Supra Soft Continuity Via Supra Soft Omega Open Sets

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Abstract

This paper presents four new types of continuity in the context of supra-soft topological spaces: supra-soft ω-continuity, supra-soft ω-irresoluteness, supra-soft contra-continuity, and supra-soft contra-ω-continuity. The main contribution is the clear definitions and detailed study of these concepts, which helps us better understand how they work and how they are interconnected. We carefully examine how these new concepts connect among themselves and with analogous concepts in traditional supra-topological spaces. We also demonstrate how these different forms of continuity behave under common mathematical operations, such as composition and restriction. To make everything easier to understand, we introduce several examples that emphasize how these new concepts compare with existing, well-known concepts, giving a better picture of how continuity works in a more generalized topological settings.

Keywords: Supra ω-open sets; Supra-soft continuity; Supra-soft irresoluteness; Supra generated soft topology

1 Introduction and Preliminaries

Molodtsov [1] developed the notion of soft sets in 1999 as a novel technique to cope with uncertain data when modeling real-world situations in various fields, including data science, engineering, economics, and health sciences. Numerous academics have applied soft set theory as a mathematical tool to real-world situations (see [2, 3]). Shabir and Naz [4] pioneered the framework of soft topology and researched numerous related subjects. Then, numerous academics interested in abstract structures tried to broaden topological notions to encompass soft-topological space.

Mashhour et al. [5] defined supra-topological spaces by omitting the finite intersection constraint from the classical topology definition. Many topological scholars studied topological concepts using supra-topologies to investigate their properties [6-9]. The authors of [10] employed supra-topologies to create new rough set models for modeling information systems. Furthermore, the authors of [11] exploited supra-topologies for digital image processing.

The concept of supra-soft topological spaces, introduced in 2014, generalizes crisp mathematical structures to soft ones. It includes concepts like continuity [12], compactness [13], separability [14, 15], and separation axioms [16, 17]. Research in the field of supra-soft topologies remains vibrant and active.

This paper is aimed at augmenting the field of soft topology by introducing four new types of supra-soft continuity. The motivation behind it is to generalize the available continuity concepts within the flexible soft set structure and to provide a better description through various characterizations. Through exploring the connections with supra-topological spaces and studying their preservation properties, this paper aims to link different topological structures. Also, certain examples will clarify how these new ideas extend previous ones,

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resulting in an even better understanding of continuity in generalized topological spaces. Last but not least, the study aims to advance the theory of soft topology and present suggestions for future research.

"Assume that M is a non-empty set and A is a set of parameters. A soft set over M relative to A is a function $G : A \longrightarrow \mathcal{P}(M)$. $SS(M, A)$ will denote the family of all soft sets over M relative to A. The null soft set and the absolute soft set are denoted by 0_A and 1_A, respectively. Let $G \in SS(M, A)$. If $G(a) = U$ for all $a \in A$, then G is denoted by C_U. If $G(a) = U$ and $G(b) = \emptyset$ for all $b \in A - \{a\}$, then G is denoted by a_U. If $G(a) = \{x\}$ and $G(b) = \emptyset$ for all $b \in A - \{a\}$, then G is called a soft point over M relative to A and denoted by a_x. $SP(M, A)$ will denote the family of all soft points over M relative to A. If $G \in SS(M, A)$ and $a_x \in SP(M, A)$, then a_x is said to belong to G (notation: $a_x \tilde{\in} G$) if $x \in G(a)$. If $\{G_\alpha : \alpha \in \Delta\} \subseteq SS(M, A)$, then the soft union and soft intersection of $\{G_\alpha : \alpha \in \Delta\}$ are denoted by $\tilde{\cup}_{\alpha\in\Delta} G_\alpha$ and $\tilde{\cap}_{\alpha\in\Delta} G_\alpha$, respectively, and defined by

$(\tilde{\cup}_{\alpha\in\Delta} G_\alpha)(a) = \cup_{\alpha\in\Delta} G_\alpha(a)$ and $(\tilde{\cap}_{\alpha\in\Delta} G_\alpha)(a) = \cap_{\alpha\in\Delta} G_\alpha(a)$ for all $a \in A$.

Let $SS(M, A)$ and $SS(N, B)$ be two families of soft sets, and $p : M \longrightarrow N$, $u : A \longrightarrow B$ be two maps. Then a soft map $g_{pu} : SS(M, A) \longrightarrow SS(N, B)$ is defined as follows: For each $H \in SS(M, A)$ and $K \in SS(N, B)$, $(g_{pu}(H))(b) = \emptyset$ if $u^{-1}(b) = \emptyset$, $(g_{pu}(H))(b) = \cup_{a\in u^{-1}(b)} p(H(a))$ if $u^{-1}(b) \neq \emptyset$, and $\left(g_{pu}^{-1}(K)\right)(a) = p^{-1}(K(u(a)))$ for all $a \in A$."

For concepts and expressions not described here, we refer readers to [18, 19]; supra-soft topological space (Supra-STS) and supra-topological space (supra-TS) are utilized in this paper.

The sequel will utilize the following definitions:

Definition 1.1. Let $(M, \aleph)$ and (N, Im) be two supra-TSs, and let $h : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ be a map. Then

(a) [20] h is called supra continuous at $x \in X$ if for every $V \in \mathrm{Im}$ such that $h(x) \in V$, there exists $U \in \aleph$ such that $x \in U$ and $h(U) \subseteq V$. If h is supra continuous at each $x \in X$, then h is called supra continuous.

(b) [21] h is called supra ω-continuous at $x \in X$ if for every $V \in \mathrm{Im}$ such that $h(x) \in V$, there exists $U \in \aleph_\omega$ such that $x \in U$ and $h(U) \subseteq V$. If h is supra ω-continuous at each $x \in X$, then h is called supra ω-continuous.

(c) [22] h is called supra ω-irresolute at $x \in X$ if for every $V \in \mathrm{Im}_\omega$ such that $h(x) \in V$, there exists $U \in \aleph_\omega$ such that $x \in U$ and $h(U) \subseteq V$. If h is supra ω-irresolute at each $x \in X$, then h is called supra ω-irresolute.

(d) [23] h is called supra contra-continuous if $h^{-1}(V) \in \aleph^c$ for every $V \in \mathrm{Im}$.

(e) [22] h is called supra contra-ω-continuous if $h^{-1}(V) \in (\aleph_\omega)^c$ for every $V \in \mathrm{Im}$.

Definition 1.2. [12] Let (M, ρ, A) and (N, λ, B) be two supra-STSs. A soft map $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is called supra-soft continuous at a soft point $a_x \in SP(M, A)$ if for every $G \in \lambda$ such that $g_{pu}(a_x) \tilde{\in} G$, there exists $H \in \rho$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. If g_{pu} is supra-soft continuous at each soft point $a_x \in SP(M, A)$, then g_{pu} is called supra-soft continuous.

2 Supra-Soft ω-Continuity

In this section, we introduce supra-soft ω-continuous maps. We introduce several characterizations of them. Also, we introduce the correspondence between this concept and its analogous one in general topology. Moreover, we demonstrate how this distinct form of continuity behaves under common mathematical operations, such as composition and restriction.

Definition 2.1. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. A soft map $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is called supra-soft ω-continuous at a soft point $a_x \in SP(M, A)$ if for every $G \in \lambda$ such that $g_{pu}(a_x) \tilde{\in} G$, there exists $H \in \rho_\omega$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. If g_{pu} is supra-soft ω-continuous at each soft point $a_x \in SP(M, A)$, then g_{pu} is called supra-soft ω-continuous.

Theorem 2.2. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous at $a_x \in SP(M, A)$, then $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-continuous at x.

Proof. Let $V \in \lambda_{u(a)}$ such that $p(x) \in V$. Choose $G \in \lambda$ such that $G(u(a)) = V$. Then $g_{pu}(a_x) = (u(a))_{p(x)} \tilde{\in} G$. Since $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous at a_x, then there exists $H \in \rho_\omega$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. Thus, we have $x \in H(a) \in (\rho_\omega)_a$ and $p(H(a)) = (g_{pu}(H))(u(a)) \subseteq G(u(a)) = V$. On the other hand, by [, $(\rho_\omega)_a = (\rho_a)_\omega$. This shows that $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-continuous at x.

Corollary 2.3. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous, then $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-continuous.

Theorem 2.4. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$, and $u : A \longrightarrow B$, where u is a bijection and let $a_x \in SP(M, A)$. Then $g_{pu} : (M, \otimes_{a \in A} \rho_a, A) \longrightarrow (N, \otimes_{b \in B} \lambda_b, B)$ is supra-soft ω-continuous at a_x iff $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-continuous at x.

Proof. *Necessity*. Let $g_{pu} : (M, \otimes_{a \in A} \rho_a, A) \longrightarrow (N, \otimes_{b \in B} \lambda_b, B)$ be supra-soft ω-continuous at a_x. Then by Theorem 2.2, $p : (M, (\otimes_{a \in A} \rho_a)_a) \longrightarrow \left(N, (\otimes_{b \in B} \lambda_b)_{u(a)}\right)$ is supra ω-continuous at x. On the other hand, by Theorem 2.5 of [19], $(\otimes_{a \in A} \rho_a)_a = \rho_a$ and $(\otimes_{b \in B} \lambda_b)_{u(a)} = \lambda_{u(a)}$. It follows that $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-continuous at x.

Sufficiency. Let $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ be supra ω-continuous at x. Let $G \in \otimes_{b \in B} \lambda_b$ such that $g_{pu}(a_x) = (u(a))_{p(x)} \tilde{\in} G$. Then $p(x) \in G(u(a)) \in \lambda_{u(a)}$. So, there exists $U \in (\rho_a)_\omega$ such that $x \in U$ and $p(U) \subseteq G(u(a))$. By Theorem 3.10 of [19], $a_U \in (\otimes_{a \in A} \rho_a)_\omega$. Therefore, we have $a_x \tilde{\in} a_U \in (\otimes_{a \in A} \rho_a)_\omega$ and $g_{pu}(a_U) \tilde{\subseteq} G$. This proves that $g_{pu} : (M, \otimes_{a \in A} \rho_a, A) \longrightarrow (N, \otimes_{b \in B} \lambda_b, B)$ is supra-soft ω-continuous at a_x.

