A Study on Subsequent Expansions of the Operators $S(A)$, Algebraic Sum, and Geometric Product over IVTNFSS

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Abstract

In this work, subsequent expansions of the operators $S(A)$, algebraic sum and geometric product over IVTNFs. The first of $S(\alpha,\beta)$ is called the shrinking operator and the second, which is an extension of the first, is called (α,β) - shrink operator. The membership values & the values of non-membership be not completion our for all time possible, although in the branch of IVTNFS, it plays an additional significant character at this time, since the interval valued temporal neutrosophic fuzzy sets provides the best solution for finding the shortest distance in deciding one's career, judgment making and image processing and many more areas. Especially in medical diagnosis, when using this concept, there is a real chance that there will be a non-zero fraction of hesitation at any point in the assessment.

Keywords: Interval valued intuitionistic fuzzy sets; Neutrosophic sets; Interval neutrosophic fuzzy sets; Fuzzy sets

1. Introduction

Initially describe the notion of IFS-Intuitionistic fuzzy sets [1-3] through Krassimir T. Atanassov in 1986, which be an extension of fuzzy subsets and an excellent generalization of fuzzy sets. Because the inception of the IFS, numerous researchers contain exposed their concentration in theoretical concepts as well as applied in numerous areas such as recognition of pattern, machine education, similarity dealing out, decision-making, and more. Furthermore, the term IVIFS namely interval valued intuitionistic fuzzy sets [6] be earliest invited by Atanassov with Gargov in 1989, which be an excess of sweeping statement of both Intuitionistic Fuzzy Sets and Interval Valued Fuzzy Sets. The hesitant idea was used to interval valued intuitionistic hesitant fuzzy sets by Broumi and Smarandache in [4]. Chun-Hsiao Chu and associates. Presented two similarity metrics, created a strategy based on them, and demonstrated that our approach might resolve the pattern recognition issues in [5]. Parvathi and Geetha [7] covered several characteristics of temporal intuitionistic fuzzy sets.

In [8-9] Rajesh, the second type of Intuitionistic Fuzzy Sets with Intervals (IVIFSST) is a new extension of IVIFS and established some of their features. In this research work, a new shrinking operator, and the subsequent expansion of the operator $S(\alpha,\beta)(A)$ are introduced and several of its properties are established. The values of member and the values of non-member not fulfillment certain situations, but in the IVNFS satisfied all the situation, it plays a most significant responsibility at this juncture, since the second type of interval of Neutrosophic Sets provides the most excellent result for finding the smallest distance in decision making, career choice making, and many more. Xu and Zeshui [10] made decisions on interval-valued intuitionistic fuzzy sets by using an aggregating operator. Kaviyarasu et al. recently presented connection indices and a few of its derivatives in graph

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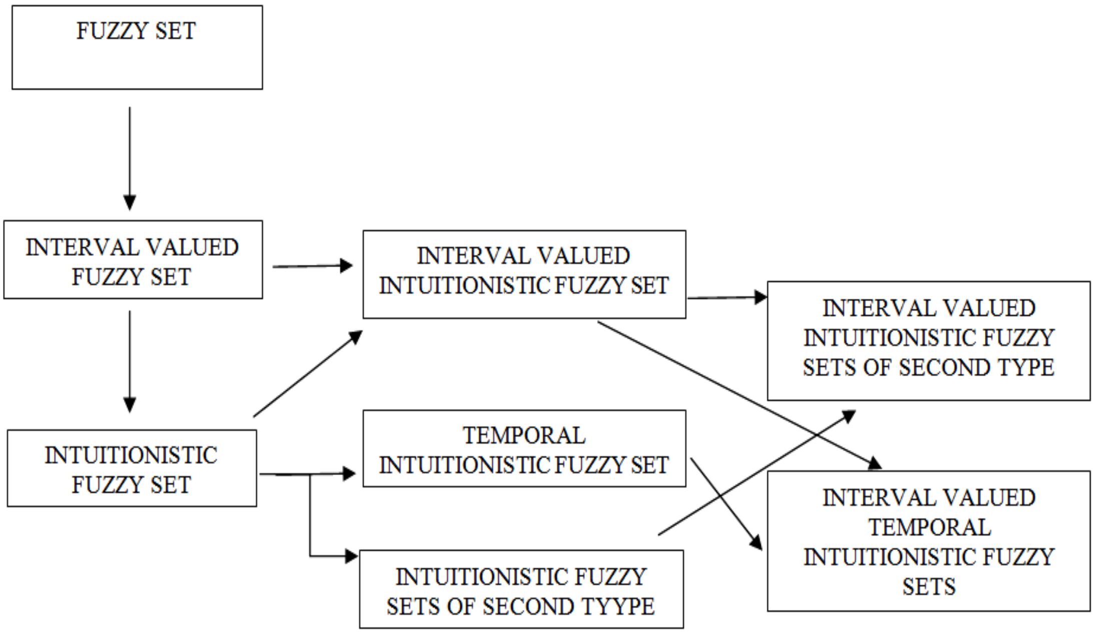
theory for neutrosophic fuzzy sets [11-12]. In this research paper will be prepared as follow: in the 2nd section, some basic definitions are presented, in the 3rd Section, we present the shrinking operator & the subsequent expansion of S(A) operator on the temporal neutrosophic fuzzy sets with intervals and explore a few of their basic properties. Section 4 concludes this document.



