Finite-Time Stability in the Discrete Sel'kov-Schnakenberg Reaction-Diffusion Model: Analytical Analysis and Numerical Simulations

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Abstract

This study investigates the finite-time stability (FTS) of the discrete Sel'kov-Schnakenberg reaction-diffusion (SSRD) system, a mathematical model capturing the interplay between local reactions and spatial diffusion. A novel discretization framework based on finite difference methods (FDM) is developed to transform the continuous reaction-diffusion (RD) system into a discrete counterpart, enabling detailed computational analysis. Sufficient conditions for FTS are derived using Lyapunov functions (LF) and eigenvalue-based methods, ensuring precise predictions of the system's behavior. Numerical simulations validate theoretical findings, demonstrating the proposed methods' practical applicability to scenarios such as chemical reactions, biological processes, and technological systems. The influence of system parameters, boundary conditions, and initial conditions on the dynamic behavior is systematically analyzed. This study contributes to the broader understanding of RD systems, addressing key challenges in stability analysis and offering a computationally efficient framework with implications for science and engineering.

Keywords: Finite-time stabilit; Reaction-diffusion systems; Sel'kov-Schnakenberg model; Finite difference methods; Lyapunov functions

1 Introduction

The RD systems, modeled by partial differential equations (PDEs), are crucial for describing dynamic processes in chemistry, biology, ecology, and physics.[1] These systems capture the interplay between local reactions and diffusion-driven spatial transport, enabling the analysis of phenomena such as pattern formation and oscillatory behavior. Analytical approaches, including port-Hamiltonian representations and colored Petri nets, have been developed to study RD systems.[2] While RD PDEs have traditionally been applied to large-population scenarios, recent studies have connected them to particle-based stochastic models, validating their

use as asymptotic approximations under specific conditions.[3] Beyond chemistry, RD systems are used to model biological processes such as population dynamics and migration, and technological phenomena such as innovation diffusion.[4] Their ability to integrate spatial and temporal dimensions makes RD systems indispensable for studying complex processes across various disciplines.

In this manuscript, we analyze the following RD system:

$$\begin{cases} u_t(x,t) = \alpha(p - u + \beta v + u^2 v) + \Delta u, & (x,t) \in \Omega \times \mathbb{R}_+^*, \\ v_t(x,t) = \alpha(q - \beta v - u^2 v) + \zeta \Delta v, & (x,t) \in \Omega \times \mathbb{R}_+^*, \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}_+^*, \\ u(x,0) = \psi(x),\ v(x,0) = \varphi(x), & x \in \Omega. \end{cases} \tag{1}$$

Here, the domain $\Omega \subset \mathbb{R}^n$ has a smooth boundary $\partial\Omega$, and Δ represents the Laplacian. u and v denote reactant concentrations, while $p, \beta, q, \alpha > 0$ are constants, and ζ is the scaled diffusion coefficient.

The Sel'kov-Schnakenberg (SS) model is a prominent example of RD systems, describing the autocatalytic chemical and biochemical reactions.[5] Known for its oscillatory behavior, the SS model has significant applications in biological processes.[6] Researchers have employed diverse mathematical techniques to study this model, including fractional-order derivatives and the φ6-model expansion method.[7] These investigations have explored a priori bounds, global existence, and the regularity of solutions in physically relevant dimensions. Additionally, semi-analytical approaches, such as the Galerkin method, have analyzed stability, bifurcations, and the effects of varying parameters.[8] These efforts have provided critical insights into the dynamics of SS systems and their applications in real-world chemical and biological contexts.

A defining feature of the SS system is its rich dynamical behavior, which includes steady states, limit cycles, and chaotic attractors.[9] The model incorporates linear and nonlinear autocatalysis, accounting for catalyst decay and potential reversibility.[10,11] Oscillatory behavior, constrained by matter conservation in closed systems, has been extensively studied. Recent analyses have focused on reversible Schnakenberg equations, demonstrating the presence of global attractors under minimal solutions and their upper semi-continuity concerning reverse reaction rates.[12]

Originating from Sel'kov's work on glycolysis,[13] the SS model has become a foundation for studying oscillatory chemical reactions. Subsequent research has established criteria for spatial pattern formation,[14] examined nonlinear stability using Lyapunov functionals,[15] and derived amplitude equations to interpret structural transitions and Turing patterns.[16] These studies have elucidated the role of diffusion rates and activator-inhibitor interactions in generating complex patterns.