Corollary 2.5. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$ and $u : A \longrightarrow B$, where u is a bijection. Then $g_{pu} : (M, \otimes_{a \in A} \rho_a, A) \longrightarrow (N, \otimes_{b \in B} \lambda_b, B)$ is supra-soft ω-continuous iff $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-continuous for all $a \in A$.

Corollary 2.6. Let $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ be a map between two supra-TSs and let $u : A \longrightarrow B$ be a bijective map. Let $a_x \in SP(M, A)$. Then $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ is supra ω-continuous at x iff $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), B)$ is supra-soft ω-continuous at a_x.

Proof. For each $a \in A$ and $b \in B$, let $\aleph_a = \aleph$ and $\mathrm{Im}_b = \mathrm{Im}$. Then $\mu(\aleph) = \otimes_{a \in A} \aleph_m$ and $\tau(\mathrm{Im}) = \otimes_{b \in B} \mathrm{Im}_b$. Theorem 2.4 ends the proof.

Corollary 2.7. Let $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ be a map between two supra-TSs and let $u : A \longrightarrow B$ be a bijective map. Then $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ is supra ω-continuous iff $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), B)$ is supra-soft ω-continuous.

Theorem 2.8. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous at $a_x \in SP(M, A)$, then $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra continuous at x.

Proof. Let $V \in \lambda_{u(a)}$ such that $p(x) \in V$. Choose $G \in \lambda$ such that $G(u(a)) = V$. Then $g_{pu}(a_x) = (u(a))_{p(x)} \tilde{\in} G$. Since $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous at a_x, then there exists $H \in \rho$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. Thus, we have $x \in H(a) \in \rho_a$ and $p(H(a)) = (g_{pu}(H))(u(a)) \subseteq G(u(a)) = V$. This shows that $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra continuous at x.

Corollary 2.9. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous, then $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra continuous.

Theorem 2.10. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$ and $u : A \longrightarrow B$, where u is a bijection and let $a_x \in SP(M, A)$. Then $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ is supra-soft continuous at a_x iff $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra continuous at x.

Proof. *Necessity*. Let $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ be supra-soft continuous at a_x. Then by Theorem 2.8, $p : (M, (\otimes_{a\in A}\rho_a)_a) \longrightarrow \left(N, (\otimes_{b\in B}\lambda_b)_{u(a)}\right)$ is supra continuous at x. On the other hand, by Theorem 2.5 of [19], $(\otimes_{a\in A}\rho_a)_a = \rho_a$ and $(\otimes_{b\in B}\lambda_b)_{u(a)} = \lambda_{u(a)}$. It follows that $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra continuous at x.

Sufficiency. Let $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ be supra ω-continuous at x. Let $G \in \otimes_{b\in B}\lambda_b$ such that $g_{pu}(a_x) = (u(a))_{p(x)} \tilde{\in} G$. Then $p(x) \in G(u(a)) \in \lambda_{u(a)}$. So, there exists $U \in \rho_a$ such that $x \in U$ and $p(U) \subseteq G(u(a))$. By Theorem 3.10 of [19], $a_U \in \otimes_{a\in A}\rho_a$. Therefore, we have $a_x \tilde{\in} a_U \in \otimes_{a\in A}\rho_a$ and $g_{pu}(a_U) \tilde{\subseteq} G$. This proves that $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ is supra-soft continuous at a_x.

Corollary 2.11. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$ and $u : A \longrightarrow B$, where u is a bijection. Then $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ is supra-soft continuous iff $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra continuous for all $a \in A$.

Corollary 2.12. Let $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ be a map between two supra-TSs and let $u : A \longrightarrow B$ be a bijective map. Let $a_x \in SP(M, A)$. Then $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ is supra continuous at x iff $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), B)$ is supra-soft continuous at a_x.

Proof. For each $a \in A$ and $b \in B$, let $\aleph_a = \aleph$ and $\mathrm{Im}_b = \mathrm{Im}$. Then $\mu(\aleph) = \otimes_{a\in A}\aleph_m$ and $\tau(\mathrm{Im}) = \otimes_{b\in B}\mathrm{Im}_b$. Theorem 2.10 ends the proof.

Corollary 2.13. Let $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ be a map between two supra-TSs and let $u : A \longrightarrow B$ be a bijective map. Then $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ is supra continuous iff $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), B)$ is supra-soft continuous.

Theorem 2.14. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between two supra-STSs and let $a_x \in SP(M, A)$. If g_{pu} is supra-soft continuous at a_x, then g_{pu} is supra-soft ω-continuous at a_x.

Proof. Let $G \in \lambda$ such that $g_{pu}(a_x) \tilde{\in} G$. Since g_{pu} is supra-soft continuous at a_x, there exists $H \in \rho$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. Since $\rho \subseteq \rho_\omega$, then $H \in \rho_\omega$. It follows that g_{pu} is supra-soft ω-continuous at a_x.

Corollary 2.15. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between two supra-STSs and let $a_x \in SP(M, A)$. If g_{pu} is supra-soft continuous, then g_{pu} is supra-soft ω-continuous.

The following example will show that the converses of Theorem 2.14 and Corollary 2.15 are not true in general:

Example 2.16. Let $M = N = \mathbb{R}$, $\aleph = \{\varnothing\} \cup \{U \subseteq M : U \text{ is infinite}\}$, $\text{Im} = \{\varnothing, \{5\}, \mathbb{R}\}$, and $A = [0,1]$. Define $p : (M, \aleph) \longrightarrow (N, \text{Im})$ and $u : A \longrightarrow A$ by $p(t) = t + 3$, and $u(a) = a$ for all $t \in M$ and $a \in A$. Let $V = \{5\}$. Then $V \in \text{Im}$ such that $p(2) \in V$. Conversely, for any $U \in \aleph$ with $2 \in U$, $p(U)$ is infinite and thus $p(U)$ is not a subset V. Hence, p is not supra continuous at 2. To show that p is supra ω-continuous, let $x \in M$ and let $W \in \text{Im}$ such that $p(x) \in W$. Since $\{x\} = (\mathbb{Q} \cup \{x\}) - (\mathbb{Q} - \{x\})$, $(\mathbb{Q} \cup \{x\}) \in \aleph$, and $\mathbb{Q} - \{x\}$ is countable, then $\{x\} \in \aleph_\omega$. Let $U = \{x\}$. Then $U \in \aleph_\omega$, $x \in U$, and $p(U) = \{p(x)\} \subseteq W$. It follows that $p : (M, \aleph) \longrightarrow (N, \text{Im})$ is supra ω-continuous at x. Therefore, by Corollaries 2.7 and 2.13, $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\text{Im}), A)$ is supra-soft ω-continuous but not supra-soft continuous.

Theorem 2.17. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Then the following conditions are equivalent:

(a) g_{pu} is supra-soft ω-continuous.

(b) For each $G \in \lambda$, $g_{pu}^{-1}(G) \in \rho_\omega$.

(c) For each $K \in \lambda^c$, $g_{pu}^{-1}(K) \in (\rho_\omega)^c$.

Proof. (a) $\longrightarrow$ (b): Let $G \in \lambda$ and let $a_x \tilde{\in} g_{pu}^{-1}(G)$. Then $g_{pu}(a_x) \tilde{\in} G$ and by (a), there exists $H_{a_x} \in \rho_\omega$ such that $a_x \tilde{\in} H_{a_x}$ and $g_{pu}(H_{a_x}) \tilde{\subseteq} G$. Therefore, we have $g_{pu}^{-1}(G) = \tilde{\cup}_{a_x \tilde{\in} g_{pu}^{-1}(G)} H_{a_x}$, and hence $g_{pu}^{-1}(G) \in \rho_\omega$.

(b) $\longrightarrow$ (c): Let $K \in \lambda^c$. Then $1_B - K \in \lambda$, and by (b), $g_{pu}^{-1}(1_B - K) = 1_A - g_{pu}^{-1}(K) \in \rho_\omega$. It follows that $g_{pu}^{-1}(K) \in (\rho_\omega)^c$.

(c) $\longrightarrow$ (a): Let $a_x \in SP(M, A)$ and let $G \in \lambda$ such that $g_{pu}(a_x) \tilde{\in} G$. Then $1_B - G \in \lambda^c$, and by (b), $g_{pu}^{-1}(1_B - G) = 1_A - g_{pu}^{-1}(G) \in (\rho_\omega)^c$. Let $H = g_{pu}^{-1}(G)$. Then we have $a_x \tilde{\in} H \in \rho_\omega$ and $g_{pu}(H) = g_{pu}\left(g_{pu}^{-1}(G)\right) \tilde{\subseteq} G$. It follows that g_{pu} is supra-soft ω-continuous.

Corollary 2.18. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Then $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous iff $g_{pu} : (M, \rho_\omega, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous.

Theorem 2.19. Let (M, ρ, A), (N, λ, B), and (L, γ, D) be three supra-STSs. If $g_{p_1u_1} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous and $g_{p_2u_2} : (N, \lambda, B) \longrightarrow (L, \gamma, D)$ is supra-soft continuous, then $g_{(p_2 \circ p_1)(u_2 \circ u_1)} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ is supra-soft ω-continuous.

Proof. Let $G \in \gamma$. Since $g_{p_2u_2}$ is supra-soft continuous, then $g_{p_2u_2}^{-1}(G) \in \lambda$. Since $g_{p_1u_1}$ is supra-soft ω-continuous, then by Theorem 2.17, $g_{p_1u_1}^{-1}\left(g_{p_2u_2}^{-1}(G)\right) = g_{(p_2 \circ p_1)(u_2 \circ u_1)}^{-1}(G) \in \rho_\omega$. Therefore, again by Theorem 2.17, $g_{(p_2 \circ p_1)(u_2 \circ u_1)}$ is supra-soft ω-continuous.

Theorem 2.20. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ If is supra-soft ω-continuous, then for any non-empty subset $Y \subseteq M$, $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous.

Proof. Let $G \in \lambda$. Since $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous, then $g_{pu}^{-1}(G) \in \rho_\omega$ and so $((g_{pu})|_{C_Y})^{-1}(G) = g_{pu}^{-1}(G) \tilde{\cap} C_Y \in (\rho_\omega)_Y$. Therefore, by Theorem 3.19 of [19], $((g_{pu})|_{C_Y})^{-1}(G) \in (\rho_Y)_\omega$. It follows that $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous.