2. Preliminary

Definition 2.1[1] Let K be nonempty set. As an extension of fuzzy sets, an Intuitionistic Fuzzy Sets (IFS) E within K is an object of the following form,

$$E = \{\langle s, \alpha_{\mathrm{E}}(s), \beta_{\mathrm{E}}(s)\rangle | \ s \in K\},$$

where the elements are denoted by the degrees of membership $\alpha_{\mathrm{E}}: K \to [0,1]$, and non-membership β_{E} such that $\beta_{\mathrm{E}}: \mathrm{K} \to [0,1]$. Moreover, $0 \le \alpha_{\mathrm{E}}(s) + \beta_{\mathrm{E}}(s) \le 1$, for every component $s \in K$.

Definition 2.2[7] Let K be nonempty set. The second type of Intuitionistic Fuzzy Sets (IFSST) E within K is an object of the following form,

$$E = \{\langle s, \alpha_{\mathrm{E}}(s), \beta_{\mathrm{E}}(s)\rangle | \ s \in K\},$$

where the elements are denoted by the degrees of membership $\alpha_{\mathrm{E}}: K \to [0,1]$, and non-membership β_{E} where $\beta_{\mathrm{E}}: \mathrm{K} \to [0,1]$. Moreover, $0 \le \alpha^2{}_{\mathrm{E}}(s) + \beta^2{}_{\mathrm{E}}(s) \le 1$, for every component $s \in K$.

Definition 2.3[6] An Interval Valued Intuitionistic Fuzzy Sets (IVIFS), the source E within K can be defined at the same time as

$$E = \{\langle s, M_E(s), N_E(s)\rangle | \ s \in K\},$$

wherever M_E: K → [0, 1] and N_E: K → [0, 1]. The intervals of $M_E(s)$and $N_E(s)$indicate as the quantity of membership moreover the grade of non-membership of an part s inside K, everywhere the member is denote the same as $M_E(s) = [M_{EL}(s), M_{EU}(s)]$ and non member denote as $N_E(s) = [N_{EL}(s), N_{EU}(s)]$ provided that

$$M_{EU}(s) + N_{EU}(s) \le 1 \ \forall \ s \in K$$

Definition 2.4 [6] For two IVIFSs E and F we define the subsequent associations & operation as:

i. $E \cup F = \{\langle \mathrm{s}, [\sup(\mathrm{M_{EL}}(s), \ \mathrm{M_{FL}}(s)), \sup(\mathrm{M_{EU}}(s), \ \mathrm{M_{FU}}(s))],$

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$$[\inf(\mathrm{N_{EL}}(s),\ \mathrm{N_{FL}}(s)),\inf(\mathrm{N_{EU}}(s),\ \mathrm{N_{FU}}(s))]\rangle | s \in \mathrm{K}\}$$

ii. $E \cap F = \{\langle \mathrm{s}, [\inf(\mathrm{M_{EL}}(s),\ \mathrm{M_{FL}}(s)),\inf(\mathrm{M_{EU}}(s),\ \mathrm{M_{FU}}(s))],$

$$[\sup(\mathrm{N_{EL}}(s),\ \mathrm{N_{FL}}(s)),\sup(\mathrm{N_{EU}}(s),\ \mathrm{N_{FU}}(s))]\rangle | s \in \mathrm{K}\}$$

iii. $\bar{E} = \{\langle s, [\mathrm{N_{EL}}(s), \mathrm{N_{EU}}(s)], [\mathrm{M_{EL}}(s), \mathrm{M_{EU}}(s)]\rangle |\ s \in K\}$.

Definition 2.5 [4] For every E in IVIFS we have the following information:

Operator of Necessity

$$\square \mathrm{E} = \{\langle \mathrm{s}, [\mathrm{M}_{EL}(s), \mathrm{M}_{EU}(s)], \left[\mathrm{N_{EL}}(s), \sqrt{1-\mathrm{M}_{EU}(s)}\right]\rangle |\ s \in K\}.$$

Operator of Possibility

$$\lozenge\ E = \{\langle \mathrm{s}, \left[\mathrm{M}_{EL}(s), \sqrt{1-\mathrm{N}_{EU}(s)}\right], [\mathrm{N_{EL}}(s), \mathrm{N_{EU}}(s)]\rangle |\ s \in K\}.$$

Definition 2.6[3] Let K be nonempty set. For each IVIFS E, then we define the most basic shrinking operator in the subsequent approach:

$$S(E) = \left\{\left\langle s, \left[\frac{\mathrm{M}_{EL}(s)+\mathrm{M}_{EU}(s)}{2}, \frac{\mathrm{M}_{EL}(s)+\mathrm{M}_{EU}(s)}{2}\right], \left[\frac{\mathrm{N}_{EL}(s)+\mathrm{N}_{EU}(s)}{2}, \frac{\mathrm{N}_{EL}(s)+\mathrm{N}_{EU}(s)}{2}\right]\right\rangle | s \in K\right\}.$$

Definition 2.7[3] for each IVIFS E and, let K be nonempty set. Then, the shrinking operator is defined as the following manner

$$S_{\alpha,\beta}(E) = \{\langle s, [\alpha(M_{EL}(s)+M_{EU}(s)), \alpha(M_{EL}(s)+M_{EU}(s))],$$

$$[\beta(N_{EL}(s)+N_{EU}(s)), \beta(N_{EL}(s)+N_{EU}(s))]\rangle | s \in K\}.$$

Definition 2.8 [9] Let K be nonempty set. The Interval Valued Temporal Neutrosophic Fuzzy Sets (IVTNFS) E in G is defined in the subsequent form

$$E = \{<(s,t), [\mu_{AL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$$

$$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\},$$

where the functions $\mu_E(s,t), T_E(s,t), I_E(s,t) \& F_E(s,t) : K \to]0^-, 1^+[$ and

$0 \le \mu_{AU}(s,t) + T_{AU}(s,t) + I_{AU}(s,t) + F_{AU}(s,t) \le 3$.

