{N}$. Then, there is constant C dependent on k and p such that

$${\omega_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d}\leq\frac{C(k,p)}{j}\sum_{i=0}^{j}(i+1)^{k-1}E_i\,(\mathfrak{F})_{\ell,p,d}\ .$$

Proof: Let $\mathfrak{L}$ be the best algebraic approximation polynomial of $\mathfrak{F}$ in the multi weighted space$L_{\ell,p}[-1,1]^d$.

Also, $j=1,2,3,\dots$ such that $2^j\leq i<2^{j+1}$, we have

$${\omega_k}^d(\mathfrak{F},\tfrac{1}{k})_{\ell,p,d}\leq {\omega_k}^d(\mathfrak{F}-\mathfrak{L}_{2^{j+1}},\tfrac{1}{k})_{\ell,p,d}+{\omega_k}^d(\mathfrak{L}_{2^{j+1}},\tfrac{1}{k})_{\ell,p,d}\ .$$

Since, $2^{(j+1)k}E_{2^k}(\mathfrak{F})_{\ell,p,d}\leq 2^{2k}\sum_{i=2^{k-1}+1}^{2^j}E_i(\mathfrak{F})_{\ell,p,d}.$ (14)

From proposition 5.2, we have

$${\omega_k}^d(\mathfrak{F}-\mathfrak{L}_{2^{j+1}},\frac{1}{k})_{\ell,p,d}\leq C(k,p)\left\|\mathfrak{F}(.)-\mathfrak{L}_{2^{j+1}}(.)\right\|_{\ell,p,d}=C(k,p)E_{2^{j+1}}(\mathfrak{F})_{\ell,p,d}$$

$$\leq\ C(k,p)\frac{2^{(j+1)k}}{i^k}E_{2^j}(\mathfrak{F})_{\ell,p,d}\leq\ C(k,p)\frac{2^{(j+1)k}}{i^k}2^{2k}\sum_{i=2^{k-1}+1}^{2^j}i^{k-1}E_i(\mathfrak{F})_{\ell,p,d}\ . \quad (15)$$

From another direction, using propositions 5.1& 5.2 and (14), we obtain

$${\omega_k}^d(\mathfrak{L}_{2^{j+1}},\tfrac{1}{k})_{\ell,p,d}\leq\frac{C(k,p)}{i^k}\left\|D_{u^k}^k\mathfrak{L}_{2^{j+1}}\right\|_{\ell,p,d}=\frac{C(k,p)}{i^k}\left\|D_{u^k}^k\mathfrak{L}_1(.)+\sum_{i=0}^{j}\left(D_{u^k}^k\mathfrak{L}_{2^{i+1}}-D_{u^k}^k\mathfrak{L}_{2^i}\right)\right\|_{\ell,p,d}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\|D^k_{u^k}\mathfrak{L}_1(.)\right\|_{\ell,p,d}\ \tfrac{C(k,p)}{i^k}+\left\|\textstyle\sum_{i=0}^{j}\left(D^k_{u^k}\mathfrak{L}_{2^{i+1}}-D^k_{u^k}\mathfrak{L}_{2^i}\right)\right\|_{\ell,p,d,}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{\|\mathfrak{L}_1(.)\|_{\ell,p,d}+\left\|\textstyle\sum_{i=0}^{j}\left(\mathfrak{L}_{2^{i+1}}-\mathfrak{L}_{2^i}\right)\right\|_{\ell,p,d,}\right\}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{E_0(\mathfrak{F})_{\ell,p,d}+\left\|\textstyle\sum_{i=0}^{j}2^{(i+1)k}E_{2^i}(\mathfrak{F})_{\ell,p,d}\right\|_{\ell,p,d,}\right\}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{E_0(\mathfrak{F})_{\ell,p,d}+2^kE_1(\mathfrak{F})_{\ell,p,d}+\left\|\textstyle\sum_{i=0}^{j}2^{(i+1)k}E_{2^i}(\mathfrak{F})_{\ell,p,d}\right\|_{\ell,p,d,}\right\}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{E_0(\mathfrak{F})_{\ell,p,d}+2^kE_1(\mathfrak{F})_{\ell,p,d}+\textstyle\sum_{i=1}^{j}\sum_{m=2^{i-1}+1}^{2^i}m^{(i+1)k}E_m(\mathfrak{F})_{\ell,p,d}\right\}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{E_0(\mathfrak{F})_{\ell,p,d}+\textstyle\sum_{m=1}^{2^j}m^{(i+1)k}E_m(\mathfrak{F})_{\ell,p,d}\right\} \qquad (16)$$

From (14), (15) and (16), we get

$$\omega_k{}^d\left(\mathfrak{L}_{2^{j+1}},\tfrac{1}{k}\right)_{\ell,p,d}\leq \tfrac{C(k,p)}{i^k}\left\{\textstyle\sum_{m=2^{i-1}+1}^{2^i}m^{(i+1)k}E_m(\mathfrak{F})_{\ell,p,d}+E_0(\mathfrak{F})_{\ell,p,d}+\sum_{m=1}^{2^j}m^{(i+1)k}E_m(\mathfrak{F})_{\ell,p,d}\right\}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{\textstyle\sum_{m=1}^{2^i}m^{(i-1)}E_m(\mathfrak{F})_{\ell,p,d}+E_0(\mathfrak{F})_{\ell,p,d}\right\}$$

$$\leq \tfrac{C(k,p)}{i^k}\left\{\textstyle\sum_{m=1}^{2^i}m^{(i-1)}E_m(\mathfrak{F})_{\ell,p,d}\right\}.$$

The proof of theorem followed.

6. Conclusion

The concentration of linear operators was unpretentious to prove in the measurable space $L_p[0,1]$ but there were few works in weighted space. In this work, we included characteristics of the approximation of unrestrained functions in measured space by linear operators via direct and converse approximation theorems. We also explored the relationship between the modulus of softness and the K- functional, where we proved that these tools were equivalent. For more future work, these provided tools can be combined with both [10-11].

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DOI: https://doi.org/10.54216/IJNS.260416