Lemma 2.21. Let (M, ρ, A) be a supra-STS and let Y be a non-empty subset of M. Then $H \in (\rho_Y)^c$ iff there exists $K \in \rho^c$ such that $H = K \tilde{\cap} C_Y$.

Proof. $H \in (\rho_Y)^c$ iff $1_A - H \in \rho_Y$, which is true iff there is $T \in \rho$ such that $1_A - H = T \tilde{\cap} C_Y$, but in this case $1_A - T \in \rho^c$ and $H = (1_A - T) \tilde{\cap} C_Y$.

Theorem 2.22. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between two supra-STSs and let $M = Y \cup Z$, where $\{C_Y, C_Z\} \subseteq \rho^c$. If $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ and $(g_{pu})|_{C_Z} : (Z, \rho_Z, A) \longrightarrow (N, \lambda, B)$ are supra-soft continuous, then $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous.

Proof. Let $K \in \lambda^c$. Then

$$\begin{aligned} g_{pu}^{-1}(K) &= g_{pu}^{-1}(K) \tilde{\cap} 1_A \\ &= g_{pu}^{-1}(K) \tilde{\cap} (C_Y \tilde{\cup} C_Z) \\ &= \left(g_{pu}^{-1}(K) \tilde{\cap} C_Y\right) \tilde{\cup} \left(g_{pu}^{-1}(K) \tilde{\cap} C_Z\right) \\ &= \left(((g_{pu})|_{C_Y})^{-1}(K)\right) \tilde{\cup} \left(((g_{pu})|_{C_Z})^{-1}(K)\right). \end{aligned}$$

Since $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ and $(g_{pu})|_{C_Z} : (Z, \rho_Z, A) \longrightarrow (N, \lambda, B)$ are supra-soft continuous, then $\left(((g_{pu})|_{C_Y})^{-1}(K)\right) \in (\rho_Y)^c$ and $\left(((g_{pu})|_{C_Z})^{-1}(K)\right) \in (\rho_Z)^c$. By Lemma 2.21, $((g_{pu})|_{C_Y})^{-1}(K)$, $((g_{pu})|_{C_Z})^{-1}(K) \in \rho^c$. It follows that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous.

Theorem 2.23. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between two supra-STSs and let $M = Y \cup Z$, where $\{C_Y, C_Z\} \subseteq (\rho_\omega)^c$. If $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ and $(g_{pu})|_{C_Z} : (Z, \rho_Z, A) \longrightarrow (N, \lambda, B)$ are supra-soft ω-continuous, then $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous.

Proof. We are going to apply Theorem 2.17. Let $K \in \lambda^c$. Then

$$\begin{aligned} g_{pu}^{-1}(K) &= g_{pu}^{-1}(K) \tilde{\cap} 1_A \\ &= g_{pu}^{-1}(K) \tilde{\cap} (C_Y \tilde{\cup} C_Z) \\ &= \left(g_{pu}^{-1}(K) \tilde{\cap} C_Y\right) \tilde{\cup} \left(g_{pu}^{-1}(K) \tilde{\cap} C_Z\right) \\ &= \left(((g_{pu})|_{C_Y})^{-1}(K)\right) \tilde{\cup} \left(((g_{pu})|_{C_Z})^{-1}(K)\right). \end{aligned}$$

Since $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ and $(g_{pu})|_{C_Z} : (Z, \rho_Z, A) \longrightarrow (N, \lambda, B)$ are supra-soft ω-continuous, then $\left(((g_{pu})|_{C_Y})^{-1}(K)\right) \in ((\rho_Y)_\omega)^c$ and $\left(((g_{pu})|_{C_Z})^{-1}(K)\right) \in ((\rho_Z)_\omega)^c$. Since by Theorem 3.19 of [19], $(\rho_Y)_\omega = (\rho_\omega)_Y$ and $(\rho_Z)_\omega = (\rho_\omega)_Z$, then $\left(((g_{pu})|_{C_Y})^{-1}(K)\right) \in ((\rho_\omega)_Y)^c$ and $\left(((g_{pu})|_{C_Z})^{-1}(K)\right) \in ((\rho_\omega)_Z)^c$. Therefore, by Lemma 2.21, $((g_{pu})|_{C_Y})^{-1}(K)$, $((g_{pu})|_{C_Z})^{-1}(K) \in (\rho_\omega)^c$. It follows that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous.

Lemma 2.24. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. Consider the projections $\pi_M : M \times N \longrightarrow M$, $\pi_N : M \times N \longrightarrow N$, $\pi_A : A \times B \longrightarrow A$, and $\pi_B : A \times B \longrightarrow B$ on M, N, A, and B, respectively. Then $g_{(\pi_M)(\pi_A)} : (M \times N, pr(\rho \times \lambda), A \times B) \longrightarrow (M, \rho, A)$ and $g_{(\pi_N)(\pi_B)} : (M \times N, pr(\rho \times \lambda), A \times B) \longrightarrow (N, \lambda, B)$ are supra-soft continuous.

Proof. Let $G \in \rho$ and $H \in \lambda$. Then $g_{(\pi_M)(\pi_A)}^{-1}(G) = G \times 1_B \in pr(\rho \times \lambda)$ and $g_{(\pi_M)(\pi_A)}^{-1}(H) = 1_A \times H \in pr(\rho \times \lambda)$. Hence, $g_{(\pi_M)(\pi_A)}$ and $g_{(\pi_N)(\pi_B)}$ are supra-soft continuous maps.

Theorem 2.25. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ and $g_{qv} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ be two soft maps between supra-STSs. Define $s : M \longrightarrow N \times L$ and $r : A \longrightarrow B \times D$ by $s(m) = (p(m), q(m))$ and $r(a) = (u(a), v(a))$. If $g_{sr} : (M, \rho, A) \longrightarrow (N \times L, pr(\lambda \times \gamma), B \times D)$ is supra-soft continuous, then both g_{pu} and g_{qv} are supra-soft continuous.

Proof. Assume that g_{sr} is a supra-soft continuous map. Since $g_{pu} = g_{(\pi_N \circ s)(\pi_B \circ r)}$ and $g_{qv} = g_{(\pi_L \circ s)(\pi_D \circ r)}$, then by Lemma 2.24, we conclude that g_{pu} and g_{qv} are supra-soft continuous.

Theorem 2.26. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ and $g_{qv} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ be two soft maps between supra-STSs. Define $s : M \longrightarrow N \times L$ and $r : A \longrightarrow B \times D$ by $s(m) = (p(m), q(m))$ and

$r(a) = (u(a), v(a))$. If $g_{sr} : (M, \rho, A) \longrightarrow (N \times L, pr(\lambda \times \gamma), B \times D)$ is supra-soft ω-continuous, then both g_{pu} and g_{qv} are supra-soft ω-continuous.

Proof. Assume that g_{sr} is supra-soft ω-continuous. Since $g_{pu} = g_{(\pi_N \circ s)(\pi_B \circ r)}$ and $g_{qv} = g_{(\pi_L \circ s)(\pi_D \circ r)}$, then by Lemma 2.24 and Theorem 2.19, we conclude that g_{pu} and g_{qv} are supra-soft ω-continuous.

Theorem 2.27. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs, and let $\emptyset \neq Y \subseteq M$ such that $\rho_Y \subseteq \rho$. If there is $a_y \in SP(Y, A)$ such that $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous at a_y, then g_{pu} is supra-soft continuous at a_y.

Proof. Let $G \in \lambda$ such that $g_{pu}(a_y) \widetilde{\in} G$. Since $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous at a_y, there exists $H \in \rho_Y$ such that $a_y \widetilde{\in} H$ and $g_{pu}(H) \widetilde{\subseteq} G$. Since by assumption $\rho_Y \subseteq \rho$, then $H \in \rho$. This shows that g_{pu} is supra-soft continuous at a_y.

Corollary 2.28. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Let $\{Y_\alpha : \alpha \in \triangle\}$ be a cover of M such that for each $\alpha \in \triangle, \rho_{Y_\alpha} \subseteq \rho$ and $g_{pu}|_{C_{Y_\alpha}}$ is supra-soft continuous at each soft point $d_z \in SP(Y_\alpha, A)$. Then g_{pu} is supra-soft continuous.

Proof. Let $a_x \in SP(M, A)$. We show that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous at a_x. Since $\{Y_\alpha : \alpha \in \triangle\}$ is a cover of M, then there exists $\alpha_\circ \in \triangle$ such that $x \in Y_{\alpha_\circ}$ and so $a_x \in SP(Y_{\alpha_\circ}, A)$. Since $\rho_{Y_{\alpha_\circ}} \subseteq \rho$, then by Theorem 2.27, it follows that g_{pu} is supra-soft continuous at a_x.

Theorem 2.29. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs and let $\emptyset \neq Y \subseteq M$ such that $(\rho_Y)_\omega \subseteq \rho_\omega$. If there is $a_y \in SP(Y, A)$ such that $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous at a_y, then g_{pu} is supra-soft ω-continuous at a_y.

Proof. Let $G \in \lambda$ such that $g_{pu}(a_y) \widetilde{\in} G$. Since $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous at a_y, there exists $H \in (\rho_Y)_\omega$ such that $a_y \widetilde{\in} H$ and $g_{pu}(H) \widetilde{\subseteq} G$. Since by assumption $(\rho_Y)_\omega \subseteq \rho_\omega$, then $H \in \rho_\omega$. This shows that g_{pu} is supra-soft ω-continuous at a_y.

Corollary 2.30. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Let $\{Y_\alpha : \alpha \in \triangle\}$ be a cover of M such that for each $\alpha \in \triangle$, $(\rho_{Y_\alpha})_\omega \subseteq \rho_\omega$ and $(g_{pu})|_{C_{Y_\alpha}}$ is supra-soft ω-continuous at each soft point $d_z \in SP(Y_\alpha, A)$. Then g_{pu} is supra-soft ω-continuous.

