Asymptotically Stability Concept for Perturbed Bilinear Time Varying Controlled Differential-algebraic Systems and Applications under Neutrosophic Environment

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Abstract

Starting from semi-explicit perturbed bilinear time varying neutrosophic differential – algebraic equations (PBTVDAs). We develop a method for the stabilization of this controlled bilinear time varying neutrosophic differential – algebraic equations and prove that the controlled perturbed system can be stabilized by putting specific conditions on the proposed control. This method transfers the system to standard canonical form and uses the exponential stability concept. Therefore, the stabilization of this system is achieved finally; we present numerical results for the battery model, which confirm the theoretical results.

Keywords: Bilinear; Neutrosophic equation; Differential equation; Algebraic equation; Exponential stability

1. Introduction

Bilinear systems (Bs) are important subclasses of nonlinear systems with many applications in engineering, biology and economics. The control of Bs has been extensively studied in [1]–[4] . The stabilization of bilinear systems by the use of linear feedback control was discussed by [5, 6]. The stabilization of bilinear systems with respect to a constant nonlinear feedback control has been derived by [7, 8]. For bilinear system, a quadratic state feedback control has been considered to guarantee global asymptotically stability of the closed -loop system [9, 10]. The problem of linear feedback design for bilinear control system guaranteeing their conditional closed- loop stability is proposed by [11, 12]. Most papers dealing with time invariant continuous bilinear systems with linear feedback and stability problems has been studied using a sufficient condition for the existence of a feedback control [13, 14].

The singular bilinear system often consists of a set of differential and algebraic equations. Since singular bilinear system is a special case of nonlinear descriptor system, so it has been studied in [15, 16]. A new set of sufficient conditions is derived via the continuous state feedback the guarantees the global asymptotically stabilization of the closed loop system of singular bilinear systems [17–19].

A bilinear time varying differential – algebraic equations have been designed to be stabilized by finding a robust controller using exponential stabilization approach via logarithmic norm and finite time stability concept.

The paper is organized as follows. The problem formulation is defined in section 2. Section 3 gives the necessary time varying background on the non-homogenous time varying DAs. Sections 4-6 show the way to find stability for this model of systems. Finally, a practical implementation for this work presented in section 7.

2. Problem Description

Let Introduce perturbed bilinear time varying Differential -algebraic Systems as follows:

$$\dot{w}_1 = \Big(\big(A_1(t) + \delta A_1(t)\big) + (A_2 + \delta A_2)u(t)\Big)\, w_1(t) \qquad \dots(1)$$

$$0 = (A_3 + \delta A_3)w_1(t) + (A_4 + \delta A_4)w_2(t) \qquad \dots(2)$$

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With

1) $w_1(t) \in \mathbb{R}^{n_1}$, $w_2(t) \in \mathbb{R}^{n_2}$ with $n = n_1 + n_2$

2) the matrices: $A_1(t) \in \mathbb{R}^{n_1 \times n_1}, A_2 \in \mathbb{R}^{n_1 \times 1}, A_3 \in \mathbb{R}^{(n-n_1)\times n_1}, A_4$ nonsingular matrix with $A_4 \in \mathbb{R}^{n_2 \times n_2}$

3) the constant perturbation matrices $\delta A_2, \delta A_3, \delta A_4$ are bounded, i.e. $\|\delta A_2\| \le a_2, \|\delta A_3\| \le a_3, \|\delta A_4\| \le a_4$, where a_2, a_3, a_4, constants.

3. Concepts an Overview

Proposition 1:[20, 21]

Consider the nonhomogeneous time varying linear differential algebraic initial value problems

$$E(t)w'(t) = A(t)w(t) + g(t) \qquad \dots (3)$$

Where $(E(t), A(t), g(t)) \in C((I,\infty), \mathbb{R}^{n\times n})^2 \times C((I,\infty), \mathbb{R}^n)$

$n \in \mathbb{N}, I \in [-\infty, \infty), a \ge I$

1) The system (3) is transferable to standard canonical form by some $(p(t), s(t)) \in C^n(I; GL_n(\mathbb{R}))^2$.