Definition 2.9 [9] Let E be a nonempty set and A is defined over Interval Valued Temporal Neutrosophic Fuzzy Sets. The Necessity operator is defined in the following form

$$\square \mathrm{E} = \{<(s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$$

$$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), 1-T_{EU}(s,t)] > |(s,t) \in K \times T\}.$$

Example 2.10 Consider an interval temporal neutrosophic fuzzy set A on G specified by

E= $\{< [a, 0.3, 0.4], [0.5, 0.3], [0.4, 0.5], [0.2.0.3] >, < [b, 0.4, 0.6], [0.5, 0.6], [0.1, 0.4], [0.4, 0.7] >\}$.

Then,

$$\square E = \{< [a, 0.3, 0.4], [0.5, 0.3], [0.4, 0.5], [0.2.0.7] >, < [b, 0.4, 0.6], [0.5, 0.6], [0.1, 0.4], [0.4, 0.3] >\}$$

Definition 2.11[9] Let G be a nonempty set, and A is defined over interval valued temporal neutrosophic fuzzy sets then, possibility operator is defined in the following form

$$\lozenge\ E = \{<(s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), 1-T_{EU}(s,t)],$$

$$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\}.$$

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Example 2.12 Consider an interval valued temporal neutrosophic fuzzy set A on G specified by

$$E = \{\langle [a, 0.3, 0.5], [0.4, 0.2], [0.4, 0.5], [0.2.0.6]\rangle, \langle [b, 0.3, 0.4], [0.3, 0.6], [0.1, 0.8], [0.5, 0.4]\rangle\}$$

Then,

$\lozenge\, E = \{\langle [a, 0.3, 0.5], [0.4, 0.8], [0.4, 0.5], [0.2.0.6]\rangle, \langle [b, 0.3, 0.4], [0.3, 0.4], [0.1, 0.8], [0.5, 0.4]\rangle\}$.

3. Subsequent Expansions of the Operator S(A), Algebraic Sum and Geometric Product Over IVTNFSS

In this phase, the subsequent expansions of the operator $S(A)$, algebraic sum and geometric product over IVTNFSs and establish a few of their property.

Definition 3.1 Assume that K be nonempty set, and for each IVTNFS E, define the simplest shrinking operator in the following manner.

$$S(E) = \left\{\left\langle (s,t), \left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}\right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}\right]\right.\right.$$
$$\left.\left.\left[\frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}, \frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}\right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}\right]\right\rangle \middle| s \in K \,\&\, t \in T\right\}.$$

For each IVTNFS A, and nonempty set K, we define the shrinking operator in the following manner.

Definition 3.2 For every IVTNFS E and $\alpha, \beta \in [0, 0.5]$. Then we define (α, β) - shrinking operator in the following manner

$$S_{\alpha,\beta}(E) = \{\langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)],$$
$$[\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)],$$
$$[\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))]\rangle | s \in K \,\&\, t \in T\}$$

Definition 3.3 Assume that K is nonempty set, & E and F are in IVTNFSs. We define the operator known as algebraic sum in the following manner:

$$E \oplus F = \{\langle (s,t), [\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EU}(s,t).\mu_{FU}(s,t)],$$
$$\begin{bmatrix} T_{EL}(s,t) + T_{FL}(s,t) - T_{EL}(s,t).T_{FL}(s,t), \\ T_{EU}(s,t) + T_{FU}(s,t) - T_{EU}(s,t).T_{FU}(s,t) \end{bmatrix}, [I_{EL}(s,t).I_{FL}(s,t), I_{EU}(s,t).I_{FU}(s,t)],$$
$$[F_{EL}(s,t).F_{FL}(s,t), F_{EU}(s,t).F_{FU}(s,t)]\rangle | s \in K \,\&\, t \in T\}.$$

Definition 3.4 Assume that K is nonempty set, & E and F are in IVTNFSs. We define the operator for the geometric product in the following manner

$$E \otimes F = \left\{\langle (s,t), \begin{bmatrix} \mu_{EL}(s,t) + \mu_{FL}(s,t) - \mu_{EL}(s,t).\mu_{FL}(s,t), \\ \mu_{EU}(s,t) + \mu_{FU}(s,t) - \mu_{EU}(s,t).\mu_{FU}(s,t) \end{bmatrix},\right.$$
$$[T_{EL}(s,t).T_{FL}(s,t), T_{EU}(s,t).T_{FU}(s,t)], \begin{bmatrix} I_{EL}(s,t) + I_{FL}(s,t) - I_{EL}(s,t).I_{FL}(s,t), \\ I_{EU}(s,t) + I_{FU}(s,t) - I_{EU}(s,t).I_{FU}(s,t) \end{bmatrix},$$
$$\left.\begin{bmatrix} F_{EL}(s,t) + F_{FL}(s,t) - F_{EL}(s,t).F_{FL}(s,t), \\ F_{EU}(s,t) + F_{FU}(s,t) - F_{EU}(s,t).F_{FU}(s,t) \end{bmatrix}\rangle | s \in K \,\&\, t \in T\right\}. \quad \Leftarrow$$

Example 3.4 Assume that $E = \{\langle [0.6, 0.2], [0.7, 0.3], [0.4, 0.5], [0.4, 0.6]\rangle, \langle [0.5, 0.2], [0.5, 0.3],$ $[0.5, 0.5], [0.5, 0.4]\rangle\}$ and $F = \{\langle [0.6, 0.1], [0.5, 0.4], [0.3, 0.7], [0.5, 0.4]\rangle, \langle [0.7, 0.3], [0.6, 0.3],$ $[0.6, 0.3], [0.5, 0.4]\rangle\}$ are two IVTNFSs. Then we have

$E \oplus F = \{\langle [0.36, 0.02], [0.85, 0.68], [0.12, 0.35], [0.2, 0.24]\rangle,$

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$\langle[0.35, 0.6], [0.8, 0.51], [0.3, 0.15], [0.25, 0.16]\rangle\}$.