FTS of the SSRD system has also been rigorously analyzed. Full-domain and boundary controllers have been developed to achieve finite-time stabilization, with sufficient conditions derived for systems with Neumann and mixed boundary conditions.[17,18] Extensions to robust FTS for uncertain systems have further enhanced these models' theoretical and practical applicability.[19]

This manuscript examines the FTS of the discrete SSRD system, a mathematical framework representing the interaction between local reactions and spatial diffusion. The objectives of this study include:

- Develop a discretization framework using finite difference methods to transform the continuous RD system into a discrete system, enabling computational analysis.
- Derive sufficient conditions for FTS using LF and eigenvalue-based methods, ensuring accurate predictions of system behavior.
- Validate the theoretical findings through numerical simulations, demonstrating the real-world relevance and applicability of the proposed methods in practical situations such as chemical reactions, biological processes, and technological systems.
- Investigate the effects of system parameters, and boundary conditions on the dynamic behavior of the model, offering insights into optimizing performance in practical applications.

This work contributes to the broader understanding of RD systems by addressing key challenges in stability and synchronization analysis, with implications for various scientific and engineering fields. The following outlines the structure of the paper: Section 2 presents the discretization process and model formulation, presenting the transformation of the continuous system into its discrete counterpart. Section 3 discusses the FTS analysis, including the derivation of sufficient conditions using LF and eigenvalue-based methods. Section 4 focuses on numerical applications, validating theoretical findings through simulations and exploring parameter effects on system dynamics. Finally, Section 5 concludes with a summary of key contributions and outlines future research directions.

2 Discritization process and model

The discretization of the SSRD system has been extensively studied, with various numerical methods employed to analyze its dynamics, stability, and bifurcation behavior. For instance,[16] utilized the Galerkin method to investigate stability and bifurcation properties, while[20] introduced a cellular automata approach, enhancing computational efficiency through discretized variables and lookup tables. Ersoy[21] implemented a finite element method combined with Crank-Nicolson time integration, utilizing cubic B-spline basis functions to achieve higher accuracy. More recently,[22] proposed a linearized finite difference algorithm based on a three-level linearization scheme, demonstrating the solution's existence, uniqueness, and boundedness. These studies underscore the flexibility and effectiveness of numerical methods in addressing the computational challenges of RD systems while preserving critical features such as Turing patterns.

FDM remains a prominent approach for discretizing PDEs, particularly in spatially extended systems. This technique approximates spatial and temporal derivatives using difference formulas over a discrete grid. Typically, a uniform spatial mesh with grid spacing Δx is employed, where grid points are defined as $x_i = i\Delta x$.

The time derivative is approximated using the forward Euler method, resulting in the following expression:

$$\Delta_h \mathfrak{Q}_i(t) = \frac{\mathfrak{Q}_i(t+h) - \mathfrak{Q}_i(t)}{\Delta x}, \tag{2}$$

Central difference formulas are commonly used to approximate second-order spatial derivatives, such as:

$$\mathfrak{Q}_{xx}(x,t) \approx \frac{\mathfrak{Q}_{i+1}(t) - 2\mathfrak{Q}_i(t) + \mathfrak{Q}_{i-1}(t)}{(\Delta x)^2}. \tag{3}$$

This discretization reduces the original PDE to a system of ODEs for the discrete variables $\chi(x_i, t)$, which can then be solved using methods like implicit Euler integration.[23] The popularity of finite difference schemes arises from their simplicity and computational efficiency.[24]

