Estimation of the stress–strength Reliability for Benktander Distribution

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Abstract

This work focuses on the estimation reliability function $(R = p(x < y))$, where x and y are two independent Benktander distributions. The greatest likelihood's asymptotic distribution is found. The maximum likelihood estimator, the moment method estimator, and the approximate maximum likelihood estimator of $R$are proposed. We obtain the asymptotic distribution of R's maximum likelihood estimate. The R confidence interval can be found using the asymptotic distribution.

Keywords: Benktander; Maximum likelihood; Asymptotic distribution; Fisher information matrix

1. Introduction

An important topic of interest, particularly in the field of system dependability, is the estimation of the reliability function $(R = p(x < y))$ the derivation of an explicit expression for R is the subject of numerous articles in the statistical literature. The algebraic form of R has been determined for a few well-known distributions. [3] has taken into consideration the estimation of R when X and Y are normal. [4] examined R estimation in the case of exponential variables X and Y. When X and Y are from gamma distributions with known shape parameters, R was estimated by [1]. [5] and [2] Estimated the $Pr[X > Y]$ for Gamma Distributions. It should be noted that the statistical literature frequently uses the R estimate. For instance, if X is a system's strength under stress Y, then R is a system performance metric that appears in the context of a system's mechanical dependability. Only when the applied stress exceeds the system's strength does the system fail. Although [8] considered this specific issue, the approach was incomplete. He primarily determined the MLE and confidence interval of a specific transformation of R. It should be noted that the statistical literature has made extensive use of related problems. [9] have examined the maximum likelihood estimator (MLE) of R when X and Y have bivariate exponential distributions .[10-13]considered the estimation of R when X and Y are normally distributed. The reliability is examined for both logistic distribution and Laplace distribution by [6]. [7] studied the stability of coupled sequential fractional differential equations with boundary. When X and X are independent random variables, the problem of predicting the probability that one random variable would surpass the other, or $(x > y)$, has continuously drawn interest. The parameter R represents the reliability parameter. In traditional stress-strength reliability, the question of whether a component's random strength (X) exceeds the stress (Y) it is subjected to arises; if $x \leq y$, the component fails or the system that uses the component may malfunction. The Benktander distribution is a sensible option for modeling limited heavy-tailed data because of its special combination of finite moments, tail flexibility, and parametric simplicity. This is especially true in reliability and actuarial situations where conventional distributions are inadequate.

2. Estimation of R

Assume that the random variable X has a Benktander distribution with parameters α and, β where $\alpha > 0$ and $\beta > 0$.We indicate it with $X \sim$Benk (α, β). The following is its probability density function:

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$$f(x) = e^{\frac{\alpha(1-x)}{\beta}}\left(\alpha x^{\beta} - \beta + 1\right), \quad x > 1, \alpha > 0, \qquad 0 < \beta \le 1.$$

Here we will consider $\beta = 1$ then the probability density function become

$$f(x) = \alpha e^{\alpha(1-x)}, \quad x > 1, \alpha > 0 \qquad (2.1)$$

In this section, we consider the problem of estimating $R = p(x < y)$

2.1. Maximum likelihood estimator (MLE) of *R*

Let $X \sim \mathrm{Benk}(\alpha_1, 1)$ and $Y \sim \mathrm{Benk}(\alpha_2, 1)$ where X and Y are independent random variable then

$$R = p(x < y) = \int_1^{\infty} \alpha_2 e^{\alpha_2(1-y)} dy \int_1^{y} \alpha_1 e^{\alpha_1(1-x)}\, dx,$$

$$R = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \qquad (2.1.1)$$

Let $x_1, x_2, \dots, x_n$ be a random sample from $\mathrm{Benk}(\alpha_1, 1)$ and $y_1, y_2, \dots, y_m$ be a r.s from $\mathrm{Benk}(\alpha_2, 1)$ then the joint probability density function of (X, Y) is

$$f(x, y) = \alpha_1 \alpha_2 e^{\alpha_1(1-x)} e^{\alpha_2(1-y)}, \quad x, y > 1, \qquad (2.1.2)$$

Then the likelihood function for (2.1.1) is

$$L(\alpha_1, \alpha_2) = {\alpha_1}^n {\alpha_2}^m e^{\alpha_1 \sum_{i=1}^{n}(1-x_i)} e^{\alpha_2 \sum_{j=1}^{m}(1-y_j)},$$

$$lnl = nln(\alpha_1) + mln(\alpha_2) + \alpha_1 \sum_{i=1}^{n}(1 - x_i) + \alpha_2 \sum_{j=1}^{m}(1 - y_j). \qquad (2.1.3)$$