$E \otimes F = \{\langle[0.84, 0.28], [0.35, 0.12], [0.58,0.85], [0.7, 0.76]\rangle,$

$\langle[0.85, 0.44], [0.3, 0.09], [0.8, 0.65], [0.75, 0.64]\rangle\}$.

Theorem 3.1 ***Let K be a non - unfilled set and we have the following for each IVTNFS E,***

i. $\boldsymbol{S(S(E)) = S(E)}$, **ii)** $\boldsymbol{\sim S(\sim E) = S(E)}$.

Proof: (i). Let

$$E = \{ < (s,t), [\mu_{AL}(s,t)\,, \mu_{EU}(s,t)]\,, [T_{EL}(s,t)\,, T_{EU}(s,t)], [I_{EL}(s,t)\,, I_{EU}(s,t)]\,, [F_{EL}(s,t)\,, F_{EU}(s,t)] > |(s,t) \in K \times T\}.$$

Then

$$S(E) = \left\{\left\langle\left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}\right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}\right] \left[\frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}, \frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}\right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}\right]\right\rangle |(s,t) \in K \times T\right\}.$$

Applying the shrinking operator we have

$$= S\left\{\left\langle\left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}\right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}\right] \left[\frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}, \frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}\right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}\right]\right\rangle |s \in K \,\&\, t \in T\right\}$$

$$= \left\{\left\langle\left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}\right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}\right] \left[\frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}, \frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}\right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}\right]\right\rangle |s \in K \,\&\, t \in T\right\}$$

$= S(E)$.

(ii). Let $E = \{ < (s,t), [\mu_{AL}(s,t)\,, \mu_{EU}(s,t)]\,, [T_{EL}(s,t)\,, T_{EU}(s,t)], [I_{EL}(s,t)\,, I_{EU}(s,t)]\,, [F_{EL}(s,t)\,, F_{EU}(s,t)] > |(s,t) \in K \times T\}$.

Then

$$\sim E = \{ < (s,t), [\mu_{AL}(s,t)\,, \mu_{EU}(s,t)]\,, [F_{EL}(s,t)\,, F_{EU}(s,t)]\,, [1 - I_{EL}(s,t)\,, 1 - I_{EU}(s,t)], [T_{EL}(s,t)\,, T_{EU}(s,t)] > |(s,t) \in K \times T\}.$$

Applying the shrinking operator we have

$$= \left\{\left\langle\left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}\right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}\right] \left[\frac{1 - I_{EL}(s,t) + 1 - I_{EU}(s,t)}{2}, \frac{1 - I_{EL}(s,t) + 1 - I_{EU}(s,t)}{2}\right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}\right]\right\rangle |s \in K \,\&\, t \in T\right\}$$

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$$= \left\{ \left\langle \left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2} \right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2} \right], \left[\frac{I_{EU}(s,t) + I_{EL}(s,t)}{2}, \frac{I_{EU}(s,t) + I_{EL}(s,t)}{2} \right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2} \right] \right\rangle | s \in K \,\&\, t \in T \right\}$$

Applying the complement operator we have

$$= \sim \left\{ \left\langle \left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2} \right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2} \right], \left[\frac{I_{EU}(s,t) + I_{EL}(s,t)}{2}, \frac{I_{EU}(s,t) + I_{EL}(s,t)}{2} \right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2} \right] \right\rangle | s \in K \,\&\, t \in T \right\}$$

$$= \left\{ \left\langle \left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2} \right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2} \right], \left[1 - \frac{I_{EU}(s,t) + I_{EL}(s,t)}{2}, 1 - \frac{I_{EU}(s,t) + I_{EL}(s,t)}{2} \right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2} \right] \right\rangle | s \in K \,\&\, t \in T \right\}$$

$$= \left\{ \left\langle \left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2} \right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2} \right] \left[\frac{I_{EL}(s,t) + I_{EU}(s,t)}{2}, \frac{I_{EL}(s,t) + I_{EU}(s,t)}{2} \right], \left[\frac{F_{EL}(s,t) + F_{EU}(s,t)}{2}, \frac{F_{EL}(s,t) + F_{EU}(s,t)}{2} \right] \right\rangle | s \in K \,\&\, t \in T \right\}$$

$= S(E)$.

This completes the proof of the theorem.

Theorem 3.2 if K is a nonempty set, and then we can write the following for each IVTNFS E.

i. $\boldsymbol{S(\Box E) = S(E)}$,

ii. $\boldsymbol{S(\Diamond\, E) = S(E)}$.

Proof: (i). Let $E = \{ < (s,t), [\mu_{AL}(s,t)\, , \mu_{EU}(s,t)]\, , [T_{EL}(s,t)\, , T_{EU}(s,t)],$
$[I_{EL}(s,t)\, , I_{EU}(s,t)]\, , [F_{EL}(s,t)\, , F_{EU}(s,t)] > |(s,t) \in K \times T\}$.