Using the second-order difference operator $\Delta^2\chi$, defined as:

$$\Delta^2 \mathfrak{Q}_i(t) = \mathfrak{Q}_{i+2}(t) - 2\mathfrak{Q}_{i+1}(t) + \mathfrak{Q}_i(t), \tag{4}$$

The 2^{nd} derivative may be represented as:

$$\frac{\partial^2 \mathfrak{Q}(x,t)}{\partial x^2} \approx \frac{1}{(\Delta x)^2} \Delta^2 \mathfrak{Q}_{i-1}(t). \tag{5}$$

Building on this framework, the discrete-time version of the SSRD system is formulated as detailed below:

$$\begin{cases} \Delta_h u_i(t) = \alpha\big(p - u_i(t) + \beta v_i(t) + u_i^2(t)v_i(t)\big) + \dfrac{\Delta^2 u_{i-1}(t)}{(\Delta x)^2}, \\ \Delta_h v_i(t) = \alpha\big(q - \beta v_i(t) - u_i^2(t)v_i(t)\big) + \dfrac{\zeta \Delta^2 v_{i-1}(t)}{(\Delta x)^2}. \end{cases} \tag{6}$$

Periodic boundary conditions (PBCs) are imposed to ensure consistency:

$$\begin{cases} u_0(t) = u_N(t), \quad u_1(t)) = u_{N+1}(t), \\ v_0(t) = v_N(t), \quad v_1(t)) = v_{N+1}(t), \end{cases} \tag{7}$$

With initial conditions (ICs) given as:

$$u_i(0) = \psi_i > 0, \quad v_i(0) = \varphi_i > 0. \tag{8}$$

This discretized model forms the framework for analyzing the system's dynamics, stability, and bifurcations in discrete spatial and temporal domains.

This study leverages the discretization of the SSRD system to investigate FTS. Key contributions include:

- Discretizing the system converts complex PDEs into manageable ODEs, enabling the derivation of conditions for FTS through discrete dynamics.
- The second-order FDM ensures accurate spatial derivative approximations, preserving the system's inherent dynamics and facilitating stability analysis using LF and related tools.
- Techniques like implicit Euler integration and central difference schemes enhance numerical robustness, avoiding artifacts that could compromise FTS insights.
- Consistent with,[22] the boundedness, existence, and uniqueness of solutions provide a solid basis for analyzing FTS.
- Periodic boundary conditions maintain the physical integrity of the system, preventing artificial boundary effects that might distort stability analyses.
- The discretized model supports numerical simulations, enabling the observation of finite-time convergence and validation of theoretical predictions, including estimates of settling times.

This study provides a computationally robust framework and facilitates a deeper understanding of FTS within RD systems.

Having established the discretization framework, we now focus on its application to FTS analysis.

3 The Analytical Result

FTS represents a critical concept in studying dynamic systems, where the system's state converges to equilibrium within a predetermined limited duration. This property has gained substantial attention due to its applicability in systems requiring rapid stabilization, such as biological models, chemical reactions, and engineering applications. In this section, we investigate the FTS of the discrete SSRD system, extending classical stability notions to capture finite-time behavior. By leveraging discretization methods and stability tools, we derive sufficient conditions that ensure convergence within a finite duration. The implications of these results are further validated through numerical simulations, reinforcing their theoretical and practical significance.

First, we analyze the system's equilibrium points and the solutions that meet specific conditions. These steady-state solutions correspond to the equations' non-negative values.

$$\begin{cases} \alpha\big(p - u^* + \beta v^* + u^2 v^*\big) = 0, \\ \alpha\big(q - \beta v^* - u^{*2} v^*\big) = 0. \end{cases} \tag{9}$$

System (9) yields the equilibrium points:

$$u^* = p + q, \quad v^* = \frac{q}{(p+q)^2 + \beta}. \tag{10}$$

Theorem 3.1. [25] *Consider functions $\vartheta, \varsigma \in \mathbb{R}$, and let $m, N \in \mathbb{N}$ such that $m < N$. The formula for summation by parts is expressed as:*

$$\sum_{i=m}^{N-1} \vartheta_i \Delta \varsigma_i = \vartheta_i \varsigma_i \Big|_m^N - \sum_{i=m}^{N-1} \varsigma_{i+1} \Delta \vartheta_i. \tag{11}$$