Taking partial derivatives of (2.1.2) with respect to α_1 and α_2 and equating to 0, we get The estimator of α_1 and α_2 by method of Maximum likelihood is as follows:

$$\frac{dlnl}{d\alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n}(1 - x_i) = 0 \Longrightarrow \hat{\alpha}_1 = \frac{-n}{\sum_{i=1}^{n}(1 - x_i)},$$

$$\frac{dlnl}{d\alpha_2} = \frac{m}{\alpha_1} + \sum_{j=1}^{m}(1 - y_j) = 0 \Longrightarrow \hat{\alpha}_2 = \frac{-m}{\sum_{j=1}^{m}(1 - y_j)}.$$

Therefore, the MLE of R is

$$\hat{R}_{\mathrm{mle}} = \frac{\frac{n}{\sum_{i=1}^{n}(1-x_i)}}{\frac{n}{\sum_{i=1}^{n}(1-x_i)} + \frac{m}{\sum_{j=1}^{m}(1-y_j)}}.$$

2.2. Asymptotic distribution

This section first obtains the asymptotic distribution of $\hat{\alpha}=(\alpha_1, \alpha_2)$, from which the asymptotic distribution of $\hat{R}_{\mathrm{mle}}$ is determined. We derive the asymptotic confidence interval of R from the asymptotic distribution of $\hat{R}_{\mathrm{mle}}$. The Fisher information matrix of $I(\alpha)$ can be represented as follows:

$$I(\alpha) = -\begin{bmatrix} E\left(\frac{d^2 l}{d{\alpha_1}^2}\right) & E\left(\frac{d^2 l}{d\alpha_1 \alpha_2}\right) \\ E\left(\frac{d^2 l}{d\alpha_1 \alpha_2}\right) & E\left(\frac{d^2 l}{d{\alpha_2}^2}\right) \end{bmatrix},$$

$$E\left(\frac{d^2 l}{d{\alpha_1}^2}\right) = \frac{-n}{{\alpha_1}^2},$$

$$E\left(\frac{d^2 l}{d{\alpha_2}^2}\right) = \frac{-m}{{\alpha_2}^2},$$

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$$E\left(\frac{d^2 l}{d\alpha_1\alpha_2}\right) = E\left(\frac{d^2 l}{d\alpha_2\alpha_1}\right) = 0.$$

Then Fisher information matrix become

$$I(\alpha) = \begin{bmatrix} \frac{n}{{\alpha_1}^2} & 0 \\ 0 & \frac{m}{{\alpha_2}^2} \end{bmatrix},$$

thus

$$I^{-1}(\alpha) = \begin{bmatrix} \frac{{\alpha_1}^2}{n} & 0 \\ 0 & \frac{{\alpha_2}^2}{m} \end{bmatrix}.$$

Theorem 2.2.1 As $n \to \infty$ and $m \to \infty$ the Mle asymptotically normal

$$\sqrt{n}\begin{pmatrix} \widehat{\alpha_1} - \alpha_1 \\ \widehat{\alpha_2} - \alpha_2 \end{pmatrix} \xrightarrow{d} N\left(0, I^{-1}(\alpha)\right),$$

Proof.

The central limit theorem and the asymptotic characteristics of MLEs provide the proof.If $(\widehat{\alpha_1}, \widehat{\alpha_2})$ are asymptotically normal then any differentiable function R of the estimator $\hat{R}$ is also asymptotically normal

$$\sqrt{n}\left(\hat{R} - R\right) \xrightarrow{d} N(0, V),$$

where V is asymptotic variance

$$V = \nabla R^t I^{-1}(\theta) \nabla R\,,$$

$$\nabla = \begin{bmatrix} \frac{dR}{d\alpha_1} \\ \frac{dR}{d\alpha_2} \end{bmatrix} = \begin{bmatrix} \frac{\alpha_2}{(\alpha_1 + \alpha_2)^2} \\ \frac{-\alpha_1}{(\alpha_1 + \alpha_2)^2} \end{bmatrix}.$$