Proof. Let $a_x \in SP(M, A)$. We show that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous at a_x. Since $\{Y_\alpha : \alpha \in \triangle\}$ is a cover of M, then there exists $\alpha_\circ \in \triangle$ such that $x \in Y_{\alpha_\circ}$ and so $a_x \in SP(Y_{\alpha_\circ}, A)$. Since $\left(\rho_{Y_{\alpha_\circ}}\right)_\omega \subseteq \rho_\omega$, then by Theorem 2.29, it follows that g_{pu} is supra-soft ω-continuous at a_x.

Theorem 2.31. Let (M, ρ, A) and (N, λ, B) be two supra-STSs with A and B are countable, and let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be supra-soft ω-continuous and surjective. If (M, ρ, A) is supra-soft Lindelof, then (N, λ, B) is supra-soft Lindelof.

Proof. Since $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous, then by Corollary 2.18, $g_{pu} : (M, \rho_\omega, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous. Since (M, ρ, A) is supra-soft Lindelof, then by Theorem 3.18 of [19], (M, ρ_ω, A) is supra-soft Lindelof. Proposition 4.15 of [24], ends the proof.

Definition 2.32. A soft map $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is called supra-soft ω-closed if $g_{pu}(F) \in (\lambda_\omega)^c$ for each $F \in \rho^c$.

Theorem 2.33. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a supra-soft ω-closed map such that for each $b_y \in SP(N, B)$, $g_{pu}^{-1}(b_y)$ is a supra-soft Lindelof subset of (M, ρ, A). If (N, λ, B) is supra-soft Lindelof, then (M, ρ, A) is supra-soft Lindelof.

Proof. Let $\mathcal{G} \subseteq \rho$ such that $1_A = \tilde{\cup}_{G \in \mathcal{G}} G$. For each $b_y \in SP(N,B)$, $g_{pu}^{-1}(b_y)$ is a supra-soft Lindelof subset of (M, ρ, A) and there exists a countable subcollection $\mathcal{G}_1(b_y)$ of $\mathcal{G}$ such that $g_{pu}^{-1}(b_y) \tilde{\subseteq} \tilde{\cup}_{G \in \mathcal{G}_1(b_y)} G$. For each $b_y \in SP(N,B)$, put $G(b_y) = \tilde{\cup}_{G \in \mathcal{G}_1(b_y)} G$ and $H(b_y) = 1_B - g_{pu}(1_A - G(b_y))$. Since g_{pu} is supra-soft ω-closed, then for each $b_y \in SP(N,B)$, $H(b_y) \in (\lambda_\omega)^c$ with $b_y \tilde{\in} H(b_y)$ and $g_{pu}^{-1}(H(b_y)) \tilde{\subseteq} G(b_y)$. Since $H(b_y) \in (\lambda_\omega)^c$, there exists $K(b_y) \in \lambda$ such that $b_y \tilde{\in} K(b_y)$ and $K(b_y) - H(b_y)$ is a countable soft set. For each $b_y \in SP(N,B)$, we have $K(b_y) \tilde{\subseteq} (K(b_y) - H(b_y)) \tilde{\cup} H(b_y)$ and so

$$\begin{aligned} g_{pu}^{-1}(K(b_y)) &\tilde{\subseteq} g_{pu}^{-1}(K(b_y) - H(b_y)) \tilde{\cup} g_{pu}^{-1}(H(b_y)) \\ &\tilde{\subseteq} g_{pu}^{-1}(K(b_y) - H(b_y)) \tilde{\cup} G(b_y). \end{aligned}$$

Since $K(b_y) - H(b_y)$ is a countable soft set and $g_{pu}^{-1}(b_y)$ is a supra-soft Lindelof subset of (M, ρ, A), there exists a countable a countable subcollection $\mathcal{G}_2(b_y)$ of $\mathcal{G}$ such that $g_{pu}^{-1}(K(b_y) - H(b_y)) \tilde{\subseteq} \tilde{\cup} \{G : G \in \mathcal{G}_2(b_y)\}$ and hence $g_{pu}^{-1}(K(b_y)) \tilde{\subseteq} \tilde{\cup} \{G : G \in \mathcal{G}_2(b_y)\} \tilde{\cup} [G(b_y)]$

Since $\{K(b_y) : b_y \in SP(N,B)\} \subseteq \lambda$, $1_B = \tilde{\cup}_{b_y \in SP(N,B)} K(b_y)$, and (N, λ, B) is supra-soft Lindelof, there exists a countable subset $S \subseteq SP(N,B)$ such that $1_B = \tilde{\cup} \{K(b_y) : b_y \in S\}$. Therefore,

$$\begin{aligned} 1_A &= \tilde{\cup} \left\{ g_{pu}^{-1}(K(b_y)) : b_y \in S \right\} \\ &= \tilde{\cup}_{b_y \in S} \left[\tilde{\cup} \{G : G \in \mathcal{G}_2(b_y)\} \right] \tilde{\cup} \left[\tilde{\cup} \{G : G \in \mathcal{G}_1(b_y)\} \right] \\ &= \tilde{\cup} \{G : G \in \mathcal{G}_1(b_y) \cup \mathcal{G}_2(b_y), b_y \in S\}. \end{aligned}$$

This shows that (M, ρ, A) is supra-soft Lindelof.

3 Supra-Soft ω-Irresoluteness

In this section, we introduce supra-soft ω-irresoluteness maps. We introduce several characterizations of them. Also, we introduce the correspondence between this concept and its analogous one in general topology. Moreover, we demonstrate how this distinct form of continuity behaves under common mathematical operations, such as composition and restriction.

Definition 3.1. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. A soft map $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is called supra-soft ω-irresolute at a soft point $a_x \in SP(M,A)$ if for every $G \in \lambda_\omega$ such that $g_{pu}(a_x) \tilde{\in} G$, there exists $H \in \rho_\omega$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. If g_{pu} is supra-soft ω-irresolute at each soft point $a_x \in SP(M,A)$, then g_{pu} is called supra-soft ω-irresolute.

Theorem 3.2. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute at $a_x \in SP(M,A)$, then $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-irresolute at x.

Proof. Let $V \in (\lambda_{u(a)})_\omega$ such that $p(x) \in V$. By Theorem 3.8 of [19], $V \in (\lambda_\omega)_{u(a)}$. Choose $G \in \lambda_\omega$ such that $G(u(a)) = V$. Then $g_{pu}(a_x) = (u(a))_{p(x)} \tilde{\in} G$. Since $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute at a_x, then there exists $H \in \rho_\omega$ such that $a_x \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. Thus, we have $x \in H(a) \in (\rho_\omega)_a$ and $p(H(a)) = (g_{pu}(H))(u(a)) \subseteq G(u(a)) = V$. On the other hand, by Theorem 3.8 of [19], $H(a) \in (\rho_a)_\omega$. This shows that $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-irresolute at x.

Corollary 3.3. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute, then $p : (M, \rho_a) \longrightarrow (N, \lambda_{u(a)})$ is supra ω-irresolute.

Theorem 3.4. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$ and $u : A \longrightarrow B$, where u is a bijection, and let $a_x \in SP(M,A)$. Then g_{pu} :

$(M,\otimes_{a\in A}\rho_a,A) \longrightarrow (N,\otimes_{b\in B}\lambda_b,B)$ is supra-soft ω-irresolute at a_x iff $p:(M,\rho_a) \longrightarrow \left(N,\lambda_{u(a)}\right)$ is supra ω-irresolute at x.

Proof. *Necessity*. Let $g_{pu}:(M,\otimes_{a\in A}\rho_a,A) \longrightarrow (N,\otimes_{b\in B}\lambda_b,B)$ be supra-soft ω-irresolute at a_x. Then by Theorem 3.2, $p:(M,(\otimes_{a\in A}\rho_a)_a) \longrightarrow \left(N,(\otimes_{b\in B}\lambda_b)_{u(a)}\right)$ is supra ω-irresolute at x. On the other hand, by Theorem 2.5 of [19], $(\otimes_{a\in A}\rho_a)_a=\rho_a$ and $(\otimes_{b\in B}\lambda_b)_{u(a)}=\lambda_{u(a)}$. It follows that $p:(M,\rho_a) \longrightarrow \left(N,\lambda_{u(a)}\right)$ is supra ω-irresolute at x.

Sufficiency. Let $p:(M,\rho_a) \longrightarrow \left(N,\lambda_{u(a)}\right)$ is supra ω-irresolute at x. Let $G\in(\otimes_{b\in B}\lambda_b)_\omega$ such that $g_{pu}(a_x)=(u(a))_{p(x)}\widetilde{\in}G$. Then by Theorem 3.10 of [19], $G\in\otimes_{b\in B}(\lambda_b)_\omega$. So, $p(x)\in G(u(a))\in(\lambda_{u(a)})_\omega$. Hence, there exists $U\in(\rho_a)_\omega$ such that $x\in U$ and $p(U)\subseteq G(u(a))$. By Theorem 3.10 of [19], $a_U\in(\otimes_{a\in A}\rho_a)_\omega$. Therefore, we have $a_x\widetilde{\in}a_U\in(\otimes_{a\in A}\rho_a)_\omega$ and $g_{pu}(a_U)\widetilde{\subseteq}G$. This proves that $g_{pu}:(M,\otimes_{a\in A}\rho_a,A) \longrightarrow (N,\otimes_{b\in B}\lambda_b,B)$ is supra-soft ω-irresolute at a_x.

Corollary 3.5. Let $\{(M,\rho_a):a\in A\}$ and $\{(N,\lambda_b):b\in B\}$ be two families of supra-TSs. Consider the maps $p:M\longrightarrow N$ and $u:A\longrightarrow B$, where u is a bijection. Then $g_{pu}:(M,\otimes_{a\in A}\rho_a,A) \longrightarrow (N,\otimes_{b\in B}\lambda_b,B)$ is supra-soft ω-irresolute iff $p:(M,\rho_a) \longrightarrow \left(N,\lambda_{u(a)}\right)$ is supra ω-irresolute for all $a\in A$.

Corollary 3.6. Let $p:(M,\aleph)\longrightarrow(N,\mathrm{Im})$ be a map between two supra-TSs and let $u:A\longrightarrow B$ be a bijective map. Let $a_x\in SP(M,A)$. Then $p:(M,\aleph)\longrightarrow(N,\mathrm{Im})$ is supra ω-irresolute at x iff $g_{pu}:(M,\mu(\aleph),A)\longrightarrow(N,\mu(\mathrm{Im}),B)$ is supra-soft ω-irresolute at a_x.