Lemma 3.2. [25] *Let $\mathfrak{Q}(t) \in L^2([0, +\infty))$. Then, the following inequality holds:*

$$\rho_1 \sum_{i=1}^{N} |\mathfrak{Q}_i(t)|^2 \leq \sum_{i=1}^{N} |\Delta \mathfrak{Q}_i(t)|^2, \tag{12}$$

where $\rho_1 > 0$ is the eigenvalue of the system (13)-(14)*:*

$$-\Delta^2 \mathfrak{Q}_{i-1}(t) = \rho \mathfrak{Q}_i(t), \quad \textit{for } t \in \mathbb{R}_+^*, \tag{13}$$

With PBCs:

$$(\mathfrak{Q}_0(t), \mathfrak{Q}_1(t)) = (\mathfrak{Q}_N(t), \mathfrak{Q}_{N+1}(t)), \tag{14}$$

Consider the nonlinear dynamical system (15) given as follows:

$$\begin{cases} \Delta_h \mathfrak{Q}(t) = \xi\left(t, \mathfrak{Q}(t)\right), & \text{for } t \in \mathbb{R}_+^*, \\ \mathfrak{Q}(0) = \mathfrak{Q}_0, & \text{for } t \in \mathbb{R}_+^*. \end{cases} \tag{15}$$

where the nonlinear function $\xi(t, \chi(t)) : \mathbb{R}_+^* \times \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 3.3. [26] *Suppose there exists a continuous function $\mathsf{V} : D \to \mathbb{R}$, a constant $\Theta > 0$, and a neighborhood $\mho \subseteq D$ around the origin, satisfying the following conditions 1-3:*

1. $\mathsf{V}(0) = 0$,
2. $\forall t \in \mho \setminus \{0\}$, $\mathsf{V}(t) > 0$,
3. $\forall t \in \mho \setminus \{0\}$, $\Delta_h \mathsf{V}(t) \leq -\min\{\mathsf{V}(t), \Theta\}$.

Then, the equilibrium point $\mathfrak{Q}^*$ of system (15) *is FTS.*

Moreover, there exists a settling-time function $\mathfrak{t}_\mathfrak{e} : \mathbb{N} \to \mathbb{Z}^+$, satisfying the following condition:

$$\mathfrak{t}_\mathfrak{e} \leq \left\lceil \frac{\mathsf{V}(0)}{\Theta} \right\rceil. \tag{16}$$

Lemma 3.4. [27] *A constant $L > 0$ exists such that:*

$$\left| u_i^2(t) v_i(t) - u^{*2} v^* \right| < L \left(|u_i(t) - u^*| + |v_i(t) - v^*| \right). \tag{17}$$

Theorems 3.1, 3.3, and Lemmas 3.2, 3.4 are essential tools for proving the FTS in Theorem 3.5. Specifically:

- Provide discrete summation techniques and eigenvalue-based inequalities, critical for analyzing stability through Lyapunov methods.
- Establishes FTS conditions using an LF, linking these to settling time.
- Bounds nonlinear terms, ensuring the conditions in Theorem 3.5 hold under the system dynamics.

These components ensure the proof rigorously validates finite-time convergence criteria.

Theorem 3.5. *The system* (6) *is FTS at the equilibrium point* (u^*, v^*) *if:*

$$\Theta = 2\min\left\{\frac{\rho_1}{(\Delta x)^2} + \alpha\left(1 - 2L - \frac{\beta}{2}\right), \frac{\rho_2\zeta}{(\Delta x)^2} + \alpha\left(\frac{\beta}{2} - 2L\right)\right\} > 1, \tag{18}$$

Where the settling time is estimated as:

$$\mathfrak{t}_\epsilon \leq \left\lceil \frac{\mathsf{V}(0)}{\Theta} \right\rceil, \tag{19}$$

and

$$\mathsf{V}(0) = \frac{1}{2}\sum_{i=1}^{N}\left((\varphi_i - u^*)^2 + (\psi_i - v^*)^2\right). \tag{20}$$