Then,

$$V = \begin{bmatrix} \frac{\alpha_2}{(\alpha_1 + \alpha_2)^2} & \frac{-\alpha_1}{(\alpha_1 + \alpha_2)^2} \end{bmatrix} \begin{bmatrix} \frac{{\alpha_1}^2}{n} & 0 \\ 0 & \frac{{\alpha_2}^2}{m} \end{bmatrix} \begin{bmatrix} \frac{\alpha_2}{(\alpha_1 + \alpha_2)^2} \\ \frac{-\alpha_1}{(\alpha_1 + \alpha_2)^2} \end{bmatrix},$$

$$V = \frac{{\alpha_1}^2 {\alpha_2}^2}{n(\alpha_1 + \alpha_2)^2} + \frac{{\alpha_1}^2 {\alpha_2}^2}{m(\alpha_1 + \alpha_2)^2},$$

$$\sqrt{\mathrm{n}}\left(\hat{\mathrm{R}} - \mathrm{R}\right) \xrightarrow{\mathrm{d}} \left(0, \frac{{\alpha_1}^2 {\alpha_2}^2}{\mathrm{n}(\alpha_1 + \alpha_2)^2} + \frac{{\alpha_1}^2 {\alpha_2}^2}{\mathrm{m}(\alpha_1 + \alpha_2)^2}\right).$$

An approximate confidence interval for R is obtained from this result as

$\hat{R} \pm Z_{\frac{\alpha}{2}} \sqrt{V}$.

2.3 Moment estimator of *R*

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a Benktander Distribution with population parameter $\alpha_1$with probability density function

$$f(x) = \alpha e^{\alpha(1-x)}, x > 1\,.$$

The expected value of x and y are

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$$E(x) = \int_1^\infty \alpha_1 x e^{\alpha_1(1-x)}\, dx = 1 + \frac{1}{\alpha_1},$$

$$E(y) = \int_1^\infty \alpha_2 y e^{\alpha_2(1-y)}\, dy = 1 + \frac{1}{\alpha_2},$$

By equating the sample mean with the corresponding population mean, we get

The estimator of α_1 and α_2 by method of moment is as follows:

$$1 + \frac{1}{\alpha_1} = \bar{x} \Longrightarrow \hat{\alpha}_1 = \frac{1}{\bar{x}-1} \quad , \qquad (2.3.1)$$

and

$$1 + \frac{1}{\alpha_2} = \bar{y} \Longrightarrow \hat{\alpha}_2 = \frac{1}{\bar{y}-1}. \qquad (2.3.2)$$

By substituting equation (2.3.1) and (2.3.2) in equation (2.1.1) we get

$$\hat{R}_{mom} = \frac{\frac{1}{\bar{x}-1}}{\frac{1}{\bar{x}-1} + \frac{1}{\bar{y}-1}}.$$

3. Simulation Study

In this section, we utilized the R software to conduct a simulation study in which we generated 1000 samples from the Benktander distribution with varying sample sizes $n = m = 20, 30, 100$, with $\alpha_1 = (1, 1.5)$ and $\alpha_2 = (1, 2)$, in order to examine the behavior of the mean square error (Mse) of the MLE reliability estimator and determine the 95% confidence interval of R. In this section, we studied the behavior of the Mse of the MLE reliability estimator and finding 95% confidence interval (CI) of R using the R software program to run a simulation study, in which we generate 1000 samples from Benktander distribution with different simple sizes $n = m = 20, 30, 100$, with $\alpha_1 = (1, 1.5)$ and $\alpha_2 = (1, 2)$.

Table 1: MSE and confidence interval of the MLE of R, when $\alpha_1 = (1, 1.5)$, $\alpha_2 = (1, 2)$

SN	α_1	α_2	S.size	R	$\hat{R}$	Mse	CI-lower	CI-lupper
1	1	1	10	0.500000	0.500309	0.011962	0.43248	0.56813
2	1	1	30	0.500000	0.499960	0.003942	0.47747	0.52244
3	1	1	100	0.500000	0.500375	0.001265	0.49339	0.50735
4	1	2	10	0.333333	0.340128	0.010378	0.27709	0.40316
5	1	2	30	0.333333	0.334214	0.003157	0.31411	0.35431
6	1	2	100	0.333333	0.333907	0.000992	0.32772	0.34008
7	1.5	1	10	0.600000	0.595136	0.012018	0.52722	0.66305
8	1.5	1	30	0.600000	0.595700	0.004002	0.57310	0.61829
9	1.5	1	100	0.600000	0.600257	0.001130	0.59366	0.60685
10	1.5	2	10	0.428571	0.432463	0.011377	0.36636	0.49856
11	1.5	2	30	0.428571	0.428455	0.003987	0.40584	0.45106
12	1.5	2	100	0.428571	0.427657	0.001167	0.42095	0.43435

