Jordan Endo Bi-AntiDerivation of 2-Torison Free Rings and Neutrosophic Rings

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Abstract

Let $\mathcal{M}$ be the direct product of an associative ring Ʀ. In the work the concepts of Endo Bi-Antiderivation, Jordan Endo Bi-Antiderivation and Quasi Endo Bi-Antiderivation on a ring $\mathcal{M}$ are introduced, furthermore the relations between these bi-additive mappings are given. As essential point, we searched for appropriate conditions that make equivalence between Jordan Endo Bi-Antiderivation and Quasi Endo Bi-Antiderivation. Also, we prove the same results for the generalized case of neutrosophic rings.

Keywords: Direct product of ring; Prime rings; Bi-additive mapping; Neutrosophic ring

1. Introduction

In present work, we continue the series of papers concerning new bi-additive mapping of prime rings. For any $u, \omega \in$ Ʀ, the commutator $u\omega - \omega u$ symbolized by $[u, \omega]$ [1]. An m-torsion free ring R is a ring with the characteristic that $m\omega = 0$, for $\omega \in$ R implies that $\omega = 0$, where m is a positive integer [2]. Ʀ is known as prim -ring if $u, \omega \in$ Ʀ such that uƦ$\omega = (0)$ this implies either $u = 0$ or $\omega = 0$ [3].Take note that the statements below are equivalent:

- Ʀ is prime – ring.
- The zero ideal of Ʀ is a prime – ideal in the non-abelian sense.
- Ʀ is one where the right annihilator of non-– zero ideal is merely $\{0\}$.

Ʀ is known as semi primeness if u Ʀ $u = (0)$ that is $u = 0$ [3]. An additive map ρ:Ʀ ⟼ Ʀ is known as a relatively commuting if $[\rho(u), u]^n = 0$, $\forall u \in$ Ʀ, $n \geq 2$ and $n \in$ Z [4] While if $(n = 1)$, then ρ is known as commuting on Ʀ [5]. We are frequent uses of the commutator identities $[u\,\omega, \mathfrak{r}] = [u, \mathfrak{r}]\,\omega + u\,[\omega, \mathfrak{r}]$, and $[u, \omega\,\mathfrak{r}] = [u, \mathfrak{r}] + \mathfrak{r}[u, \omega]$[6]. Additive map $\mathfrak{D}$: Ʀ ⟼ Ʀ is known derivation if $\mathcal{D}(u\mathfrak{r}) = \mathcal{D}(u)\,\mathfrak{r} + u\,\mathcal{D}(\mathfrak{r})$ $\forall$ $u, \mathfrak{r} \in$ Ʀ[7] . Posner [7] gave the first key result on concerning the centralizing $\mathfrak{V}$ commuting maps . He demonstrates that the existences of nontrivial centralizing derivation of a prime ring Ʀ requires Ʀ must be commutative. A biadditive map $\mathcal{L}$: Ʀ × Ʀ ⟶ Ʀ is known as Symmetric if $\mathcal{L}(u, v) = \mathcal{L}(v, u)$ correct $\forall u, v \in$ Ʀ [8] . $\mathfrak{D}$: Ʀ × Ʀ ⟶ Ʀ is known symmetric biderivation if $\mathfrak{D}(u\,\omega, v) = \mathfrak{D}(u, v)\,\omega + u\,\mathfrak{D}(\omega, v)$ satisfy $\forall$ $u, v, \omega \in$ Ʀ [9]. Note that any commuting mapping ρ give space to define a symmetric biderivation given by $\mathfrak{D}(u, v) = [\rho(u), v]$, for each $u, v \in$ R [10].

Neutrosophic rings as generalizations of classical rings defined and studied by many authors [12-13]. The main idea of a neutrosophic ring is to extend the classical ring R by adding a logical element refers to indeterminacy [14].

The main goal in the present research is to introduce some new double- acting bi-additive mappings related with derivations and ant derivations. In addition, the same results will be discussed on the generalized case of neutrosophic rings.

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2. Preliminaries

Definition 2 .1: Let $\mathcal{M}$ be the direct product of a ring $\mathcal{R}$. A bi-additive map $g : \mathcal{M} \longrightarrow \mathcal{M}$ is called Endo Bi-Antiderivation if the following holds:

$$\mathcal{T}(v\omega, st) = \mathcal{T}(\omega, t)(v, s) + (\omega, t)\mathcal{T}(v, s),$$

for all $s, t, v, \omega \in \mathcal{R}$.

A map $\mathcal{T}$ is known as Jordan Endo Bi-Antiderivation on $\mathcal{M}$ if the following holds:

$$\mathcal{T}(\omega^2, s^2) = \mathcal{T}(\omega, s)(\omega, s) + (\omega, s)\mathcal{T}(\omega, s), \text{ for all } \omega, s \in \mathcal{R}.$$

Example 2.2: Let $\mathcal{M}$ be the direct product of the ring $\mathcal{R}$, where $\mathcal{R}=\left\{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in Q\right\}$.

Then the bi-additive $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ define by:

$$\mathcal{T}\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}\right), \text{ for all } \left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) \in \mathcal{M}.$$

is an Endo Bi-Antiderivation on $\mathcal{M}$.