Proof. Let the LF be defined as:

$$\mathsf{V}(t) = \frac{1}{2}\sum_{i=1}^{N}\mathfrak{m}_i^2(t) + \mathfrak{r}_i^2(t). \tag{21}$$

Substituting:

$$\mathfrak{m}_i(t) = u_i(t) - u^*, \quad \mathfrak{r}_i(t) = v_i(t) - v^*. \tag{22}$$

We aim to show that the discrete derivative of this functional, $\Delta_h\mathsf{V}(t)$, is negative definite. This can be demonstrated as follows:

$$\begin{aligned}
\Delta_h\mathsf{V}(t) &= \sum_{i=1}^{N}\left[\mathfrak{m}_i(t)\Delta_h\mathfrak{m}_i(t) + \mathfrak{r}_i(t)\Delta_h\mathfrak{r}_i(t)\right] \\
&= \sum_{i=1}^{N}\Bigg[\mathfrak{m}_i(t)\Bigg(\alpha\big(-\mathfrak{m}_i(t) + \beta\mathfrak{r}_i(t) + (u_i^2(t)r_i(t) - u^{*2}r^*)\big) \\
&\quad + \frac{1}{(\Delta x)^2}\Delta^2\mathfrak{m}_{i-1}(t)\Bigg) + \mathfrak{r}_i(t)\Bigg(\alpha\big(-\beta\mathfrak{r}_i(t) - (u_i^2(t)r_i(t) - u^{*2}r^*)\big) \\
&\quad + \frac{\zeta}{(\Delta x)^2}\Delta^2\mathfrak{r}_{i-1}(t)\Bigg)\Bigg] \\
&\leq -\sum_{i=1}^{N}\left(\alpha\mathfrak{m}_i^2(t) + \alpha\beta\mathfrak{r}_i^2(t) - \alpha\beta\left|\mathfrak{m}_i(t)\right|\left|\mathfrak{r}_i(t)\right|\right) \\
&\quad + \alpha\sum_{i=1}^{N}\left|u_i^2(t)v_i(t) - u^{*2}v^*\right|\left(|\mathfrak{m}_i(t)| + |\mathfrak{r}_i(t)|\right) \\
&\quad + \frac{1}{(\Delta x)^2}\sum_{i=1}^{N}\mathfrak{m}_i(t)\Delta^2\mathfrak{m}_{i-1}(t) + \frac{\zeta}{(\Delta x)^2}\sum_{i=1}^{N}\mathfrak{r}_i(t)\Delta^2\mathfrak{r}_{i-1}(t).
\end{aligned} \tag{23}$$

By applying Lemmas 3.2-3.4 and Theorem 3.1, we derive:

$$\begin{aligned}
\Delta_h\mathsf{V}(t) &\leq -\alpha\sum_{i=1}^{N}\left[\left(1 - 2L - \frac{\beta}{2}\right)\mathfrak{m}_i^2(t) + \left(\frac{\beta}{2} - 2L\right)\mathfrak{r}_i^2(t)\right] \\
&\quad - \frac{1}{(\Delta x)^2}\sum_{i=1}^{N}|\Delta\mathfrak{m}_i(t)|^2 - \frac{\zeta}{(\Delta x)^2}\sum_{i=1}^{N}|\Delta\mathfrak{r}_i(t)|^2 \\
&\leq -2\min\left\{\frac{\rho_1}{(\Delta x)^2} + \alpha\left(1 - 2L - \frac{\beta}{2}\right), \frac{\rho_2\zeta}{(\Delta x)^2} + \alpha\left(\frac{\beta}{2} - 2L\right)\right\}\mathsf{V}(t).
\end{aligned} \tag{24}$$