Such that $E(t) = diag\left(I_{n_1}, N(t)\right), A(t) = diag(J(t), I_{n_2})$ …(4)

Where $N(t) \in C^n(I; \mathbb{R}^{n\times n})$ be pointwise lower triangular.

2) The initial value problem (3) has a solution if and only if

$$\left\{w_0 + s(t_0)\begin{bmatrix} 0 \\ I_{n_2}\end{bmatrix} \sum_{k=0}^{n_2-1} \left(N(t)\right)^k \left(\left[0, I_{n_2}\right]p(t)g(t)\right)^{(k)} |_{t=t_0}\right\} \in ims(t_0)\begin{bmatrix} I_{n_1} \\ 0\end{bmatrix} \qquad \dots (5)$$

3) Any solution of (3) such that (5) can be uniquely extended into a global solution $w(t)$, and this solution satisfies, for the generalized transition matrix $\mathcal{Y}(t, t_0)$ of $\left(E(t), A(t)\right)$ for $t \in I$,

$$w(t) = \mathcal{Y}(t, t_0)w_0 + \int_{t_0}^{t} \mathcal{Y}(t,\tau)s(\tau)p(\tau)g(\tau)d\tau$$

$$-s(t)\begin{bmatrix} 0 \\ I_{n_2}\end{bmatrix} \sum_{k=0}^{n_2-1} \left(N(t)\right)^k \left(\left[0, I_{n_2}\right]p(t)g(t)\right)^{(k)}. \qquad \dots (6)$$

Where $\mathcal{Y}(t, t_0) = s(t)\begin{bmatrix} \emptyset_{A_1}(t,t_0) & 0 \\ 0 & 0\end{bmatrix} s^{-1}(t)$ and $\emptyset_{A_1}(t, t_0)$ denotes the transition matrix of $w'(t) = A(t)w$

4) The initial value problem

$$N(t)w'(t) = w + g(t), w(t_0) = w_0 . \qquad \dots (7)$$

Has a solution if and only if

$$\sum_{k=0}^{n_2-1} \left(N(t)\right)^k g(t)^{(k)}|_{t=t_0} = w_0 . \qquad \dots (8)$$

And any solution of (7) can be uniquely extended to a global solution $w(t)$, and this solution satisfies

$$w(t) = -\sum_{k=0}^{n_2-1} \left(N(t)\right)^k g(t)^{(k)} \qquad \dots (9)$$

Proposition 2:[20, 21]

Suppose the time varying system $\left(E(t), A(t)\right) \in C((I,\infty), \mathbb{R}^{n\times n})^2$ is transferable into SCF and let $\mathcal{Y}(.,.)$ denotes the generalized transition matrix of $\left(E(t), A(t)\right)$, then:

$\left(E(t), A(t)\right)$ is exponentially stable if and only if $\exists \alpha, \beta > 0, \forall (t_0, w_0) \in \mathcal{W}_{E,A}, \forall t > t_0$:

$$\|\mathcal{Y}(t, t_0)w_0\| \le \alpha e^{-\beta(t-t_0)}\|w_0\|$$

Where $\mathcal{W}_{E,A}$ is the set of all consistent initial condition.

4. Assumption

For the bilinear controlled time varying Differential -algebraic Systems (1) - (2) with parametric uncertainty, let we consider:

1- $\delta A_1(t)$ be a time varying perturbation such that $(A_1 + \delta A_1)(t)$ is stable.

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2-$\|A_4^{-1}\delta A_4\| < 1$, with non-singular matrix A_4 .

3-consider the control $u(t) = \frac{f(w_1(t))}{\|w_1(t_0)\|^2}$, $w_1(t_0) \neq 0$ and $f(w_1(t))$ is a vector of nonlinear integral functions with $\|f(w_1(t))\| \leq f(t)\|w_1(t)\|^2$.