Example 2.3: If ρ is an Antiderivation of ring $\mathcal{R}$ and $\mathcal{M}$ is the direct product of $\mathcal{R}$ and $\mathcal{T}$ be a bi- additive mappings on $\mathcal{S}$ defined by

$$\mathcal{T}(r, s) = (\rho(r), \rho(s)), \text{ for all } (r, s) \in \mathcal{S}.$$

Then $\mathcal{T}$ is Endo Bi-Antiderivation on $\mathcal{M}$.

Remark 2.4:

Endo Bi-Antiderivation

Jordan Endo Bi-Antiderivation.

In general, the opposite is not true.

Example 2.5:

Let $\mathcal{R}=\left\{\begin{pmatrix} f & g \\ \alpha & h \end{pmatrix}, f, g, h \in \mathcal{N}, \text{and } \alpha \in I\right\}$, where $\mathcal{N}= \mathbb{Q}[x]$ is the polynomial ring with property that $s^2=0$, and I be an ideal of $\mathcal{N}$ generated by s.

When $\mathcal{M}$ direct product of $\mathcal{R}$ and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ be a bi-additve mapping define by:

$$\mathcal{T}\left(\begin{pmatrix} f & g \\ \alpha & h \end{pmatrix}, \begin{pmatrix} p & q \\ \beta & k \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right).$$

Then $\mathcal{T}$ is a Jordan Endo Bi-Antiderivation but not Endo Bi-Antiderivation.

Definition 2.6: Let $\mathcal{M}$ be the direct product of a ring $\mathcal{R}$. A bi-additive map $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ is called a Quasi Endo Bi-Antiderivation if the following holds:

$$\mathcal{T}(v\omega, n^2) = \mathcal{T}(\omega, n)(v, n) + (\omega, n)\mathcal{T}(v, n),$$

for all $n, v, \omega \in \mathcal{R}$.

Remarks 2.7:

Endo Bi-Antiderivation

Quasi Endo Bi-Antiderivation

Jordan Endo Bi-Antiderivation

In general, the converse is not true

Lemma 2.8: [11] If $\mathcal{R}$ is 2- torsion free semiprime ring and $a, b, \omega \in \mathcal{R}$ the relation

$$a\omega b + b\omega a = 0$$

holds, hence $a\omega b = b\omega a = 0$ is fulfilled , $\forall \omega \in \mathcal{R}$.

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Lemma 2.9: Suppose $\mathcal{M}$ is the direct product of a ring Ʀ and $\mathcal{T}$ is a Jordan Endo Ɓi-Antiderivation on $\mathcal{M}$, then for all a ,$б$, $ɥ$, v, z ∈ Ʀ.

i- $\mathcal{T}(ɥ^2, aб + бa) = \mathcal{T}(ɥ, б)(ɥ, a) + (ɥ, б)\mathcal{T}(ɥ, a) + \mathcal{T}(ɥ, a)(ɥ, б) + (ɥ, a)\mathcal{T}(ɥ, б)$

ii- $\mathcal{T}(aб + бa, ɥ^2) = \mathcal{T}(б, ɥ)(a, ɥ) + (б, ɥ)\mathcal{T}(a, ɥ) + \mathcal{T}(a, ɥ)(б, ɥ) + (a, ɥ)\mathcal{T}(б, ɥ)$

iii- $\mathcal{T}(ɥv + vɥ, aб + бa) = \mathcal{T}(v, б)(ɥ, a) + (v, б)\mathcal{T}(ɥ, a) + \mathcal{T}(ɥ, б)(v, a) + (ɥ, б)\mathcal{T}(v, a) + \mathcal{T}(ɥ, a)(v, б) + (ɥ, a)\mathcal{T}(v, б) + \mathcal{T}(v, a)(ɥ, б) + (v, a)\mathcal{T}(ɥ, б)$

iv- $\mathcal{T}(aбa, ɥ^3) = \mathcal{T}(a, ɥ)(б, ɥ)(a, ɥ) + (a, ɥ)\mathcal{T}(б, ɥ)(a, ɥ) + (a, ɥ)(б, ɥ)\mathcal{T}(a, ɥ)$

v- $\mathcal{T}(ɥ^3, aбa) = \mathcal{T}(ɥ, a)(ɥ, б)(ɥ, a) + (ɥ, a)\mathcal{T}(ɥ, б)(ɥ, a) + (a, ɥ)(ɥ, б)\mathcal{T}(ɥ, a)$

vi- $\mathcal{T}(ɥ^3, azб + бza) = \mathcal{T}(ɥ, б)(ɥ, z)(ɥ, a) + (ɥ, б)\mathcal{T}(ɥ, z)(ɥ, a) + (ɥ, б)(ɥ, z)\mathcal{T}(ɥ, a) + \mathcal{T}(ɥ, a)(ɥ, z)(ɥ, б) + (ɥ, a)\mathcal{T}(ɥ, z)(ɥ, б) + (ɥ, a)(ɥ, z)\mathcal{T}(ɥ, б)$.

vii- $\mathcal{T}(azб + бza, ɥ^3) = \mathcal{T}(б, ɥ)(z, ɥ)(a, ɥ) + (б, ɥ)\mathcal{T}(z, ɥ)(a, ɥ) + (б, ɥ)(z, ɥ)\mathcal{T}(a, ɥ) + \mathcal{T}(a, ɥ)(z, ɥ)(б, ɥ) + (a, ɥ)\mathcal{T}(z, ɥ)(б, ɥ) + (a, ɥ)(z, ɥ)\mathcal{T}(б, ɥ)$