Proof. For each $a\in A$ and $b\in B$, let $\aleph_a=\aleph$ and $\mathrm{Im}_b=\mathrm{Im}$. Then $\mu(\aleph)=\otimes_{a\in A}\aleph_a$ and $\tau(\mathrm{Im})=\otimes_{b\in B}\mathrm{Im}_b$. Theorem 3.4 ends the proof.

Corollary 3.7. Let $p:(M,\aleph)\longrightarrow(N,\mathrm{Im})$ be a map between two supra-TSs and let $u:A\longrightarrow B$ be a bijective map. Then $p:(M,\aleph)\longrightarrow(N,\mathrm{Im})$ is supra ω-irresolute iff $g_{pu}:(M,\mu(\aleph),A)\longrightarrow(N,\mu(\mathrm{Im}),B)$ is supra-soft ω-irresolute.

Theorem 3.8. Let (M,ρ,A) and (N,λ,B) be two supra-STSs and let $a_x\in SP(M,A)$. Then a soft map $g_{pu}:(M,\rho,A)\longrightarrow(N,\lambda,B)$ is supra-soft ω-irresolute at a_x iff $g_{pu}:(M,\rho_\omega,A)\longrightarrow(N,\lambda_\omega,B)$ is supra-soft continuous at a_x.

Proof. Obvious.

Corollary 3.9. Let (M,ρ,A) and (N,λ,B) be two supra-STSs. Then a soft map $g_{pu}:(M,\rho,A)\longrightarrow(N,\lambda,B)$ is supra-soft ω-irresolute iff $g_{pu}:(M,\rho_\omega,A)\longrightarrow(N,\lambda_\omega,B)$ is supra-soft continuous.

Theorem 3.10. Let $g_{pu}:(M,\rho,A)\longrightarrow(N,\lambda,B)$ be a soft map between supra-STSs. Then the following conditions are equivalent:

(a) $g_{pu}:(M,\rho,A)\longrightarrow(N,\lambda,B)$ is supra-soft ω-irresolute.

(b) For each $G\in\lambda_\omega$, $g_{pu}^{-1}(G)\in\rho_\omega$.

(c) For each $F\in(\lambda_\omega)^c$, $g_{pu}^{-1}(F)\in(\rho_\omega)^c$.

(d) For each $T\in SS(M,A)$, $g_{pu}(Cl_{\rho_\omega}(T))\widetilde{\subseteq}Cl_{\lambda_\omega}(g_{pu}(T))$.

(e) For each $R\in SS(N,B)$, $Cl_{\rho_\omega}(g_{pu}^{-1}(R))\widetilde{\subseteq}g_{pu}^{-1}(Cl_{\lambda_\omega}(R))$.

Proof. (a) $\longrightarrow$ (b): Let $G\in\lambda_\omega$ and let $a_x\widetilde{\in}g_{pu}^{-1}(G)$. Then $g_{pu}(a_x)\widetilde{\in}G$. So, by (a), there exists $H\in\rho_\omega$ such that $a_x\widetilde{\in}H$ and $g_{pu}(H)\widetilde{\subseteq}G$. Therefore, we have $a_x\widetilde{\in}H\widetilde{\subseteq}g_{pu}^{-1}(g_{pu}(H))\widetilde{\subseteq}g_{pu}^{-1}(G)$. Hence, $g_{pu}^{-1}(G)\in\rho_\omega$.

(b) $\longrightarrow$ (c): Let $F \in (\lambda_\omega)^c$. Then $1_B - F \in \lambda_\omega$. So, by (b), $g_{pu}^{-1}(1_B - F) = 1_A - g_{pu}^{-1}(F) \in \rho_\omega$. Thus, $g_{pu}^{-1}(F) \in (\rho_\omega)^c$.

(c) $\longrightarrow$ (d): Let $T \in SS(M, A)$. Since $Cl_{\lambda_\omega}(g_{pu}(T)) \in (\lambda_\omega)^c$, then by (c), $g_{pu}^{-1}(Cl_{\lambda_\omega}(g_{pu}(T))) \in (\rho_\omega)^c$ and we have

$$\begin{array}{rcl} Cl_{\rho_\omega}(T) & \tilde{\subseteq} & Cl_{\rho_\omega}\left(g_{pu}^{-1}(g_{pu}(T))\right) \\ & \tilde{\subseteq} & Cl_{\rho_\omega}\left(g_{pu}^{-1}(Cl_{\lambda_\omega}(g_{pu}(T)))\right) \\ & = & g_{pu}^{-1}(Cl_{\lambda_\omega}(g_{pu}(T))). \end{array}$$

Therefore,

$$\begin{array}{rcl} g_{pu}(Cl_{\rho_\omega}(T)) & \tilde{\subseteq} & g_{pu}(g_{pu}^{-1}(Cl_{\lambda_\omega}(g_{pu}(T)))) \\ & \tilde{\subseteq} & Cl_{\lambda_\omega}(g_{pu}(T)). \end{array}$$

(d) $\longrightarrow$ (e): Let $R \in SS(N, B)$. By (d),

$$g_{pu}\left(Cl_{\rho_\omega}\left(g_{pu}^{-1}(R)\right)\right) \tilde{\subseteq} Cl_{\lambda_\omega}\left(g_{pu}\left(g_{pu}^{-1}(R)\right)\right) \tilde{\subseteq} Cl_{\lambda_\omega}(R).$$

Thus,

$$Cl_{\rho_\omega}\left(g_{pu}^{-1}(R)\right) \tilde{\subseteq} g_{pu}^{-1}(g_{pu}\left(Cl_{\rho_\omega}\left(g_{pu}^{-1}(R)\right)\right)) \tilde{\subseteq} g_{pu}^{-1}(Cl_{\lambda_\omega}(R)).$$

(e) $\longrightarrow$ (a): Let $a_x \in SP(M, A)$ and let $G \in \lambda_\omega$ such that $g_{pu}(a_x) \tilde{\in} G$. Since $G \in \lambda_\omega$, then $1_B - G \in (\lambda_\omega)^c$ and $Cl_{\lambda_\omega}(1_B - G) = 1_B - G$. So, by (e),

$$\begin{array}{rcl} Cl_{\rho_\omega}(1_A - g_{pu}^{-1}(G)) & = & Cl_{\rho_\omega}(g_{pu}^{-1}(1_B - G)) \\ & \tilde{\subseteq} & g_{pu}^{-1}(Cl_{\lambda_\omega}(1_B - G)) \\ & = & 1_A - g_{pu}^{-1}(G). \end{array}$$

Thus, $1_A - g_{pu}^{-1}(G) \in (\rho_\omega)^c$ and hence $g_{pu}^{-1}(G) \in \rho_\omega$. Set $H = g_{pu}^{-1}(G)$. Then we have $a_x \tilde{\in} H$, $H \in \rho_\omega$, and $g_{pu}(H) = g_{pu}\left(g_{pu}^{-1}(G)\right) \tilde{\subseteq} G$. This shows that g_{pu} is supra-soft ω-irresolute at a_x.

Theorem 3.11. Every supra-soft ω-irresolute map is supra-soft ω-continuous.

Proof. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be supra-soft ω-irresolute and let $G \in \lambda \subseteq \lambda_\omega$. Then $G \in \lambda_\omega$ and $g_{pu}^{-1}(G) \in \rho_\omega$. This shows that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous.

The reverse of Theorem 3.11 is not true:

Example 3.12. Let $M = \mathbb{R}$, $N = \{a, b\}$, $\aleph = \{\emptyset, M, M - \{2\}, M - \{1, 2\}\}$, $\mathrm{Im} = \{\emptyset, \{a\}, N\}$, and $A = \mathbb{Z}$. Define $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ by $p(t) = b$ if $t \in \{1, 2\}$ and $f(t) = a$ if $t \in M - \{1, 2\}$, and consider the identity map $u : A \longrightarrow A$. Then p is supra ω-continuous. On the other hand, since $\{b\} \in \mathrm{Im}_\omega$ while $p^{-1}(\{b\}) = \{1, 2\} \notin \aleph_\omega$, then p is not supra ω-irresolute. Therefore, by Corollaries 2.7 and 3.7, $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), A)$ is supra-soft ω-continuous but not supra-soft ω-irresolute.

Theorem 3.13. Let (M, ρ, A), (N, λ, B), and (L, γ, D) be three supra-STSs. If $g_{p_1 u_1} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute and $g_{p_2 u_2} : (N, \lambda, B) \longrightarrow (L, \gamma, D)$ is supra-soft ω-continuous, then $g_{(p_2 \circ p_1)(u_2 \circ u_1)} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ is supra-soft ω-continuous.

Proof. Let $G \in \gamma$. Since $g_{p_2 u_2}$ is supra-soft ω-continuous, then $g_{p_2 u_2}^{-1}(G) \in \lambda_\omega$. Since $g_{p_1 u_1}$ is supra-soft ω-irresolute, then by Theorem 3.10, $g_{p_1 u_1}^{-1}\left(g_{p_2 u_2}^{-1}(G)\right) = g_{(p_2 \circ p_1)(u_2 \circ u_1)}^{-1}(G) \in \rho_\omega$. Therefore, by Theorem 2.17, $g_{(p_2 \circ p_1)(u_2 \circ u_1)}$ is supra-soft ω-continuous.

Theorem 3.14. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute, then for any non-empty subset $Y \subseteq M$, $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute.

Proof. Let $G \in \lambda_\omega$. Since $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute, then $g_{pu}^{-1}(G) \in \rho_\omega$ and so $((g_{pu})|_{C_Y})^{-1}(G) = g_{pu}^{-1}(G) \tilde{\cap} C_Y \in (\rho_\omega)_Y$. Therefore, by Theorem 3.19 of [19], $((g_{pu})|_{C_Y})^{-1}(G) \in (\rho_Y)_\omega$. It follows that $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute.

Theorem 3.15. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between two supra-STSs and let $M = Y \cup Z$, where $\{C_Y, C_Z\} \subseteq (\rho_\omega)^c$. If $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ and $(g_{pu})|_{C_Z} : (Z, \rho_Z, A) \longrightarrow (N, \lambda, B)$ are supra-soft ω-irresolute, then $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute.