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Table 2: MSE and confidence interval of the moment method estimator of R, when $\alpha_1 = (1, 1.5)$, $\alpha_2 = (1, 2)$

SN	α_1	α_2	S.size	R	$\hat{R}$	Mse	CI-lower	CI-lupper
1	1	1	10	0.50000	0.497501	0.01254	0.42805	0.56695
2	1	1	30	0.50000	0.500140	0.00415	0.47707	0.52320
3	1	1	100	0.50000	0.499429	0.00132	0.49228	0.50657
4	1	2	10	0.33333	0.338678	0.09465	0.27843	0.39892
5	1	2	30	0.33333	0.337815	0.00329	0.31732	0.35830
6	1	2	100	0.33333	0.335625	0.00096	0.32956	0.34168
7	1.5	1	10	0.60000	0.595378	0.01111	0.53007	0.66068
8	1.5	1	30	0.60000	0.596298	0.00359	0.57486	0.61773
9	1.5	1	100	0.60000	0.599524	0.00121	0.59270	0.60634
10	1.5	2	10	0.42857	0.432570	0.01125	0.36683	0.49830
11	1.5	2	30	0.42857	0.433468	0.00388	0.41122	0.45571
12	1.5	2	100	0.42857	0.429454	0.00120	0.42266	0.43624

In **tables 1** and **2**.It is found that the variance of $\hat{R}$ and the lengths of the confidence intervals decrease as the sample size increases in both method.

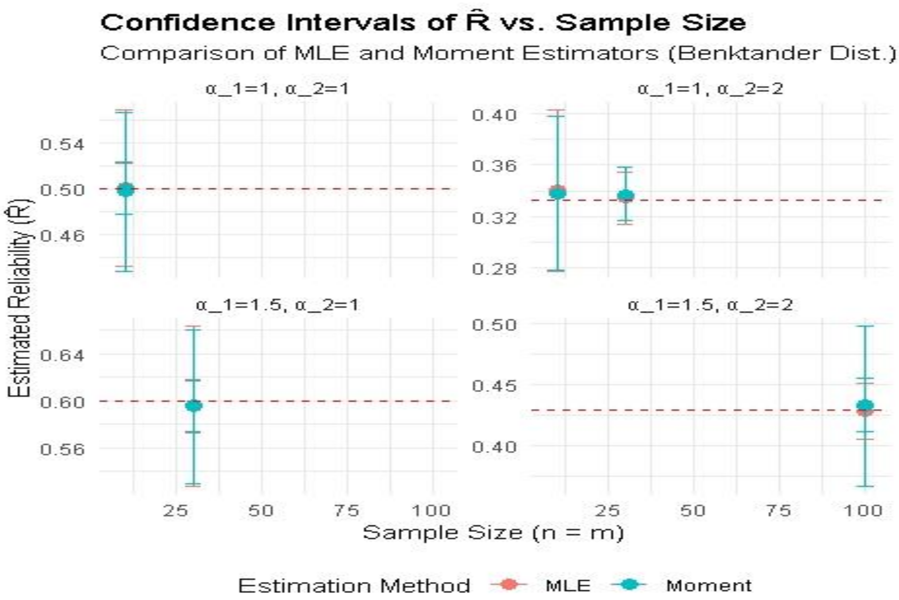

Figure 1. Confidence Intervals of $\hat{R}$ ˆplotted vs. Sample Size

This figure will make it easier to compare the MLE and moment estimation techniques across various parameter scenarios and will clearly demonstrate how the confidence intervals get smaller as sample size grows.

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4. Conclusion

In this work, we examine various approaches for estimating $R = P(X < Y)$ in the case of independent Benktander random variables X and Y with equal form parameters. Keep in mind that our approach is easily extensible for predicting $p(x < y)$ in which X and Y are independent. It is noted that the MLE performs admirably. Even with a relatively small sample size, we advised using the Bootstrap confidence interval based on the simulation findings. Making use of the MLE's asymptotic distribution to build Confidence intervals are ineffective. We handle statistical inferences using a methodical computational technique. This method could be useful in situations when the sample distributions are very complex or difficult to derive. Simulation investigations on the asymptotic and computational approach test powers demonstrate that both tests function satisfactorily. We may suggest research issues to investigate further in the fields of fuzzy soft set theory and fixed point theory. For instance, we are hoping to compare our work with the works such as [14-17]

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