Proof: **(i)** For $\mathcal{T}$ is a Jordan Endo Ɓi-Antiderivation, we have:

$\mathcal{T}(ɥ^2, (a + б)^2) = \mathcal{T}(ɥ, a + б)(ɥ, a + б) + (ɥ, a + б)\mathcal{T}(ɥ, a + б)$

$= \mathcal{T}(ɥ, a)(ɥ, a) + \mathcal{T}(ɥ, a)(ɥ, б) + \mathcal{T}(ɥ, б)(ɥ, a) + \mathcal{T}(ɥ, б)(ɥ, б) + (ɥ, a)\mathcal{T}(ɥ, a) + (ɥ, б)\mathcal{T}(ɥ, a) + (ɥ, a)\mathcal{T}(ɥ, б) + (ɥ, б)\mathcal{T}(ɥ, б)$ …….. (1)

From another standpoint

$\mathcal{T}(ɥ^2, (a + б)^2) = \mathcal{T}(ɥ^2, a^2 + aб + бa + б^2) = \mathcal{T}(ɥ^2, a^2) + \mathcal{T}(ɥ^2, aб + бa) + \mathcal{T}(ɥ^2, б^2)$

$= \mathcal{T}(ɥ^2, a^2) + \mathcal{T}(ɥ^2, aб + бa) + \mathcal{T}(ɥ^2, б^2)$

$= \mathcal{T}(ɥ, a)(ɥ, a) + (ɥ, a)\mathcal{T}(ɥ, a) + \mathcal{T}(ɥ^2, aб + бa) + \mathcal{T}(ɥ, б)(ɥ, б) + (ɥ, б)\mathcal{T}(u, б)$

With an instant consideration to the last two relations we can conclude that:

$\mathcal{T}(ɥ^2, aб + бa) = \mathcal{T}(ɥ, б)(ɥ, a) + (ɥ, б)\mathcal{T}(ɥ, a) + \mathcal{T}(ɥ, a)(ɥ, б) + (ɥ, a)\mathcal{T}(ɥ, б)$

(iii)- A linearization of (i) gives:

$\mathcal{T}((v + \omega)^2, aб + бa) = \mathcal{T}(v + \omega, б)(v + \omega, a) + (v + \omega, a)\mathcal{T}(v + \omega, б) + \mathcal{T}(v + \omega, a)(v + \omega, б) + (v + \omega, б)\mathcal{T}(v + \omega, a)$

$= \mathcal{T}(v, б)(v, a) + \mathcal{T}(v, б)(\omega, a) + \mathcal{T}(\omega, б)(v, a) + \mathcal{T}(\omega, б)(\omega, a) + (v, a)\mathcal{T}(v, б) + (v, a)\mathfrak{D}(\omega, б) + (\omega, a)\mathfrak{D}(v, б) + (\omega, a)\mathfrak{D}(\omega, б) + \mathfrak{D}(v, a)(v, б) + \mathfrak{D}(v, a)(\omega, б) + \mathcal{T}(\omega, a)(v, б) + \mathcal{T}(\omega, a)(\omega, б) + (v, б)\mathcal{T}(v, a) + (v, б)\mathcal{T}(\omega, a) + (\omega, б)\mathcal{T}(v, a) + (\omega, б)\mathcal{T}(\omega, a)$ …..(2)

In a second view, we have:

$\mathcal{T}((v + \omega)^2, aб + бa) = \mathcal{T}(v^2 + v\omega + \omega v + \omega^2, aб + бa)$

$= \mathcal{T}(v^2, aб + бa) + \mathcal{T}(v\omega + \omega v, aб + бa) + \mathcal{T}(\omega^2, aб + бa)$

Depending on the part (i), the last relation becomes

$= \mathcal{T}(v, б)(v, a) + (v, б)\mathcal{T}(v, a) + \mathcal{T}(v, a)(v, б) + (v, a)\mathcal{T}(v, б) + \mathcal{T}(v\omega + \omega v, aб + бa) + \mathcal{T}(\omega, б)(\omega, a) + (\omega, б)\mathcal{T}(\omega, a) + \mathcal{T}(\omega, a)(\omega, б) + (\omega, a)\mathcal{T}(\omega, б)$ (3)

The out putting of relations (2) and (3) are for same expression, this leads us to:

$\mathcal{T}(v\omega + \omega v, aб + бa) = \mathcal{T}(v, б)(\omega, a) + \mathcal{T}(\omega, б)(v, a) + (v, б)\mathcal{T}(\omega, a) + (\omega, a)\mathcal{T}(v, б) + \mathcal{T}(v, a)(\omega, б) + \mathcal{T}(\omega, a)(v, б) + (v, a)\mathcal{T}(\omega, б) + (\omega, б)\mathcal{T}(v, a)$

(vi)- Putting $((aб + бa), ɥ^2)$ instead of $(б, ɥ)$ in (ii) gives:

$\mathcal{T}(a(aб + бa) + (aб + бa)a, ɥ^3) = \mathcal{T}((aб + бa), ɥ^2)(a, ɥ) + ((aб + бa), ɥ^2)\mathcal{T}(a, ɥ) + \mathcal{T}(a, ɥ)((aб + бa), ɥ^2) + (a, ɥ)\mathcal{T}((aб + бa), ɥ^2)$