5. Stability of Perturbed Bilinear Time Varying Controlled Differential -algebraic Systems

Let us recall the perturbed bilinear time varying Differential -algebraic equations (1) and (2)

$$\dot{w}_1 = \Big(\big(A_1(t) + \delta A_1(t)\big) + (A_2 + \delta A_2)u(t)\Big)\, w_1(t) \qquad \ldots (10)$$

$$0 = (A_3 + \delta A_3)w_1(t) + (A_4 + \delta A_4)w_2(t) \qquad \ldots (11)$$

With consistent initial condition $w_0(t) = \begin{bmatrix} w_1(t_0) \\ w_2(t_0) \end{bmatrix} \in \mathcal{N}([A_3 + \delta A_3 \ \ A_4 + \delta A_4])$ …(12)

The null space of matrix $[A_3 + \delta A_3 \ \ A_4 + \delta A_4]$

Where

$$\dot{w}_1 = (A_1(t) + \delta A_1(t))w_1(t) + (A_2 + \delta A_2)u(t)w_1(t) \qquad \ldots (13)$$

$$\dot{w}_1 - \big(A_1(t) + \delta A_1(t)\big)w_1(t) = (A_2 + \delta A_2)u(t)w_1(t) \qquad \ldots (14)$$

$$w_1(t) = w_1(t_0)\emptyset_{A_1(t)+\delta A_1(t)}(t,t_0) + \int_{t_0}^{t}(A_2 + \delta A_2)u(\tau)\, w_1(\tau)\emptyset_{A_1(t)+\delta A_1(t)}(t,\tau)d\tau \qquad \ldots (15)$$

Where $\emptyset_{A_1(t)+\delta A_1(t)}$ denotes the transition matrix of $\dot{w}_1 = (A_1(t) + \delta A_1(t))w_1(t)$

Define $w_1(t) = p(t)^{-1}x_1(t)$, $p(t)$ nonsingular time varying matrix

$$x_1 = p(t)\emptyset_{A_1(t)+\delta A_1(t)}(t,t_0)p^{-1}(t_0)x_1(t_0)$$

$$+p(t)\int_{t_0}^{t}(A_2 + \delta A_2)u(\tau)\, p^{-1}(t_0)\, \emptyset_{A_1(t)+\delta A_1}(t,\tau)x_1(\tau)d\tau \qquad \ldots (16)$$

Because the generalized transition matrix $\mathcal{Y}(t,\tau) = p(t)\, \emptyset_{A_1(t)+\delta A_1(t)}(t,\tau)\, p^{-1}(\tau)$ …(17)

And since any solution of problem (10-12) with $(t_0, x_0) \in \mathcal{W}_{E,A}$

Extending uniquely to a global solution

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \mathcal{Y}(t,t_0)\begin{bmatrix} w_1(t_0) \\ w_2(t_0) \end{bmatrix} = p(t)\begin{bmatrix} \emptyset_{A_1(t)+\delta A_1(t)} & 0 \\ 0 & 0 \end{bmatrix} p^{-1}(t)\begin{bmatrix} w_1(t_0) \\ w_2(t_0) \end{bmatrix} \qquad \ldots (18)$$

From properties of transition matrix see [22, 23], then

$$x_1 = \mathcal{Y}(t,t_0)\, x_1(t_0) + \int_{t_0}^{t}(A_2 + \delta A_2)u(\tau)\, \mathcal{Y}(t,\tau)\, x_1(\tau)d\tau \qquad \ldots (19)$$