Assuming:

$$\Theta = 2\min\left\{\frac{\rho_1}{(\Delta x)^2} + \alpha\left(1 - 2L - \frac{\beta}{2}\right), \frac{\rho_2\zeta}{(\Delta x)^2} + \alpha\left(\frac{\beta}{2} - 2L\right)\right\} > 1,$$

We find:

$$\Delta_h \mathsf{V}(t) \leq -\mathsf{V}(t). \tag{25}$$

Given that $\mathsf{V}(t)$ is decreasing for $0 < t < t_{\max}$, it follows that $\mathsf{V}(0) > \mathsf{V}(t) > \mathsf{V}(t_{\max})$. Thus:

$$\Delta_h \mathsf{V}(t) \leq -\Theta\mathsf{V}(t) < -\Theta < -\Theta\mathsf{V}(t_{\max}) \to 0, \tag{26}$$

And consequently:

$$\Delta_h \mathsf{V}(t) \leq -\min\left\{\mathsf{V}(t), \Theta\right\}. \tag{27}$$

Therefore, by Theorem 3.3, the zero solution (u^*, v^*) of system (6) is FTS. Moreover, the settling time $\mathfrak{t}_\mathfrak{e}$ satisfies:

$$\mathfrak{t}_\mathfrak{e} \leq \left\lceil \frac{\mathsf{V}(0)}{\Theta} \right\rceil.$$

□

4 Numerical applications

In this part, we focus on the numerical application of the theoretical findings derived in earlier parts of this study. The analysis is centered on validating the FTS of the discrete SSRD system using numerical simulations.

The primary objectives include:

- Demonstrating the practical implementation of the discretized model, particularly through the FDM, which transforms the RD PDEs into a solvable discrete framework.
- Investigating the temporal and spatial dynamics of the system under various parameter settings and initial conditions to verify theoretical predictions.
- Highlighting the role of system parameters, boundary conditions, and numerical schemes in influencing stability, convergence rates, and transient behaviors.

The numerical experiments elucidate system behavior under controlled conditions, validating a computational framework for analyzing FTS phenomena. This bridges theory and practice, ensuring derived conditions align with observed dynamics.

4.1 Simulation

To study the spatial and temporal behaviors of the SS system and validate the derived theoretical findings, the subsequent methodology is utilized:

- The domain $x \in [0, 10]$ is partitioned into $\mathbb{N} = 115$ points with a spacing of $\Delta x = 0.086956521739130$. Using central difference approximations for the second-order spatial derivatives, the system is converted into algebraic equations as represented in Equation (5).

- ICs are defined as follows:

$$\begin{aligned} \psi_i &= 1.17 - 1.17 \sin\left(0.086956521739130\pi i\right), \\ \varphi_i &= 1 - \cos\left(0.086956521739130\pi i\right),\ i = 0, \ldots, 115, \end{aligned} \tag{28}$$

Ensuring physiological relevance. PBCs are also imposed for continuity across the spatial domain, as outlined in Equation (29).

$$(u_0(t), v_1(t)) = (u_1(t), v_1(t)), \tag{29}$$

- The key system parameters are chosen as:

$$\zeta = 1,\ \alpha = 1,\ p = 1,\ \beta = 0.01,\ q = 1.35275,\ \rho_1 = 0.75\, \rho_2 = 1,\ L = 4.898820650427711. \tag{30}$$

To maintain numerical stability and precision, the time step is set as $h = 0.008695652173913$. Temporal integration is performed using the explicit forward Euler method, updating the variables $u_i(t)$ and $v_i(t)$ iteratively as specified in Equation (6).

- FTS is assessed using the LF:

$$\mathsf{V}(t) = \frac{1}{2} \sum_{i=1}^{N} \left((u_i(t) - 2.35275)^2 + (v_i(t) - 0.2439394916003)^2 \right), \tag{31}$$

The initial values are calculated as:

$$\mathsf{V}(0) = 180.5627801461430. \tag{32}$$

The derivative of $\mathsf{V}(t)$ confirms stability by ensuring a negative $\Delta_h V(t)$, as required by Theorem 3.5, with:

$$\Theta = 180.7697173982892, \quad \text{settling time: } \mathfrak{t}_\epsilon \leq \lceil \mathsf{V}(0)/\Theta \rceil = 0.998855243814481\,\text{s}. \tag{33}$$

we obtain:

$$\mathsf{V}(0.998855243814481) = 0. \tag{34}$$

Figure 1 illustrates the Lyapunov function's progression over time.