$= \mathcal{T}(б, ɥ)(a, ɥ)(a, ɥ) + (б, ɥ)\mathcal{T}(a, ɥ)(a, ɥ) + \mathcal{T}(a, ɥ)(б, ɥ)(a, ɥ) + (a, ɥ)\mathcal{T}(б, ɥ)(a, ɥ) + (a, ɥ)(б, ɥ)\mathcal{T}(a, ɥ) + (б, ɥ)(a, ɥ)\mathcal{T}(a, ɥ) + \mathcal{T}(a, ɥ)(a, ɥ)(б, ɥ) + \mathcal{T}(a, ɥ)(б, ɥ)(a, ɥ) + (a, ɥ)\mathcal{T}(б, ɥ)(a, ɥ) + (a, ɥ)(б, ɥ)\mathcal{T}(a, ɥ) + (a, ɥ)\mathcal{T}(a, ɥ)(б, ɥ) + (a, ɥ)(a, ɥ)\mathcal{T}(б, ɥ)$ (4)

However, the left side of the above equation can be expressed in the form:

$\mathcal{T}(a(aб + бa) + (aб + бa)a, ɥ^3) = \mathcal{T}(a^2б + 2aбa + бa^2, ɥ^3)$

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$= \mathcal{T}(a^2б + бa^2 , ц^3) + 2\,\mathcal{T}(aбa , ц^3)$ (5)

The substitution $(a^2, ц^2)$ instead of $(a, ц)$ in (ii), gives:

$\mathcal{T}(a^2б + бa^2 , ц^3) = \mathcal{T}(б, ц)(a^2, ц^2) + (б , ц)\ \mathcal{T}(a^2, ц^2) + \mathcal{T}(a^2, ц^2)\,(б, ц) + (a^2, ц^2)\,\mathcal{T}(б, ц)$

$= \mathcal{T}(б, ц)(a, ц)\,(a, ц) + (б, ц)\ \mathcal{T}(a, ц)\,(a, ц) + (б, ц)\,(a, ц)\,\mathcal{T}(a, ц) + \mathcal{T}(a, ц)\,(a, ц)\,(б, ц) + (a, ц)\,\mathcal{T}(a, ц)\,(б, ц) + (a, ц)\,(a, ц)\ \mathcal{T}(б, ц)$

In view of above equation, the relation (4) can be given as follows:

$\mathcal{T}(a(a\, б + б\, a) + (a\, б + б\, a)\, a, ц^3) = 2\,\mathcal{T}(aбa , ц^3) + \mathcal{T}(б, ц)(a, ц)(a, ц) + (б, ц)\ \mathcal{T}(a, ц)(a, ц) + (б, ц)(a, ц)\ \mathcal{T}(a, ц) + \mathcal{T}(a, ц)(a, ц)(б, ц) + (a, ц)\,\mathcal{T}(a, ц)\,(б, ц) + (a, ц)\,(a, ц)\,\mathcal{T}(б , ц)$ …….(6)

Again, since out putting of relations (4) and (6) are for same expression, we arrive at:

$2\,\mathcal{T}(aбa , ц^3) = 2\,\mathcal{T}(a , ц)\,(б, ц)\,(a, ц) + 2\,(a, ц)\ \mathcal{T}(б, ц)\,(a, ц) + 2(a , ц)\ (б, ц)\ \mathcal{T}(a, ц)$

That is $\mathcal{T}(aбa , ц^3) = \mathcal{T}(a, ц)\ (б, ц)(a, ц) + (a , ц)\ \mathcal{T}(б, ц)\,(a, ц) + (a , ц)\,(б, ц)\ \mathcal{T}(a, ц)$

(iv)-According to the part (v), we have:

$\mathcal{T}(ц^3, a\omega a) = \mathcal{T}(ц, a)(ц , \omega)(ц , a) + (ц , a)\,\mathcal{T}(ц, \omega)(ц, a) + (ц, a)(ц, \omega)\,\mathcal{T}(ц, a)$.

The linearization of above relation gives:

$\mathcal{T}\,(ц^3, (a + б)\ \omega\ (a + б)) = \mathcal{T}\,(ц , (a + б))\,(ц , \omega)\,(ц , (a + б)) + (ц , (a + б))\,\mathcal{T}(ц , \omega)\ (ц, (a + б)) + (ц, (a + б))\ (ц, \omega)\ \mathcal{T}\,(ц, (a + б))$.

$= \mathcal{T}\,(ц , a)\ (ц , \omega)\ (ц,\ a) + \mathcal{T}\,(ц , a)(ц , \omega)(ц, б) + \mathcal{T}\,(ц, б)\ (ц, \omega)\ (ц,\ a) + + \mathcal{T}\,(ц, б)(ц , \omega)(ц, б) + (ц, a)\ \mathcal{T}\,(ц, \omega)\,(ц, a) + (ц, a)\ \mathcal{T}\,(ц, \omega)\,(ц, б) + (ц, б)\ \mathcal{T}\,(ц, \omega)\,(ц, a) + (ц, б)\ \mathcal{T}\,(ц, \omega)\,(ц, б) + (ц, a)\,(ц, \omega)\ \mathcal{T}\,(ц, a) + (ц, a)\,(ц, \omega)\,\mathcal{T}\,(ц, б) + (ц, б)\,(ц, \omega)\ \mathcal{T}\,(ц, a) + (ц, б)\,(ц, \omega)\ \mathcal{T}\,(ц, б)$ ….(7)