Then

$\Box E = \{ < (s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$
$[I_{EL}(s,t), I_{EU}(s,t)]\, , [F_{EL}(s,t), 1 - T_{EU}(s,t)] > |(s,t) \in K \times T\}$

$\Diamond\, E = \{ < (s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), 1 - T_{EU}(s,t)],$
$[I_{EL}(s,t), I_{EU}(s,t)]\, , [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\}$,

and

$$S(E) = \left\{ \left\langle \left[\frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2}, \frac{\mu_{EL}(s,t) + \mu_{EU}(s,t)}{2} \right], \left[\frac{T_{EL}(s,t) + T_{EU}(s,t)}{2}, \frac{T_{EL}(s,t) + T_{EU}(s,t)}{2} \right] \right.\right.$$

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$$\left.\begin{matrix}\left[\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2},\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2}\right],\\ \left[\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2},\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2}\right]\end{matrix}\right\rangle |s\in K\ \&\ t\in T\Bigg\}.$$

Applying the shrinking operator over $\square E$

$S(\square E)=S\{<(s,t),[\mu_{EL}(s,t),\mu_{EU}(s,t)],[T_{EL}(s,t),T_{EU}(s,t)],$

$[I_{EL}(s,t),I_{EU}(s,t)],[F_{EL}(s,t),1-T_{EU}(s,t)]>|(s,t)\in K\times T\}$

$$=\Bigg\{\Bigg\langle\left[\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2},\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2}\right],\left[\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2},\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2}\right]$$

$$\left.\begin{matrix}\left[\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2},\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2}\right],\\ \left[\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2},1-\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2}\right]\end{matrix}\right\rangle |s\in K\ \&\ t\in T\Bigg\}$$

$$=\Bigg\{\Bigg\langle\left[\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2},\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2}\right],\left[\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2},\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2}\right]$$

$$\left.\begin{matrix}\left[\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2},\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2}\right],\\ \left[\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2},\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2}\right]\end{matrix}\right\rangle |s\in K\ \&\ t\in T\Bigg\}$$

Therefore, $S(\square A)=S(A)$.

(ii). Applying the shrinking operator over possibility operator

$S(\lozenge E)=S\{<(s,t),[\mu_{EL}(s,t),\mu_{EU}(s,t)],[T_{EL}(s,t),1-F_{EU}(s,t)],$

$[I_{EL}(s,t),I_{EU}(s,t)],[F_{EL}(s,t),F_{EU}(s,t)]>|(s,t)\in K\times T\}$

$$=\Bigg\{\Bigg\langle\left[\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2},\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2}\right],\left[\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2},1-\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2}\right]$$

$$\left.\begin{matrix}\left[\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2},\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2}\right],\\ \left[\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2},\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2}\right]\end{matrix}\right\rangle |s\in K\ \&\ t\in T\Bigg\}$$

$$=\Bigg\{\Bigg\langle\left[\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2},\frac{\mu_{EL}(s,t)+\mu_{EU}(s,t)}{2}\right],\left[\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2},\frac{\mathrm{T}_{EL}(s,t)+\mathrm{T}_{EU}(s,t)}{2}\right]$$

$$\left.\begin{matrix}\left[\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2},\frac{\mathrm{I}_{EL}(s,t)+\mathrm{I}_{EU}(s,t)}{2}\right],\\ \left[\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2},\frac{\mathrm{F}_{EL}(s,t)+\mathrm{F}_{EU}(s,t)}{2}\right]\end{matrix}\right\rangle |s\in K\ \&\ t\in T\Bigg\}$$

This implies that $S(\lozenge E)=S(E)$. Hence, the proof of the theorem is completed.

Theorem 3.3 Let K be a nonempty set. For each IVTNFS E and for every α, β from $[0.5, 0.5]$ we have:

(i) $\boldsymbol{S_{\alpha,\beta}(S_{\alpha,\beta}(E))=S_{\alpha^2,\beta^2}(E)}$,

(ii) $\boldsymbol{\sim S_{\alpha,\beta}(\sim E)=S(E)}$.

Proof: (i). Let $E=\{<(s,t),[\mu_{AL}(s,t),\mu_{EU}(s,t)],[T_{EL}(s,t),T_{EU}(s,t)],$

$[I_{EL}(s,t),I_{EU}(s,t)],[F_{EL}(s,t),F_{EU}(s,t)]>|(s,t)\in K\times T\}.$

Then the extension of shrinking operator can be defined in the following form

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$$S_{\alpha,\beta}(E) = \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}.$$

Applying the shrinking operator we have

$$S_{\alpha,\beta}\left(S_{\alpha,\beta}(E)\right) = S_{\alpha,\beta} \left\{ \begin{array}{l} \langle s, [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ \alpha[\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ \beta[\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha^2(T_{EL}(s,t) + T_{EU}(s,t)), \alpha^2(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta^2(F_{EL}(s,t) + F_{EU}(s,t)), \beta^2(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha^2 T_{EL}(s,t) + \alpha^2 T_{EU}(s,t), \alpha^2 T_{EL}(s,t) + \alpha^2 T_{EU}(s,t)], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta^2(F_{EL}(s,t) + F_{EU}(s,t)), \beta^2(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

$$= S_{\alpha^2,\beta^2}(E).$$

(ii). $S_{\alpha,\beta}(A) = \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

Using the complement operator over shrinking operator we have:

$$S_{\alpha,\beta}(\sim E) = \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}.$$

Again, applying the complement operator we have:

$$\sim S_{\alpha,\beta}(\sim E) = \left\{ \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \begin{bmatrix} 1 - \beta(T_{EL}(s,t) + T_{EU}(s,t)), \\ 1 - \beta(T_{EL}(s,t) + T_{EU}(s,t)) \end{bmatrix}, \right.$$

$$\left. [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \begin{bmatrix} 1 - \alpha(F_{EL}(s,t) + F_{EU}(s,t)), \\ 1 - \alpha(F_{EL}(s,t) + F_{EU}(s,t)) \end{bmatrix} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Big\}$$

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$= S_{\beta,\alpha}(A)$.