Proof. We are going to apply Theorem 3.10. Let $F \in (\lambda_\omega)^c$. Then

$$\begin{aligned} g_{pu}^{-1}(F) &= g_{pu}^{-1}(F) \tilde{\cap} 1_A \\ &= g_{pu}^{-1}(F) \tilde{\cap} (C_Y \tilde{\cup} C_Z) \\ &= \left(g_{pu}^{-1}(F) \tilde{\cap} C_Y\right) \tilde{\cup} \left(g_{pu}^{-1}(F) \tilde{\cap} C_Z\right) \\ &= \left(((g_{pu})|_{C_Y})^{-1}(F)\right) \tilde{\cup} \left(((g_{pu})|_{C_Z})^{-1}(F)\right). \end{aligned}$$

Since $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ and $(g_{pu})|_{C_Z} : (Z, \rho_Z, A) \longrightarrow (N, \lambda, B)$ are supra-soft ω-irresolute, then $\left(((g_{pu})|_{C_Y})^{-1}(F)\right) \in ((\rho_Y)_\omega)^c$ and $\left(((g_{pu})|_{C_Z})^{-1}(F)\right) \in ((\rho_Z)_\omega)^c$. Since by Theorem 3.19 of [19], $(\rho_Y)_\omega = (\rho_\omega)_Y$ and $(\rho_Z)_\omega = (\rho_\omega)_Z$, then $\left(((g_{pu})|_{C_Y})^{-1}(F)\right) \in ((\rho_\omega)_Y)^c$ and $\left(((g_{pu})|_{C_Z})^{-1}(F)\right) \in ((\rho_\omega)_Z)^c$. Therefore, by Lemma 2.21, $((g_{pu})|_{C_Y})^{-1}(F)$, $((g_{pu})|_{C_Z})^{-1}(F) \in (\rho_\omega)^c$. It follows that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute.

Theorem 3.16. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs and let $\emptyset \neq Y \subseteq M$ such that $(\rho_Y)_\omega \subseteq \rho_\omega$. If there is $a_y \in SP(Y, A)$ such that $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute at a_y, then g_{pu} is supra-soft ω-irresolute at a_y.

Proof. Let $G \in \lambda_\omega$ such that $g_{pu}(a_y) \tilde{\in} G$. Since $(g_{pu})|_{C_Y} : (Y, \rho_Y, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute at a_y, there exists $H \in (\rho_Y)_\omega$ such that $a_y \tilde{\in} H$ and $g_{pu}(H) \tilde{\subseteq} G$. Since by assumption $(\rho_Y)_\omega \subseteq \rho_\omega$, then $H \in \rho_\omega$. This shows that g_{pu} is supra-soft ω-irresolute at a_y.

Corollary 3.17. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Let $\{Y_\alpha : \alpha \in \triangle\}$ be a cover of M such that for each $\alpha \in \triangle$, $(\rho_{Y_\alpha})_\omega \subseteq \rho_\omega$ and $(g_{pu})|_{C_{Y_\alpha}}$ is supra-soft ω-irresolute at each soft point $d_z \in SP(Y_\alpha, A)$. Then g_{pu} is supra-soft ω-irresolute.

Proof. Let $a_x \in SP(M, A)$. We show that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-irresolute at a_x. Since $\{Y_\alpha : \alpha \in \triangle\}$ is a cover of M, then there exists $\alpha_\circ \in \triangle$ such that $x \in Y_{\alpha_\circ}$ and so $a_x \in SP(Y_{\alpha_\circ}, A)$. Since $\left(\rho_{Y_{\alpha_\circ}}\right)_\omega \subseteq \rho_\omega$, then by Theorem 3.16, it follows that g_{pu} is supra-soft ω-irresolute at a_x.

4 Supra-Soft Contra-Continuity and Contra-ω-Continuity

In this section, we introduce supra-soft contra-continuity and contra-ω-continuity. We introduce several characterizations of each of them. Also, we introduce the correspondence between this concept and its analogous one in general topology. Moreover, we demonstrate how this distinct form of continuity behaves under common mathematical operations, such as composition.

Definition 4.1. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. A soft map $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is called

(a) supra-soft contra-continuous if for every $G \in \lambda$, $g_{pu}^{-1}(G) \in \rho^c$.

(b) supra-soft contra-ω-continuous if for every $G \in \lambda$, $g_{pu}^{-1}(G) \in (\rho_\omega)^c$.

Theorem 4.2. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft contra-continuous, then $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-continuous.

Proof. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be supra-soft contra-continuous. Let $V \in \lambda_{u(a)}$. Choose $G \in \lambda$ such that $G(u(a)) = V$. Then $g_{pu}^{-1}(G) \in \rho^c$, and so, $p^{-1}(V) = (g_{pu}^{-1}(G))(a) \in (\rho_a)^c$. Hence, $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-continuous.

Theorem 4.3. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$ and $u : A \longrightarrow B$, where u is a bijection and let $a_x \in SP(M, A)$. Then $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ is supra-soft contra-continuous iff $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-continuous for all $a \in A$.

Proof. *Necessity*. Let $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ be supra-soft contra-continuous. Then by Theorem 4.2, $p : (M, (\otimes_{a\in A}\rho_a)_a) \longrightarrow \left(N, (\otimes_{b\in B}\lambda_b)_{u(a)}\right)$ is supra contra-continuous. On the other hand, by Theorem 2.5 of [19], $(\otimes_{a\in A}\rho_a)_a = \rho_a$ and $(\otimes_{b\in B}\lambda_b)_{u(a)} = \lambda_{u(a)}$. It follows that $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-continuous.

Sufficiency. Let $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ be supra contra-continuous for all $a \in A$. Let $G \in \otimes_{b\in B}\lambda_b$. We will show that $(g_{pu}^{-1}(G))(a) \in (\rho_a)^c$ for all $a \in A$. Let $a \in A$. Since $G \in \otimes_{b\in B}\lambda_b$, then $G(u(a)) \in \lambda_{u(a)}$. Since $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-continuous, then $(g_{pu}^{-1}(G))(a) = p^{-1}(G(u(a))) \in (\rho_a)^c$.

Corollary 4.4. Let $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ be a map between two supra-TSs and let $u : A \longrightarrow B$ be a bijective map. Then $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ is supra contra-continuous iff $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), B)$ is supra-soft contra-continuous.

Proof. For each $a \in A$ and $b \in B$, let $\aleph_a = \aleph$ and $\mathrm{Im}_b = \mathrm{Im}$. Then $\mu(\aleph) = \otimes_{a\in A}\aleph_a$ and $\tau(\mathrm{Im}) = \otimes_{b\in B}\mathrm{Im}_b$. Theorem 4.3 ends the proof.

Theorem 4.5. Let (M, ρ, A) and (N, λ, B) be two supra-STSs. If $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft contra-ω-continuous, then $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-ω-continuous.

Proof. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be supra-soft contra-ω-continuous. Let $V \in \lambda_{u(a)}$. Choose $G \in \lambda$ such that $G(u(a)) = V$. Then $g_{pu}^{-1}(G) \in (\rho_\omega)^c$, and so, $p^{-1}(V) = (g_{pu}^{-1}(G))(a) \in ((\rho_\omega)_a)^c = ((\rho_a)_\omega)^c$. Hence, $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-ω-continuous.

Theorem 4.6. Let $\{(M, \rho_a) : a \in A\}$ and $\{(N, \lambda_b) : b \in B\}$ be two families of supra-TSs. Consider the maps $p : M \longrightarrow N$ and $u : A \longrightarrow B$, where u is a bijection and let $a_x \in SP(M, A)$. Then $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ is supra-soft contra-ω-continuous iff $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-ω-continuous for all $a \in A$.

Proof. *Necessity*. Let $g_{pu} : (M, \otimes_{a\in A}\rho_a, A) \longrightarrow (N, \otimes_{b\in B}\lambda_b, B)$ be supra-soft contra-ω-continuous. Then by Theorem 4.5, $p : (M, (\otimes_{a\in A}\rho_a)_a) \longrightarrow \left(N, (\otimes_{b\in B}\lambda_b)_{u(a)}\right)$ is supra contra-ω-continuous. On the other hand, by Theorem 2.5 of [19], $(\otimes_{a\in A}\rho_a)_a = \rho_a$ and $(\otimes_{b\in B}\lambda_b)_{u(a)} = \lambda_{u(a)}$. It follows that $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ is supra contra-ω-continuous.

Sufficiency. Let $p : (M, \rho_a) \longrightarrow \left(N, \lambda_{u(a)}\right)$ be supra contra-continuous for all $a \in A$. Let $G \in \otimes_{b\in B}\lambda_b$. We will show that $(g_{pu}^{-1}(G))(a) \in ((\rho_a)_\omega)^c$ for all $a \in A$. Let $a \in A$. Since $G \in \otimes_{b\in B}\lambda_b$, then $G(u(a)) \in \lambda_{u(a)}$.

Since $p:(M,\rho_a)\longrightarrow\left(N,\lambda_{u(a)}\right)$ is supra contra-ω-continuous, then $(g_{pu}^{-1}(G))(a)=p^{-1}(G(u(a)))\in((\rho_a)_\omega)^c$.

Corollary 4.7. Let $p:(M,\aleph)\longrightarrow(N,\mathrm{Im})$ be a map between two supra-TSs and let $u:A\longrightarrow B$ be a bijective map. Then $p:(M,\aleph)\longrightarrow(N,\mathrm{Im})$ is supra contra-ω-continuous iff $g_{pu}:(M,\mu(\aleph),A)\longrightarrow(N,\mu(\mathrm{Im}),B)$ is supra-soft contra-ω-continuous.

Proof. For each $a\in A$ and $b\in B$, let $\aleph_a=\aleph$ and $\mathrm{Im}_b=\mathrm{Im}$. Then $\mu(\aleph)=\otimes_{a\in A}\aleph_a$ and $\tau(\mathrm{Im})=\otimes_{b\in B}\mathrm{Im}_b$. Theorem 4.6 ends the proof.

Theorem 4.8. Let $g_{pu}:(M,\rho,A)\longrightarrow(N,\lambda,B)$ be a soft map between supra-STSs. Then the following are equivalent:

(a) g_{pu} is supra-soft contra-continuous.