From another perspective, we see:

$\mathcal{T}\,(ц^3, (a + б)\ \omega(a + б)) = \mathcal{T}\,(ц^3, a\omega a + a\omega б + б\omega a + б\omega б)$

$= \mathcal{T}\,(ц^3, a\ \omega\ a) + \mathcal{T}\,(ц^3, a\omega б + б\omega a) + \mathfrak{D}\,(ц^3, б\omega б)$

$= \mathcal{T}\,(ц^3, a\omega б + б\omega б) + \mathcal{T}(ц,\ a)\,(ц , \omega)\,(ц, a) + (ц , a)\,\mathcal{T}\,(ц , \omega)\,(ц, a) + (ц, a)(ц, \omega)\ \ \mathcal{T}\,(ц, a) + \mathcal{T}\,(ц , б)\,(ц , \omega)\,(ц, б) + (ц , б)\,\mathcal{T}\,(ц , \omega)\,(ц , б) + (ц, б)\,(ц, \omega)\ \mathcal{T}\,(ц , б)$….(8)

From the equality of the equations (7) and (8), we conclude:

$\mathcal{T}\,(ц^3,\ a\ \omega\ б + бz\ б) = \mathcal{T}(ц , a)\,(ц , \omega)\,(ц, б) + (ц\ ,\ a)\ \mathcal{T}(ц, \omega)\ (ц , б) + (ц, a)\ (ц, \omega)\ \ \mathcal{T}(ц, б) + \mathcal{T}(ц, б)(ц , \omega)(ц , a) + (ц , б)\,\mathcal{T}(ц, \omega)\,(ц, a) + (ц, б)(ц, \omega)\,\mathcal{T}(ц, a)$

Definition 2.10: Let $\mathcal{M}$ be the direct product of a ring Ɽ and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ be a Jordan Ɓi-Antiderivation, we define:

$\mathcal{X}_v^{\omega} = \mathcal{T}(v\omega, ӿ^2) - \mathcal{T}(\omega, ӿ)(v, ӿ) - (\omega, ӿ)\,\mathcal{T}(v, ӿ)$

Lemma 2.11: Let $\mathcal{M}$ be the direct product of a ring Ɽ and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ be a Jordan Endo Ɓi- Antiderivation, then for all ӿ $, v, \omega,$ ц ∈ Ɽ.

(i) $\mathcal{X}_v^{\omega} + \mathcal{X}_{\omega}^{v} = 0$
(ii) $\mathcal{X}_v^{ц+\omega} = \mathcal{X}_v^{ц} + \mathcal{X}_v^{\omega}$
(iii) $(\mathcal{X} + \mathcal{Y})_v^{\omega} = \mathcal{X}_v^{\omega} + \mathcal{Y}_v^{\omega}$

Proof:(i)- For any ӿ $, v, \omega$ ∈ Ɽ , we have:

$\mathcal{X}_v^{\omega} + \mathcal{X}_{\omega}^{v} = \mathcal{T}(v\omega , ӿ^2) - \mathcal{T}(\omega, ӿ)(v, ӿ) - (\omega, ӿ)\ \mathcal{T}(v, ӿ) + \mathcal{T}(\omega v, x^2) - \mathcal{T}(v, ӿ)(\omega, ӿ)$

$- (v, ӿ)\,\mathcal{T}(\omega, ӿ)$

$= \mathcal{T}(v\omega + \omega v, ӿ^2) - \mathcal{T}(v, ӿ)(\omega, ӿ) - (v, ӿ)\ \mathcal{T}(\omega, ӿ) - \mathcal{T}(\omega, ӿ)(v, ӿ) - (\omega, ӿ)\ \mathcal{T}(v, ӿ)$

$= \mathcal{T}(v, ӿ)(\omega, ӿ) + (v, ӿ)\,\mathcal{T}(\omega, ӿ) + \mathcal{T}(\omega, ӿ)\,(\ v, ӿ) + (\omega, ӿ)\ \mathcal{T}(v, ӿ) - \mathcal{T}(v, ӿ)\,(\omega, ӿ) - (v, \eta)\ \mathcal{T}(\omega, ӿ) - \mathcal{T}(\omega, ӿ)(v, ӿ) - (\omega, ӿ)\ \mathcal{T}(v, ӿ) = 0$

(ii)- $\mathcal{X}_v^{ц+\omega} = \mathcal{T}(v\,(ц + \omega), ӿ^2) - \mathcal{T}\big((ц + \omega), ӿ\big)(v, ӿ) - ((ц + \omega), ӿ)\ \mathcal{T}(v, ӿ)$

$= \mathcal{T}(vц, ӿ^2) + \mathcal{T}(v\omega, ӿ^2) - \mathcal{T}(ц , ӿ)(v , ӿ) - \mathcal{T}(\omega, ӿ)(v, ӿ) - (ц , ӿ)\,\mathcal{T}(v, ӿ) - (\omega, ӿ)\ \mathcal{T}(v, ӿ)$

$= \mathcal{X}_v^{ц} + \mathcal{X}_v^{\omega}$

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(iii)-

$(\mathcal{X}+\mathcal{Y})_v^{\omega} = \mathcal{T}(v\omega,(x+y)^2) - \mathcal{T}(\omega,(x+y))\,(v,(x+y)) - (\omega,(x+y))\ \mathcal{T}(v,(x+y))$