That completes the proof of the theorem.

Theorem 3.4 Assume that K is a nonempty set. For each IVTNFS E and for every $\alpha, \beta \in [0.5, 0.5]$ we have the following:

i. $S_{\alpha,\beta}(\square E) = S_{\alpha,\beta}(E)$,

ii. $S_{\alpha,\beta}(\lozenge E) = S_{\alpha,\beta}(E)$.

Proof: (i). Let $E = \{ < (s,t), [\mu_{AL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$

$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\}$.

Then,

$\square E = \{ < (s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$

$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), 1 - T_{EU}(s,t)] > |(s,t) \in K \times T\}$,

$\lozenge E = \{ < (s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), 1 - T_{EU}(s,t)],$

$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\}$,

and

$$S_{\alpha,\beta}(E) = \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Bigg\}.$$

Applying the extension operator over $\square E$

$S_{\alpha,\beta}(\square E) = S_{\alpha,\beta}\{\langle (s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$

$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), 1 - T_{EU}(s,t)] > |(s,t) \in K \times T\}$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + 1 - T_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Bigg\}$$

$$= \left\{ \begin{array}{l} \langle x, [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Bigg\}$$

$= S_{\alpha,\beta}(E)$.

Therefore, $S_{\alpha,\beta}(\square E) = S_{\alpha,\beta}(E)$.

(ii). Applying the extension operator over $\lozenge$ E we have

$S_{\alpha,\beta}(\lozenge E) = S_{\alpha,\beta}\{ < (s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), 1 - T_{EU}(s,t)],$

$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\}$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + 1 - T_{EL}(s,t)), \alpha(T_{EL}(s,t) + 1 - T_{EL}(s,t))], \end{array} \right.$$

$$\left. \begin{array}{r} [I_{EL}(s,t) + I_{EU}(s,t), I_{EL}(s,t) + I_{EU}(s,t)], \\ [\beta(F_{EL}(s,t) + F_{EU}(s,t)), \beta(F_{EL}(s,t) + F_{EU}(s,t))] \end{array} \right\rangle |s \in K \,\&\, t \in T \Bigg\}$$

$$= \left\{ \begin{array}{l} \langle (s,t), [\mu_{EL}(s,t) + \mu_{EU}(s,t), \mu_{EL}(s,t) + \mu_{EU}(s,t)], \\ [\alpha(T_{EL}(s,t) + T_{EU}(s,t)), \alpha(T_{EL}(s,t) + T_{EU}(s,t))], \end{array} \right.$$

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$$\left.\begin{matrix}[I_{EL}(s,t)+I_{EU}(s,t), I_{EL}(s,t)+I_{EU}(s,t)],\\ [\beta(F_{EL}(s,t)+F_{EU}(s,t)), \beta(F_{EL}(s,t)+F_{EU}(s,t))]\end{matrix}\right\rangle |s\in K\ \&\ t\in T\bigg\}$$

$= S_{\alpha,\beta}(E)$.

Therefore, $S_{\alpha,\beta}(\lozenge E) = S_{\alpha,\beta}(E)$.

Theorem 3.5 If K is a nonempty set, then for each IVTNFS E and F we have:

i. $(\mathbf{E} \oplus \mathbf{F}) \cup (\mathbf{E} \otimes \mathbf{F}) = \mathbf{E} \oplus \mathbf{F}$,
ii. $(\mathbf{E} \oplus \mathbf{F}) \cap (\mathbf{E} \otimes \mathbf{F}) = \mathbf{E} \otimes \mathbf{F}$.

Proof: (i). Assume that K is a nonempty set & E and F are in IVTNFSs. Then,

$E = \{<(s,t), [\mu_{EL}(s,t), \mu_{EU}(s,t)], [T_{EL}(s,t), T_{EU}(s,t)],$

$[I_{EL}(s,t), I_{EU}(s,t)], [F_{EL}(s,t), F_{EU}(s,t)] > |(s,t) \in K \times T\}$,

and $F = \{<(s,t), [\mu_{FL}(s,t), \mu_{FU}(s,t)], [T_{FL}(s,t), T_{FU}(s,t)],$

$[I_{FL}(s,t), I_{FU}(s,t)], [F_{FL}(s,t), F_{FU}(s,t)] > |(s,t) \in K \times T\}$.

Hence, the algebraic sum and geometric product can be defined as

$E \oplus F = \{\langle (s,t), [\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EU}(s,t).\mu_{FU}(s,t)],$

$\begin{bmatrix} T_{EL}(s,t)+T_{FL}(s,t)-T_{EL}(s,t).T_{FL}(s,t),\\ T_{EU}(s,t)+T_{FU}(s,t)-T_{EU}(s,t).T_{FU}(s,t)\end{bmatrix}, [I_{EL}(s,t).I_{FL}(s,t), I_{EU}(s,t).I_{FU}(s,t)],$

$[F_{EL}(s,t).F_{FL}(s,t), F_{EU}(s,t).F_{FU}(s,t)]\rangle |s\in K\ \&\ t\in T\}$,

And $E \otimes F = \left\{\langle (s,t), \begin{bmatrix}\mu_{EL}(s,t)+\mu_{FL}(s,t)-\mu_{EL}(s,t).\mu_{FL}(s,t),\\ \mu_{EU}(s,t)+\mu_{FU}(s,t)-\mu_{EU}(s,t).\mu_{FU}(s,t)\end{bmatrix},\right.$