(b) For each $F\in\lambda^c$, $g_{pu}^{-1}(F)\in\rho$.

(c) For each $a_x\in SP(M,A)$ and each $F\in\lambda^c$ such that $g_{pu}(a_x)\widetilde{\in}F$, there exists $H\in\rho$ such that $a_x\widetilde{\in}H$ and $g_{pu}(H)\widetilde{\subseteq}F$.

Proof. (a) $\longrightarrow$ (b): Let $F\in\lambda^c$. Then $1_B-F\in\lambda$, and by (a), $1_A-g_{pu}^{-1}(F)=g_{pu}^{-1}(1_B-F)\in\rho^c$. Hence, $g_{pu}^{-1}(F)\in\rho$.

(b) $\longrightarrow$ (c): Let $a_x\in SP(M,A)$ and let $F\in\lambda^c$ such that $g_{pu}(a_x)\widetilde{\in}F$. Put $H=g_{pu}^{-1}(F)$. Then $a_x\widetilde{\in}H$ and, by (b), $H\in\rho$. Moreover, $g_{pu}(H)=g_{pu}\left(g_{pu}^{-1}(F)\right)\widetilde{\subseteq}F$. This ends the proof.

(c) $\longrightarrow$ (b): Let $F\in\lambda^c$. Then for each $a_x\widetilde{\in}g_{pu}^{-1}(F)$, $g_{pu}(a_x)\widetilde{\in}F$ and by (c), there exists $H_{a_x}\in\rho$ such that $a_x\widetilde{\in}H_{a_x}$ and $g_{pu}(H_{a_x})\widetilde{\subseteq}F$. Therefore, we obtain $g_{pu}^{-1}(F)=\widetilde{\cup}\left\{H_{a_x}:a_x\widetilde{\in}g_{pu}^{-1}(F)\right\}\in\rho$.

(b) $\longrightarrow$ (a): Let $G\in\lambda$. Then $1_B-G\in\lambda^c$, and by (b), $1_A-g_{pu}^{-1}(G)=g_{pu}^{-1}(1_B-G)\in\rho$. Therefore, $g_{pu}^{-1}(G)\in\rho^c$. Hence, g_{pu} is supra-soft contra-continuous.

Theorem 4.9. Let $g_{pu}:(M,\rho,A)\longrightarrow(N,\lambda,B)$ be a soft map between supra-STSs. Then the following are equivalent:

(a) g_{pu} is supra-soft contra-ω-continuous.

(b) For each $F\in\lambda^c$, $g_{pu}^{-1}(F)\in\rho_\omega$.

(c) For each $a_x\in SP(M,A)$ and each $F\in\lambda^c$ such that $g_{pu}(a_x)\widetilde{\in}F$, there exists $H\in\rho_\omega$ such that $a_x\widetilde{\in}H$ and $g_{pu}(H)\widetilde{\subseteq}F$.

Proof. (a) $\longrightarrow$ (b): Let $F\in\lambda^c$. Then $1_B-F\in\lambda$, and by (a), $1_A-g_{pu}^{-1}(F)=g_{pu}^{-1}(1_B-F)\in(\rho_\omega)^c$. Hence, $g_{pu}^{-1}(F)\in\rho_\omega$.

(b) $\longrightarrow$ (c): Let $a_x\in SP(M,A)$ and let $F\in\lambda^c$ such that $g_{pu}(a_x)\widetilde{\in}F$. Put $H=g_{pu}^{-1}(F)$. Then $a_x\widetilde{\in}H$ and, by (b), $H\in\rho_\omega$. Moreover, $g_{pu}(H)=g_{pu}\left(g_{pu}^{-1}(F)\right)\widetilde{\subseteq}F$. This ends the proof.

(c) $\longrightarrow$ (b): Let $F\in\lambda^c$. Then for each $a_x\widetilde{\in}g_{pu}^{-1}(F)$, $g_{pu}(a_x)\widetilde{\in}F$ and by (c), there exists $H_{a_x}\in\rho_\omega$ such that $a_x\widetilde{\in}H_{a_x}$ and $g_{pu}(H_{a_x})\widetilde{\subseteq}F$. Therefore, we obtain $g_{pu}^{-1}(F)=\widetilde{\cup}\left\{H_{a_x}:a_x\widetilde{\in}g_{pu}^{-1}(F)\right\}\in\rho_\omega$.

(b) $\longrightarrow$ (a): Let $G\in\lambda$. Then $1_B-G\in\lambda^c$, and by (b), $1_A-g_{pu}^{-1}(G)=g_{pu}^{-1}(1_B-G)\in\rho_\omega$. Therefore, $g_{pu}^{-1}(G)\in(\rho_\omega)^c$. Hence, g_{pu} is supra-soft contra-ω-continuous.

Theorem 4.10. Every supra-soft contra-continuous map is supra-soft contra-ω-continuous.

Proof. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be supra-soft contra-continuous and let $G \in \lambda$. Then $g_{pu}^{-1}(G) \in \rho^c \subseteq (\rho_\omega)^c$. This shows that $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft contra-ω-continuous.

The reverse of Theorem 4.10 is not true:

Example 4.11. Let $M = N = \{a, b, c, d\}$, $\aleph = \{\varnothing, \{a\}, \{d\}, \{a, d\}, M\}$, $\mathrm{Im} = \{\varnothing, \{c\}, \{b\}, \{c, b\}, N\}$, and $A = \mathbb{R}$. Consider the identity maps $p : (M, \aleph) \longrightarrow (N, \mathrm{Im})$ and $u : A \longrightarrow A$. Then $g_{pu} : (M, \mu(\aleph), A) \longrightarrow (N, \mu(\mathrm{Im}), A)$ is supra-soft contra-ω-continuous but not supra-soft contra-continuous.

Definition 4.12. Let (M, ρ, A) be a supra-STS and let $R \in SS(M, A)$. The soft set $\widetilde{\cap} \left\{H \in \rho : R \widetilde{\subseteq} H\right\}$ is called the supra soft kernel of R in (M, ρ, A), and is denoted by $ker_\rho(R)$.

Theorem 4.13. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Then the following are equivalent:

(a) g_{pu} is supra-soft contra-continuous.

(b) $g_{pu}(Cl_\rho(R)) \widetilde{\subseteq} ker_\lambda(g_{pu}(R))$ for every $R \in SS(M, A)$.

(c) $Cl_\rho\left(g_{pu}^{-1}(T)\right) \widetilde{\subseteq} g_{pu}^{-1}(ker_\lambda(T))$ for every $T \in SS(N, B)$.

Proof. (a) $\longrightarrow$ (b): Let $R \in SS(M, A)$. Let $b_y \widetilde{\notin} ker_\lambda(g_{pu}(R))$. Then, there exists $G \in \lambda$ such that $g_{pu}(R) \widetilde{\subseteq} G$ and $b_y \widetilde{\notin} G$. Let $F = 1_B - G$. Then we have $b_y \widetilde{\in} F \in \lambda^c$ and $F \widetilde{\cap} g_{pu}(R) = 0_B$. Thus, $R \widetilde{\cap} g_{pu}^{-1}(F) = 0_A$. By (a) and Theorem 4.8, $g_{pu}^{-1}(F) \in \rho$. So, $Cl_\rho(R) \widetilde{\cap} g_{pu}^{-1}(F) = 0_A$. Therefore, we obtain $g_{pu}(Cl_\rho(R)) \widetilde{\cap} F = 0_B$ and hence $b_y \widetilde{\notin} g_{pu}(Cl_\rho(R))$. This shows that $g_{pu}(Cl_\rho(R)) \widetilde{\subseteq} ker_\lambda(g_{pu}(R))$.

(b) $\longrightarrow$ (c): Let $T \in SS(N, B)$. Then by (b),

$g_{pu}(Cl_\rho\left(g_{pu}^{-1}(T)\right)) \widetilde{\subseteq} ker_\lambda\left(g_{pu}(g_{pu}^{-1}(T))\right) \widetilde{\subseteq} ker_\lambda(T)$. Thus,

$Cl_\rho\left(g_{pu}^{-1}(T)\right) \widetilde{\subseteq} g_{pu}^{-1}(g_{pu}(Cl_\rho\left(g_{pu}^{-1}(T)\right))) \widetilde{\subseteq} g_{pu}^{-1}(ker_\lambda(T))$.

(c) $\longrightarrow$ (a): Let $G \in \lambda$. Then $ker_\lambda(G) = G$ and by (c),

$Cl_\rho\left(g_{pu}^{-1}(G)\right) \widetilde{\subseteq} g_{pu}^{-1}(ker_\lambda(G)) = g_{pu}^{-1}(G)$. Therefore, $g_{pu}^{-1}(G) \in \rho^c$.

Theorem 4.14. Let $g_{pu} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ be a soft map between supra-STSs. Then the following are equivalent:

(a) g_{pu} is supra-soft contra-ω-continuous.

(b) $g_{pu}(Cl_{\rho_\omega}(R)) \widetilde{\subseteq} ker_\lambda(g_{pu}(R))$ for every $R \in SS(M, A)$.

(c) $Cl_{\rho_\omega}\left(g_{pu}^{-1}(T)\right) \widetilde{\subseteq} g_{pu}^{-1}(ker_\lambda(T))$ for every $T \in SS(N, B)$.

Proof. (a) $\longrightarrow$ (b): Let $R \in SS(M, A)$. Let $b_y \widetilde{\notin} ker_\lambda(g_{pu}(R))$. Then, there exists $G \in \lambda$ such that $g_{pu}(R) \widetilde{\subseteq} G$ and $b_y \widetilde{\notin} G$. Let $F = 1_B - G$. Then we have $b_y \widetilde{\in} F \in \lambda^c$ and $F \widetilde{\cap} g_{pu}(R) = 0_B$. Thus, $R \widetilde{\cap} g_{pu}^{-1}(F) = 0_A$. By (a) and Theorem 4.9, $g_{pu}^{-1}(F) \in \rho_\omega$. So, $Cl_{\rho_\omega}(R) \widetilde{\cap} g_{pu}^{-1}(F) = 0_A$. Therefore, we obtain $g_{pu}(Cl_{\rho_\omega}(R)) \widetilde{\cap} F = 0_B$ and hence $b_y \widetilde{\notin} g_{pu}(Cl_{\rho_\omega}(R))$. This shows that $g_{pu}(Cl_{\rho_\omega}(R)) \widetilde{\subseteq} ker_\lambda(g_{pu}(R))$.