$= \mathcal{T}(v\omega,x^2) + \mathcal{T}(v\omega,y^2) + 2\,\mathcal{T}(v\omega,xy) - \mathcal{T}(\omega,x)(v,x) - \mathcal{T}(\omega,x)(v,y) - \mathcal{T}(\omega,y)(v,y) - \mathcal{T}(\omega,y)(v,x) - (\omega,x)\,\mathcal{T}(v,x) - (\omega,x)\,\mathcal{T}(v,y) - (\omega,y)\,\mathcal{T}(v,x) - (\omega,y)\,\mathcal{T}(v,y)$

$= \mathcal{T}(v\omega,x^2) + \mathcal{T}(v\omega,y^2) - \mathcal{T}(\omega,x)(v,x) - (\omega,x)\,\mathcal{T}(v,x) - \mathcal{T}(\omega,y)(v,y) - (\omega,y)\,\mathcal{T}(v,y)$

$=\mathcal{X}_v^{\omega} + \mathcal{Y}_v^{\omega}$

Lemma 2.12: Let $\mathcal{M}$ be the direct product of a ring $\mathcal{R}$ and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ be a Jordan Endo Bi-Antiderivation ,$\forall$ x ,y , v , $\omega \in \mathcal{R}$.

$$[(v, x), (\omega, x)]\,(y, x)\,\mathcal{X}_v^{\omega} + \mathcal{X}_v^{\omega}\,(y, x)\,[(v, x), (\omega, x)]=0$$

Proof: According to the part (iv) of Lemma (2.9), we have:

$\mathcal{T}(v\omega v,x^3) = \mathcal{T}(v,x)(\omega,x)\,(v,x) + (v,x)\,\mathcal{T}(\omega,x)\,(v,x) + (v,x)(\omega,x)\,\mathcal{T}(v,x)$

Putting $(\omega m\omega, x^3)$ instead of (ω, x) in the above relation, we get:

$\mathcal{T}(v\omega m\omega v,x^5)= \mathcal{T}(v,x)(\omega m\omega, x^3)\,(v, x) + (v, x)\ \mathcal{T}(\omega m\omega, x^3, x)\,(v,x) + (v,x)\,(\omega m\omega, x^3)\ \mathcal{T}(v,x)$

Again, $\mathcal{T}(\omega v\omega,x^3)= \mathcal{T}(\omega,x)(v,x)\,(\omega,x) + (\omega,x)\ \mathcal{T}(v,x)\,\omega,x) + (\omega,x)(v,x)\ \mathcal{T}(\omega,x)$

Now, the substitution (vmv, x^3) instead of (v, x) in above relation, we arrive at:

$\mathcal{T}(\omega vmv\omega,x^5)= \mathcal{T}(\omega,x)(vmv,x^3)\,(\omega,x) + (\omega,x)\ \mathcal{T}(vmv,x^3)\,\omega,x) + (\omega,x)(vmv,x^3)\ \mathcal{T}(\omega,x)$

Set $z= v\omega m\omega v + \omega vmv\omega$, then

$\mathcal{T}(z,x^5)= \mathfrak{D}(v\omega m\omega v,x^5) + \mathcal{T}(\omega vmv\omega,x^5)$

$= \mathcal{T}(v,x)(\omega m\omega,x^3)(v,x) + (v,x)\,\mathcal{T}(\omega m\omega,x^3,x)(v,x) + (v,x)(\omega m\omega,x^3)\,\mathcal{T}(v,x) + \mathcal{T}(\omega,x)(vmv,x^3)(\omega,x) + (\omega,x)\ \mathcal{T}(vmv,x^3)\,\omega,x) + (\omega, x)(vmv, x^3)\ \mathcal{T}(\omega, x)$

$= \mathcal{T}(v, x)(\omega, x)(m, x)(\omega, x)(v, x) + (v, x)\ \mathcal{T}(\omega, x)(m, x)\,(\omega, x)(v, x) + (v, x)(\omega, x)\,\mathcal{T}(m, x)\,\omega, x)(v, x) + (v, x)(\omega, x)(m, x)\ \mathcal{T}(\omega, x)(v, x) + (v, x)(\omega, x)(m, x)(\omega, x)\,\mathcal{T}(v, x) + \mathcal{T}(\omega, x)(v, x)(m, x)(v, x)(\omega, x) + (\omega, x)\,\mathcal{T}(v, x)(m, x)\,(v, x)(\omega, x) + (\omega, x)(v, x)\,\mathcal{T}(m, x)\,(v, x)(\omega, x) + (\omega, x)(v, x)(m, x)\mathcal{T}(v, x)(\omega, x) + (\omega, x)(v, x)(m, x)\,(v, x)\ \mathcal{T}(\omega, x)\ldots$ (9)

Now, replacing (a, x) and (b, x) by $(v\omega, x^2)$ and $(\omega v, x^2)$ respectively in the part (iiv) of Lemma (2.9), we arrive at:

$\mathcal{T}(v\omega mv\omega + v\omega mv\omega, x^5) = \mathcal{T}(v\omega, x^2)(m, x)(v\omega, x^2) + (v\omega, x^2)\,\mathcal{T}(m, x)\ (v\omega, x^2) + (v\omega, x^2)(m, x)\ \mathcal{T}(v\omega, x^2) + \mathcal{T}(v\omega, x^2)(m, x)(v\omega, x^2) + (v\omega, x^2)\,\mathcal{T}(m, x)\,(v\omega, x^2) + (v\omega, x^2)\,(m, x)\,\mathcal{T}(v\omega,x)$ ……(10)