$[T_{EL}(s,t).T_{FL}(s,t), T_{EU}(s,t).T_{FU}(s,t)], \begin{bmatrix} I_{EL}(s,t)+I_{FL}(s,t)-I_{EL}(s,t).I_{FL}(s,t),\\ I_{EU}(s,t)+I_{FU}(s,t)-I_{EU}(s,t).I_{FU}(s,t)\end{bmatrix},$

$\begin{bmatrix} F_{EL}(s,t)+F_{FL}(s,t)-F_{EL}(s,t).F_{FL}(s,t),\\ F_{EU}(s,t)+F_{FU}(s,t)-F_{EU}(s,t).F_{FU}(s,t)\end{bmatrix}\rangle |s\in K\ \&\ t\in T\}$

Taking the union over the geometric product and algebraic sum

$(\mathrm{E} \oplus F) \cup (\mathrm{E} \otimes \mathrm{F}) = \{\langle (s,t), [\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EU}(s,t).\mu_{FU}(s,t)],$

$\begin{bmatrix} T_{EL}(s,t)+T_{FL}(s,t)-T_{EL}(s,t).T_{FL}(s,t),\\ T_{EU}(s,t)+T_{FU}(s,t)-T_{EU}(s,t).T_{FU}(s,t)\end{bmatrix}, [I_{EL}(s,t).I_{FL}(s,t), I_{EU}(s,t).I_{FU}(s,t)],$

$[F_{EL}(s,t).F_{FL}(s,t), F_{EU}(s,t).F_{FU}(s,t)]\} \cup \left\{\langle (s,t), \begin{bmatrix}\mu_{EL}(s,t)+\mu_{FL}(s,t)-\mu_{EL}(s,t).\mu_{FL}(s,t),\\ \mu_{EU}(s,t)+\mu_{FU}(s,t)-\mu_{EU}(s,t).\mu_{FU}(s,t)\end{bmatrix},\right.$

$[T_{EL}(s,t).T_{FL}(s,t), T_{EU}(s,t).T_{FU}(s,t)], \begin{bmatrix} I_{EL}(s,t)+I_{FL}(s,t)-I_{EL}(s,t).I_{FL}(s,t),\\ I_{EU}(s,t)+I_{FU}(s,t)-I_{EU}(s,t).I_{FU}(s,t)\end{bmatrix},$

$\begin{bmatrix} F_{EL}(s,t)+F_{FL}(s,t)-F_{EL}(s,t).F_{FL}(s,t),\\ F_{EU}(s,t)+F_{FU}(s,t)-F_{EU}(s,t).F_{FU}(s,t)\end{bmatrix}\rangle |s\in K\ \&\ t\in T\}$

$= \{[\min(\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EL}(s,t)+\mu_{FL}(s,t)-\mu_{EL}(s,t).\mu_{FL}(s,t)),$

$\min(\mu_{EU}(s,t).\mu_{FU}(s,t), \mu_{EU}(s,t)+\mu_{FU}(s,t)-\mu_{EU}(s,t).\mu_{FU}(s,t))],$

$[\max(T_{EL}(s,t)+T_{FL}(s,t)-T_{EL}(s,t).T_{FL}(s,t), T_{EL}(s,t).T_{FL}(s,t)),$

$\max(T_{EU}(s,t)+T_{FU}(s,t)-T_{EU}(s,t).T_{FU}(s,t), T_{EU}(s,t).T_{FU}(s,t))],$

$[\min(I_{EL}(s,t).I_{FL}(s,t), I_{EL}(s,t)+I_{FL}(s,t)-I_{EL}(s,t).I_{FL}(s,t)),$

$\min(I_{EU}(s,t).I_{FU}(s,t), I_{EU}(s,t)+I_{FU}(s,t)-I_{EU}(s,t).I_{FU}(s,t))],$

$[\min(F_{EL}(s,t).F_{FL}(s,t), F_{EL}(s,t)+F_{FL}(s,t)-F_{EL}(s,t).F_{FL}(s,t)),$

$\min(F_{EU}(s,t).F_{FU}(s,t), F_{EU}(s,t)+F_{FU}(s,t)-F_{EU}(s,t).F_{FU}(s,t))]\rangle |s\in K\ \&\ t\in T\}$

$= \{\langle (s,t), [\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EU}(s,t).\mu_{FU}(s,t)],$

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$$\begin{bmatrix} T_{EL}(s,t) + T_{FL}(s,t) - T_{EL}(s,t).T_{FL}(s,t), \\ T_{EU}(s,t) + T_{FU}(s,t) - T_{EU}(s,t).T_{FU}(s,t) \end{bmatrix}, [I_{EL}(s,t).I_{FL}(s,t), I_{EU}(s,t).I_{FU}(s,t)],$$

$$[F_{EL}(s,t).F_{FL}(s,t), F_{EU}(s,t).F_{FU}(s,t)]\rangle | s \in K \,\&\, t \in T$$

$= E \oplus$ F. Therefore $(\mathrm{E} \oplus \mathrm{F}) \cup (\mathrm{E} \otimes \mathrm{F}) = \mathrm{E} \oplus \mathrm{F}$.

This completes the proof for part (i) of the theorem.