(b) $\longrightarrow$ (c): Let $T \in SS(N, B)$. Then by (b),

$g_{pu}(Cl_{\rho_\omega}\left(g_{pu}^{-1}(T)\right)) \widetilde{\subseteq} ker_\lambda\left(g_{pu}(g_{pu}^{-1}(T))\right) \widetilde{\subseteq} ker_\lambda(T)$. Thus,

$Cl_{\rho_\omega}\left(g_{pu}^{-1}(T)\right) \tilde{\subseteq} g_{pu}^{-1}(g_{pu}(Cl_\rho\left(g_{pu}^{-1}(T)\right))) \tilde{\subseteq} g_{pu}^{-1}\left(ker_\lambda\left(T\right)\right)$.

(c) $\longrightarrow$ (a): Let $G \in \lambda$. Then $ker_\lambda\left(G\right) = G$ and by (c),

$Cl_{\rho_\omega}\left(g_{pu}^{-1}(G)\right) \tilde{\subseteq} g_{pu}^{-1}\left(ker_\lambda\left(G\right)\right) = g_{pu}^{-1}\left(G\right)$. Therefore, $g_{pu}^{-1}\left(G\right) \in (\rho_\omega)^c$.

Theorem 4.15. Let (M, ρ, A), (N, λ, B), and (L, γ, D) be three supra-STSs. If $g_{p_1u_1} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft contra-continuous and $g_{p_2u_2} : (N, \lambda, B) \longrightarrow (L, \gamma, D)$ is supra-soft continuous, then $g_{(p_2\circ p_1)(u_2\circ u_1)} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ is supra-soft contra-continuous.

Proof. Let $G \in \gamma$. Since $g_{p_2u_2}$ is supra-soft continuous, then $g_{p_2u_2}^{-1}(G) \in \lambda$. Since $g_{p_1u_1}$ is supra-soft contra-continuous, $g_{p_1u_1}^{-1}\left(g_{p_2u_2}^{-1}(G)\right) = g_{(p_2\circ p_1)(u_2\circ u_1)}^{-1}\left(G\right) \in \rho^c$. Therefore, $g_{(p_2\circ p_1)(u_2\circ u_1)}$ is supra-soft contra-continuous.

Theorem 4.16. Let (M, ρ, A), (N, λ, B), and (L, γ, D) be three supra-STSs. If $g_{p_1u_1} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft continuous and $g_{p_2u_2} : (N, \lambda, B) \longrightarrow (L, \gamma, D)$ is supra-soft contra-continuous, then $g_{(p_2\circ p_1)(u_2\circ u_1)} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ is supra-soft contra-continuous.

Proof. Let $G \in \gamma$. Since $g_{p_2u_2}$ is supra-soft contra-continuous, then $g_{p_2u_2}^{-1}(G) \in \lambda^c$. Since $g_{p_1u_1}$ is supra-soft continuous, then $g_{p_1u_1}^{-1}\left(g_{p_2u_2}^{-1}(G)\right) = g_{(p_2\circ p_1)(u_2\circ u_1)}^{-1}\left(G\right) \in \rho^c$. Therefore, $g_{(p_2\circ p_1)(u_2\circ u_1)}$ is supra-soft contra-continuous.

Theorem 4.17. Let (M, ρ, A), (N, λ, B), and (L, γ, D) be three supra-STSs. If $g_{p_1u_1} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft contra-ω-continuous and $g_{p_2u_2} : (N, \lambda, B) \longrightarrow (L, \gamma, D)$ is supra-soft continuous, then $g_{(p_2\circ p_1)(u_2\circ u_1)} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ is supra-soft contra-ω-continuous.

Proof. Let $G \in \gamma$. Since $g_{p_2u_2}$ is supra-soft continuous, then $g_{p_2u_2}^{-1}(G) \in \lambda$. Since $g_{p_1u_1}$ is supra-soft contra-ω-continuous, $g_{p_1u_1}^{-1}\left(g_{p_2u_2}^{-1}(G)\right) = g_{(p_2\circ p_1)(u_2\circ u_1)}^{-1}\left(G\right) \in (\rho_\omega)^c$. Therefore, $g_{(p_2\circ p_1)(u_2\circ u_1)}$ is supra-soft contra-ω-continuous.

Theorem 4.18. Let (M, ρ, A), (N, λ, B), and (L, γ, D) be three supra-STSs. If $g_{p_1u_1} : (M, \rho, A) \longrightarrow (N, \lambda, B)$ is supra-soft ω-continuous and $g_{p_2u_2} : (N, \lambda, B) \longrightarrow (L, \gamma, D)$ is supra-soft contra-continuous, then $g_{(p_2\circ p_1)(u_2\circ u_1)} : (M, \rho, A) \longrightarrow (L, \gamma, D)$ is supra-soft contra-ω-continuous.

Proof. Let $G \in \gamma$. Since $g_{p_2u_2}$ is supra-soft contra-continuous, then $g_{p_2u_2}^{-1}(G) \in \lambda^c$. Since $g_{p_1u_1}$ is supra-soft ω-continuous, then $g_{p_1u_1}^{-1}\left(g_{p_2u_2}^{-1}(G)\right) = g_{(p_2\circ p_1)(u_2\circ u_1)}^{-1}\left(G\right) \in (\rho_\omega)^c$. Therefore, $g_{(p_2\circ p_1)(u_2\circ u_1)}$ is supra-soft contra-ω-continuous.

5 Conclusion

In this paper, we introduced four different types of supra-soft continuity: supra-soft ω-continuity, supra-soft ω-irresoluteness, supra-soft contra-continuity, and supra-soft contra-ω-continuity. We characterized each of them in different ways. Moreover, we look at the relationship between these novel concepts and their analogue supra-topological concepts. Furthermore, we demonstrated how they are retained under specific compositions and restrictions. Finally, we explore some of the connections between these novel notions and well-known related concepts.

We intend to do the following in the next papers:

(i) Extend supra ω-Hausdorff spaces to include supra-STSs.

(ii) Explore how our new notions and results can be applied in digital and approximation spaces, as well as decision-making problems.

6 Author's contributions

All of the contributors wrote this article in collaboration. The final manuscript was read and approved by all writers.

7 Conflicts of interest

There are no competing interests declared by the authors.

References

[1] Molodtsov, D. Soft set theory—first results. *Comput. Math. Appl.* **1999**, *37*, 19–31.

[2] Dalkılıc, O.; Demirtas, N. Algorithms for covid-19 outbreak using soft set theory: estimation and application. *Soft Comput.* **2023**, *27*, 3203–3211.

[3] Santos-Garcıa, G.; Alcantud, J.C.R. Ranked soft sets. *Expert Syst.* **2023**, *40*, 1–24.

[4] Shabir, M.; Naz, M. On soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799.

[5] Mashhour, A.S.; Allam, A.A.; Mahmoud, F.S.; Khedr, F.H. On supra topological spaces. *Indian J. Pure Appl. Math.* **1983**, *14*, 502–510.

[6] El-Shafei, M.E.; Zakari, A.H.; Al-shami, T.M. Some applications of supra preopen sets. *J. Math.* **2020**, *2020*, 9634206.

[7] Jayanthi, D.; Rajesh, R. Soft sets in decision-making: An overview. *Soft Comput.* **2022**, *25*, 12345–12358.

[8] Kumar, A.; Gupta, V. Recent advancements in soft set theory and its applications in decision-making. *Int. J. Comput. Math.* **2022**, , *99* , 789–803.

[9] Ghaloob, E.A.; Sadek, A.R. On β*-supra topological spaces. *J. Discret. Math. Sci. Cryptogr.* **2023**, *26*, 1835–1841.

[10] Singh, R.; Yadav, S. A new approach to soft topological spaces. *Asian J. Math. Stat*. **2023**, *12*, 34–45.

[11] Kozae, A.M.; Shokry, M.; Zidan, M. Supra topologies for digital plane. AASCIT Commun. **2016**, 3, 1–10.

[12] El-Sheikh, S.A.; Abd El-Latif, A.M. Decompositions of some types of supra soft sets and soft continuity. *Int. J. Math. Trends Technol. **2014**, *9*, 37–56.

[13] El-Shafei, M.E.; Abo-Elhamayel, M.; Al-shami, T.M. Further notions related to new operators and compactness via supra soft topological spaces. *Int. J. Adv. Math.* **2019**, *1*, 44–60.

[14] Aras, C.G.; Bayramov, S. Separation axioms in supra soft topological spaces. *Filomat* **2018**, *32*, 3479–3486.

[15] Aras, C.G.; Bayramov, S. Results of some separation axioms in supra soft topological spaces. *TWMS J. Appl. Eng. Math*. **2019**, *9*, 58–63.

[16] Chen, Y.; Zhao, H. Hybrid soft set models for multi-criteria decision-making problems. *J. Math. Anal.Appl.* **2024**, *517*, 120–135.

[17] Gupta, P.; Sharma, A. Applications of soft sets in optimization problems. *J. Optim. Theory Appl* **2023**, *188*, 123–140.

[18] Al Ghour, S.; Hamed, W. On two classes of soft sets in soft topological spaces. *Symmetry* **2020**, *12*, 265.

[19] Abuzaid, D.; Al-Ghour, S. Supra soft omega-open sets and supra soft omega-regularity. *AIMS Math*. **2025**, *10*, 6636–6651.

[20] Csaszar, A. Generalized topology, generalized continuity. *Acta Math. Hung*. **2002**, 96, 351–357.

[21] Al Ghour, S.; Zareer, W. Omega open sets in generalized topological spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3010–3017.

[22] Al Ghour, S.; Zareer, W. Some mappings and separation axioms via omega open sets in generalized topological spaces. *Nonlinear Stud.* **2019**, *26*, 609–616.

[23] Jayanthi, D. Contra continuity on generalized topological spaces. *Acta Math. Hung.* **2012**, *137*, 263–271.

[24] Al-shami, T.M. Some results related to supra topological spaces. *J. Adv. Stud. Topol.* **2016**, *7*, 283–294.