According to the equality of the equations (9) and (10) implies that:

$(v, x)\,(\omega, x)\,(m, x)\,\mathcal{X}_v^{\omega} + (\omega, x)\,(v, x)\,(m, x)\mathcal{X}_{\omega}^{v} + \mathcal{X}_v^{\omega}\,(m, x)\,(\omega, x)\,(v, x) + \mathcal{X}_{\omega}^{v}\,(m, x)\,(v, x)\,(\omega, x) =0$

Since $\mathcal{X}_v^{\omega} = -\mathcal{X}_{\omega}^{v}$, then $[(v, x), (\omega, x)]\,(m, x)\ \mathcal{X}_v^{\omega} + \mathcal{X}_v^{\omega}(m, x)\ [(v, x), (\omega, x)]=0$

Theorem 2.13: Let $\mathcal{M}$ be the direct product of a 2-torsion free prime ring $\mathcal{R}$ and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ is a Jordan Endo Bi-Antiderivation, then $\mathcal{T}$ is Quasi Endo Bi-Antiderivation on $\mathcal{M}$.

Proof: Let $\mathcal{R}$ is a commutative ring, consequently so is $\mathcal{M}$, then we get:

$\mathcal{T}(sr + rs\,,t^2) = 2\,\mathcal{T}(sr,t^2)$

$= \mathcal{T}(r,t)(s,t) + (r,t)\,\mathcal{T}(s,t) + \mathcal{T}(s,t)\,(r,t) + (s,t)\,\mathcal{T}(r,t)$

$= 2\{\mathcal{T}(r,t)(s,t) + (r,t)\,\mathcal{T}(s,t)\}$, for $t, s, r \in \mathcal{R}$.

Hence by 2-torsionty free of $\mathcal{M}$

$$\mathcal{T}(sr,t^2) = \mathcal{T}(r,t)(s,t) + (r,t)\,\mathcal{T}(s,t), \text{ for } t, s, r \in \mathcal{R}.$$

Now, if $\mathcal{R}$ is a non– commutative, then using Lemma (2.12), get:

$$[(s, t), (r, t)]\,(y, t)\,\mathcal{X}_s^{r} + \mathcal{X}_s^{r}\,(y, t)\,[(s, t), (r, t)] = 0, \text{ for } t, s, r, y \in \mathcal{R}.$$

Thus, using Lemma 2.9 get: $[(s,t),(r,t)]\,(y,t)\mathcal{X}_s^{r} = 0$, $\forall x, s, r, y \in \mathcal{R}$. Since $\mathcal{R}$ is a non-commutative ring, thus primeness of $\mathcal{R}$ give

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$$\mathcal{X}_s^r = 0 \quad, \forall r, s \in R$$

That is $\mathcal{T}(sr, t^2) = \mathcal{T}(r, t)(s, t) + (r, t)\,\mathcal{T}(s, t)$.

Lemma 2.14: Let $\mathcal{M}$ be the direct product of a neutrosophic ring $\mathcal{R}$ and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ be a Jordan Endo Bi-Antiderivation ,$\forall$ x ,y , v , $\omega \in \mathcal{R}$.

$$[(v, x), (\omega, x)]\,(y, x)\,\mathcal{X}_v^\omega + \mathcal{X}_v^\omega\,(y, x)\,[(v, x), (\omega, x)]=0$$

Proof: According to the part (iv) of Lemma (2.9), we have:

$\mathcal{T}(v\omega v, x^3) = \mathcal{T}(v, x)(\omega, x)\,(v, x) + (v, x)\,\mathcal{T}(\omega, x)\,(v, x) + (v, x)(\omega, x)\,\mathcal{T}(v, x)$

Putting $(\omega m\omega, x^3)$ instead of (ω, x) in the above relation, we get:

$\mathcal{T}(v\omega m\omega v, x^5)= \mathcal{T}(v, x)(\omega m\omega, x^3)\,(v, x) + (v, x)\,\mathcal{T}(\omega m\omega, x^3, x)\,(v, x) + (v, x)\,(\omega m\omega, x^3)\,\mathcal{T}(v, x)$

Again, $\mathcal{T}(\omega v\omega, x^3)= \mathcal{T}(\omega, x)(v, x)\,(\omega, x) + (\omega, x)\,\mathcal{T}(v, x)\,\omega, x) + (\omega, x)(v, x)\,\mathcal{T}(\omega, x)$

Now, the substitution (vmv, x^3) instead of (v, x) in above relation, we arrive at:

$\mathcal{T}(\omega vmv\omega, x^5)= \mathcal{T}(\omega, x)(vmv, x^3)\,(\omega, x) + (\omega, x)\,\mathcal{T}(vmv, x^3)\,\omega, x) + (\omega, x)(vmv, x^3)\,\mathcal{T}(\omega, x)$

Set $z= v\omega m\omega v + \omega vmv\omega$, then

$\mathcal{T}(z, x^5)= \mathfrak{D}(v\omega m\omega v, x^5) + \mathcal{T}(\omega vmv\omega, x^5)$