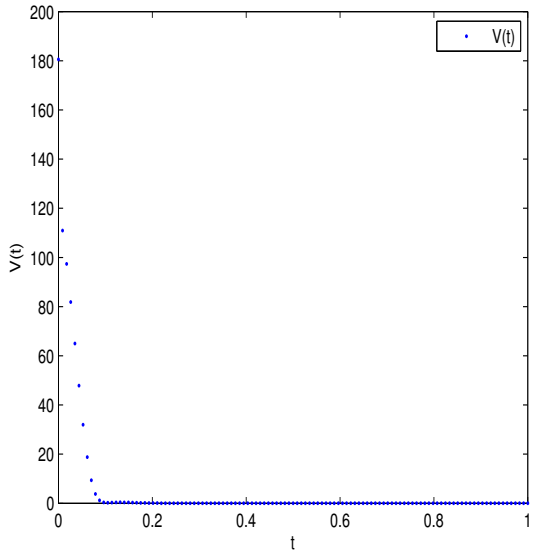

Figure 1: Lyapunov function $\mathsf{V}(t)$ evolution over time.

- Simulation results depict the evolution of $u_i(t)$ and $v_i(t)$, highlighting key patterns and transient behaviors. Figures 2 and 3 present 2D and 3D visualizations, offering insights into the system's equilibrium states and dynamic responses. Figure 1 displays the LF trends, validating stability criteria.

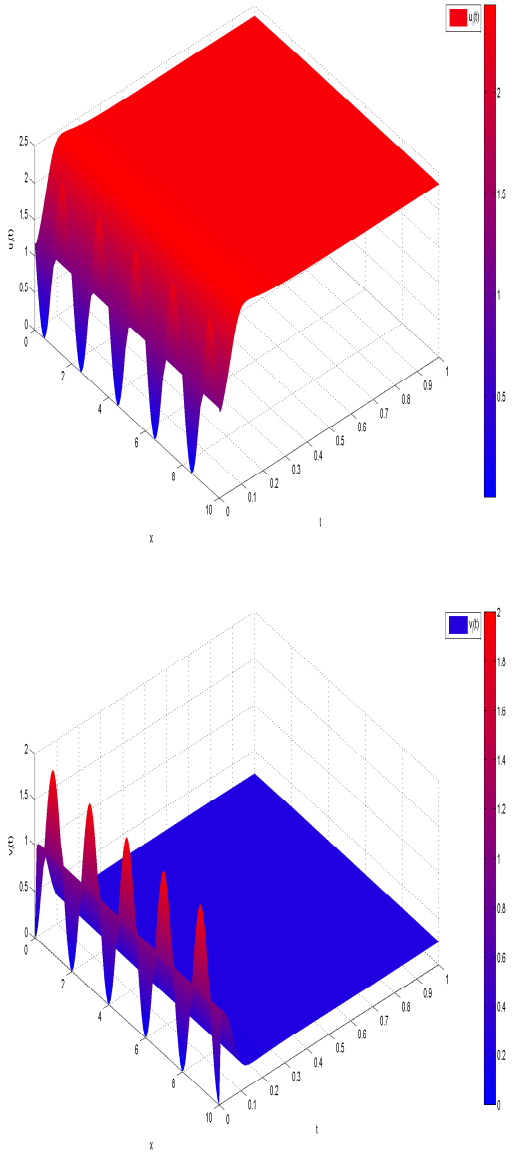

Figure 2: Spatiotemporal simulations for $u_i(t)$ and $v_i(t)$.