(ii) Taking the intersection over the geometric product and algebraic sum, we have

$$(\mathrm{E} \oplus F) \cap (\mathrm{E} \otimes \mathrm{F}) = \{\langle (s,t), [\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EU}(s,t).\mu_{FU}(s,t)],$$

$$\begin{bmatrix} T_{EL}(s,t) + T_{FL}(s,t) - T_{EL}(s,t).T_{FL}(s,t), \\ T_{EU}(s,t) + T_{FU}(s,t) - T_{EU}(s,t).T_{FU}(s,t) \end{bmatrix}, [I_{EL}(s,t).I_{FL}(s,t), I_{EU}(s,t).I_{FU}(s,t)],$$

$$[F_{EL}(s,t).F_{FL}(s,t), F_{EU}(s,t).F_{FU}(s,t)]\} \cap \left\{\langle (s,t), \begin{bmatrix} \mu_{EL}(s,t) + \mu_{FL}(s,t) - \mu_{EL}(s,t).\mu_{FL}(s,t), \\ \mu_{EU}(s,t) + \mu_{FU}(s,t) - \mu_{EU}(s,t).\mu_{FU}(s,t) \end{bmatrix}\right.,$$

$$[T_{EL}(s,t).T_{FL}(s,t), T_{EU}(s,t).T_{FU}(s,t)], \begin{bmatrix} I_{EL}(s,t) + I_{FL}(s,t) - I_{EL}(s,t).I_{FL}(s,t), \\ I_{EU}(s,t) + I_{FU}(s,t) - I_{EU}(s,t).I_{FU}(s,t) \end{bmatrix},$$

$$\begin{bmatrix} F_{EL}(s,t) + F_{FL}(s,t) - F_{EL}(s,t).F_{FL}(s,t), \\ F_{EU}(s,t) + F_{FU}(s,t) - F_{EU}(s,t).F_{FU}(s,t) \end{bmatrix}\rangle | s \in K \,\&\, t \in T\}$$

$$= \{[\max(\mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EL}(s,t) + \mu_{FL}(s,t) - \mu_{EL}(s,t).\mu_{FL}(s,t)),$$

$$\max(\mu_{EU}(s,t).\mu_{FU}(s,t), \mu_{EU}(s,t) + \mu_{FU}(s,t) - \mu_{EU}(s,t).\mu_{FU}(s,t))],$$

$$[\min(T_{EL}(s,t) + T_{FL}(s,t) - T_{EL}(s,t).T_{FL}(s,t), T_{EL}(s,t).T_{FL}(s,t)),$$

$$\min(T_{EU}(s,t) + T_{FU}(s,t) - T_{EU}(s,t).T_{FU}(s,t), T_{EU}(s,t).T_{FU}(s,t))],$$

$$[\max(I_{EL}(s,t).I_{FL}(s,t), I_{EL}(s,t) + I_{FL}(s,t) - I_{EL}(s,t).I_{FL}(s,t)),$$

$$\max(I_{EU}(s,t).I_{FU}(s,t), I_{EU}(s,t) + I_{FU}(s,t) - I_{EU}(s,t).I_{FU}(s,t))],$$

$$[\max(F_{EL}(s,t).F_{FL}(s,t), F_{EL}(s,t) + F_{FL}(s,t) - F_{EL}(s,t).F_{FL}(s,t)),$$

$$\max(F_{EU}(s,t).F_{FU}(s,t), F_{EU}(s,t) + F_{FU}(s,t) - F_{EU}(s,t).F_{FU}(s,t))]\rangle | s \in K \,\&\, t \in T\}$$

$$= \{[\mu_{EL}(s,t) + \mu_{FL}(s,t) - \mu_{EL}(s,t).\mu_{FL}(s,t), \mu_{EU}(s,t) + \mu_{FU}(s,t) - \mu_{EU}(s,t).\mu_{FU}(s,t)],$$

$$[T_{EL}(s,t).T_{FL}(s,t), T_{EU}(s,t).T_{FU}(s,t)], [I_{EL}(s,t) + I_{FL}(s,t) - I_{EL}(s,t).I_{FL}(s,t),$$

$$[I_{EL}(s,t).I_{FL}(s,t), I_{EU}(s,t) + I_{FU}(s,t) - I_{EU}(s,t).I_{FU}(s,t)],$$

$$[F_{EL}(s,t) + F_{FL}(s,t) - F_{EL}(s,t).F_{FL}(s,t), [F_{EL}(s,t).F_{FL}(s,t),$$

$$F_{EU}(s,t) + F_{FU}(s,t) - F_{EU}(s,t).F_{FU}(s,t)]\rangle | s \in K \,\&\, t \in T\}$$

$= \mathrm{E} \otimes \mathrm{F}$.

Therefore, $(\mathrm{E}\oplus \cap (\mathrm{E}\oplus\mathrm{F}) = \mathrm{E} \otimes \mathrm{F}$.

This is the proof of part (ii) of the theorem, and the proof is completed.

4. Conclusion

In this study, the subsequent expansions of the operators $S(A)$, algebraic sum and geometric product over IVTNFs and several of its properties are also established. The part of IVTNFS plays an additional significant responsibility here, since the IVTNFS provides the best solution for finding the shortest distance within the conclusion building, deciding one's career and many more. Especially in the case of medical diagnosis, when using this concept, there is a real chance that there will be a non-zero fraction of hesitation at any point in the assessment. It is still untied in the direction of classifying several operations and operators over IVTNFS and our next research work is to solve the higher complexities in engineering and science. However, it requires an intensive study of the ways for implementing methods and techniques for unanswered problems using this concept. **Future Work:** In future research, we will spotlight on studying the similarity measure and various distance method between intervals valued neutrosophic sets with time movements moreover study the real-life application of IVTNFS, such as pattern recognition, supplier selection, and image processing, carrier choosing and so on.

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