$= \mathcal{T}(v, x)(\omega m\omega, x^3)(v, x) + (v, x)\,\mathcal{T}(\omega m\omega, x^3, x)(v, x) + (v, x)(\omega m\omega, x^3)\,\mathcal{T}(v, x) + \mathcal{T}(\omega, x)(vmv, x^3)(\omega, x) + (\omega, x)\,\mathcal{T}(vmv, x^3)\,\omega, x) + (\omega, x)(vmv, x^3)\,\mathcal{T}(\omega, x)$

$= \mathcal{T}(v, x)(\omega, x)(m, x)(\omega, x)(v, x) + (v, x)\,\mathcal{T}(\omega, x)(m, x)\,(\omega, x)(v, x) + (v, x)(\omega, x)\,\mathcal{T}(m, x)\,\omega, x)(v, x) + (v, x)(\omega, x)(m, x)\,\mathcal{T}(\omega, x)(v, x) + (v, x)(\omega, x)(m, x)(\omega, x)\,\mathcal{T}(v, x) + \mathcal{T}(\omega, x)(v, x)(m, x)(v, x)(\omega, x) + (\omega, x)\,\mathcal{T}(v, x)(m, x)\,(v, x)(\omega, x) + (\omega, x)(v, x)\,\mathcal{T}(m, x)\,(v, x)(\omega, x) + (\omega, x)(v, x)(m, x)\mathcal{T}(v, x)(\omega, x) + (\omega, x)(v, x)(m, x)\,(v, x)\,\mathcal{T}(\omega, x)\ldots$ (9)

Now, replacing (a, x) and (b, x) by $(v\omega, x^2)$ and $(\omega v, x^2)$ respectively in the part (iiv) of Lemma (2.9), we arrive at:

$\mathcal{T}(v\omega mv\omega + v\omega mv\omega, x^5) = \mathcal{T}(v\omega, x^2)(m, x)(v\omega, x^2) + (v\omega, x^2)\,\mathcal{T}(m, x)\,(v\omega, x^2) + (v\omega, x^2)(m, x)\,\mathcal{T}(v\omega, x^2) + \mathcal{T}(v\omega, x^2)(m, x)(v\omega, x^2) + (v\omega, x^2)\,\mathcal{T}(m, x)\,(v\omega, x^2) + (v\omega, x^2)\,(m, x)\,\mathcal{T}(v\omega, x)$ ……(10)

According to the equality of the equations (9) and (10) implies that:

$(v, x)\,(\omega, x)\,(m, x)\,\mathcal{X}_v^\omega + (\omega, x)\,(v, x)\,(m, x)\mathcal{X}_\omega^v + \mathcal{X}_v^\omega\,(m, x)\,(\omega, x)\,(v, x) + \mathcal{X}_\omega^v\,(m, x)\,(v, x)\,(\omega, x) =0$

Since $\mathcal{X}_v^\omega = -\mathcal{X}_\omega^v$, then $[(v, x), (\omega, x)]\,(m, x)\,\mathcal{X}_v^\omega + \mathcal{X}_v^\omega(m, x)\,[(v, x), (\omega, x)]=0$

Theorem 2.15: Let $\mathcal{M}$ be the direct product of a 2-torsion free prime neutrosophic ring R and $\mathcal{T}: \mathcal{M} \longrightarrow \mathcal{M}$ is a Jordan Endo Bi-Antiderivation, then $\mathcal{T}$ is Quasi Endo Bi-Antiderivation on $\mathcal{M}$.

Proof: Let R is a commutative ring, consequently so is $\mathcal{M}$, then we get:

$\mathcal{T}(sr + rs, t^2) = 2\,\mathcal{T}(sr, t^2)$

$= \mathcal{T}(r, t)(s, t) + (r, t)\,\mathcal{T}(s, t) + \mathcal{T}(s, t)\,(r, t) + (s, t)\,\mathcal{T}(r, t)$

$= 2\{\mathcal{T}(r, t)(s, t) + (r, t)\,\mathcal{T}(s, t)\}$, for $t, s, r \in R$.

Hence by 2-torsiony free of $\mathcal{M}$

$$\mathcal{T}(sr, t^2) = \mathcal{T}(r, t)(s, t) + (r, t)\,\mathcal{T}(s, t), \text{ for } t, s, r \in R.$$

Now, if R is a non– commutative , then using Lemma (2.12), get:

$$[(s, t), (r, t)]\,(y, t)\,\mathcal{X}_s^r + \mathcal{X}_s^r\,(y, t)\,[(s, t), (r, t)] = 0, \text{ for } t, s, r, y \in R.$$

Thus , using Lemma 2.9 get: $[(s, t), (r, t)]\,(y, t)\mathcal{X}_s^r = 0$, $\forall x, s, r, y \in R$. Since R is a non commutative ring, thus primeness of R give

$$\mathcal{X}_s^r = 0 \quad, \forall r, s \in R$$

That is $\mathcal{T}(sr, t^2) = \mathcal{T}(r, t)(s, t) + (r, t)\,\mathcal{T}(s, t)$.

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3. Conclusion

In this paper, we studied the direct product of an associative ring Ɽ and the concepts of Endo Bi-Antiderivation, Jordan Endo Bi-Antiderivation and Quasi Endo Bi-Antiderivation on a ring $\mathcal{M}$ .Furthermore the relations between these bi-additive mappings are given. In addition, we searched for appropriate conditions that make equivalence between Jordan Endo Bi-Antiderivation and Quasi Endo Bi-Antiderivation. Also, we proved the same results for the generalized case of neutrosophic rings

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