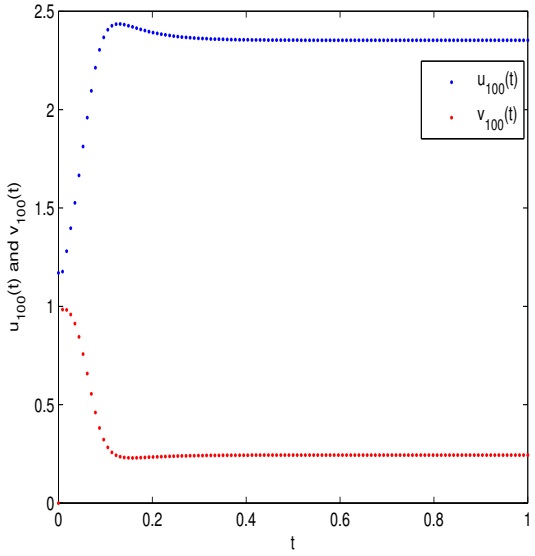

Figure 3: Evolution of variables at the boundary points $u_{115}(t)$ and $v_{115}(t)$.

4.2 Discussion

The numerical findings in Section validate the FTS analysis conducted on the discrete SSRD system. The results confirm that the derived conditions for FTS, based on LF and eigenvalue-based methodologies, are consistent with observed system behavior. Several key observations can be highlighted:

- The LF $\mathsf{V}(t)$ exhibits a strictly decreasing trend, ultimately converging to zero within a finite time, as predicted by the theoretical settling time estimates. This confirms that the proposed conditions are sufficient to ensure FTS for the SSRD system under the specified parameter settings and initial conditions.
- The FDM successfully transformed the continuous PDEs system into a discrete form that preserves the essential dynamics of the original SSRD system. The spatial and temporal discretization schemes ensured stability and accuracy, avoiding computational artifacts and maintaining the integrity of theoretical predictions.
- The simulations revealed the sensitivity of FTS to system parameters such as diffusion rates (ζ) and reaction coefficients (α, β, p, q). For instance, small perturbations in β or ζ significantly influenced the convergence and settling time rate, indicating potential avenues for optimizing these parameters in practical applications.
- The imposition of periodic PBCs demonstrated consistency across spatial domains, supporting the uniformity of system behavior and enhancing the validity of numerical simulations. This highlights the importance of carefully chosen boundary conditions in the numerical analysis of RD systems.
- The transient behaviors and equilibrium states observed in the simulations reflect the dynamics of real-world chemical and biological systems modeled by the SSRD framework. The study provides a computational basis for predicting finite-time phenomena in biochemical oscillations and spatial pattern formation scenarios.
- The proposed discrete framework and its performance align closely with existing numerical methods for RD systems, such as the Galerkin and finite element methods. However, the FDM's simplicity and computational efficiency offer a distinct advantage, particularly in scenarios requiring rapid iterations or large-scale simulations.

5 Conclusion

In this study, we investigated the FTS of the SSRD system by employing a discrete numerical framework. A FDM was developed to discretize the PDEs into a system of ODEs, enabling computational analysis. Theoretical insights were derived using Lyapunov-based methods and eigenvalue conditions, validated through numerical simulations.

The results confirmed the feasibility of achieving FTS in the SSRD system under specified conditions. Simulations showcased consistent convergence trends and dynamic behaviors reflective of real-world RD phenomena. The proposed framework's ability to preserve the system's intrinsic dynamics while addressing computational challenges highlighted its robustness.

Future work will focus on several key areas:

- Extending the current framework to include fractional derivatives, which could better capture memory and hereditary properties in complex systems.
- Exploring the robustness of FTS in systems with parameter uncertainties or external perturbations.
- Investigating alternative discretization schemes, such as spectral methods, to enhance computational efficiency and accuracy.
- Applying the proposed methodologies to practical biology, chemistry, and engineering systems, particularly those requiring rapid stabilization or pattern formation.
- Extending the FTS framework to explore finite-time synchronization in coupled RD systems.

This work contributes to the broader understanding of RD systems, offering a computationally efficient approach with implications for theoretical studies and practical applications in science and engineering.

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