From assumption

$$u(t) = \frac{f(w_1(t))}{\|w_1(t_0)\|^2}, w_1(t_0) \in \mathcal{W}_{E,A}\, , w_1(t_0) \neq 0\, , f(w_1(t)) \qquad \ldots (20)$$

vector of nonlinear functions with $\|f(w_1(t))\| \leq f(t)\|w_1(t)\|^2$

And since $w_1 = p^{-1}x_1$

Then $u(t) = \frac{f(p^{-1}(t)x_1(t))}{\|f(p^{-1}(t_0)x_1(t_0))\|^2} = \frac{f(p^{-1}(t)x_1(t))}{\|p(t_0)\|^{-2}\|x_1(t_0)\|^2}$ …(21)

one can get

$$\|x_1(t)\| \leq \|\mathcal{Y}(t,t_0)\|\, \|x_1(t_0)\| \Big[1 + \int_{t_0}^{t}\|\mathcal{Y}(t_0,\tau)\|\, \|A_2 + \delta A_2\|\ \ \|f(\tau)\|\|p(t)\|^{-2}\|p(t_0)\|^2\|x_1(t_0)\|^{-3}\|x_1(\tau)\|^3 d\tau\Big] \qquad \ldots (22)$$

Divide (22) by $\|\mathcal{Y}(t,t_0)\|\, \|x_1(t_0)\|$ one can get

$$\frac{\|x_1(t)\|}{\|\mathcal{Y}(t,t_0)\|\, \|x_1(t_0)\|} \leq 1 + \int_{t_0}^{t}\|\mathcal{Y}(t_0,\tau)\|\, \|A_2 + \delta A_2\|\ \ \|f(\tau)\|\|p(t)\|^{-2}\|p(t_0)\|^2\|x_1(t_0)\|^{-3}\|x_1(\tau)\|^3 d\tau \qquad \ldots (23)$$

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$$\frac{\|x_1(t)\|}{\|\mathcal{Y}(t,t_0)\|\,\|x_1(t_0)\|} \le 1 + \int_{t_0}^{t} \|\mathcal{Y}(t_0,\tau)\|^{-2}\,\|A_2+\delta A_2\|\,\|f(\tau)\|\|p(t)\|^{-2}\left[\frac{\|x_1(\tau)\|}{\|\mathcal{Y}(t_0,\tau)\|\|x_1(t_0)\|}\right]^3 d\tau \quad \dots (24)$$

Under assumption in section 4 and Granwall lemma [24, 25], there exist a positive integers γ,α such that $t \ge t_0, 1-2\gamma^{-2}(\|A_2\|+a_2)\int_{t_0}^{t} e^{2\alpha(t_0-\tau)}$ positive and bounded.

$$\frac{\|x_1(t)\|}{\|\mathcal{Y}(t,t_0)\|\,\|x_1(t_0)\|} \le \frac{1}{\sqrt{1-2\gamma^{-2}(\|A_2\|+a_2)\int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\,\|f(\tau)\|\|p(t)\|^{-2}d\tau}} \quad \dots(25)$$

$$\|x_1(t)\| \le \frac{\gamma e^{-2\alpha(t-t_0)}\,\|x_1(t_0)\|}{\sqrt{1-2\gamma^{-2}(\|A_2\|+a_2)\int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\,\|f(\tau)\|\|p(t)\|^{-2}d\tau}} \quad \dots (26)$$

This leads to

1)$\lim_{t\to\infty}\|x_1(t)\| \to 0$

2)the nonlinear control $u(t)$ verfy the following inequality

$$\|u(t)\| \le f(\tau)\|p(t)\|^{-2}\frac{\gamma^2 e^{-2\alpha(t-t_0)}}{1-2\gamma^{-2}(\|A_2\|+a_2)\int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\,\|f(\tau)\|\|p(t)\|^{-2}d\tau} \quad \dots (27)$$

Now for algebraic perturbed system (11) and since A_4 is non-singular matrix, then:

1) $A_4+\delta A_4 = A_4(1+A_4^{-1}\delta A_4)$ with $(1+A_4^{-1}\delta A_4)$ invertible.

2) $w_2(t) = -A_4^{-1}(1+A_4^{-1}\delta A_4)^{-1}(A_3+\delta A_3)w_1(t)$.

And since $w_1 = p^{-1}x_1$

$$x_2(t) = -A_4^{-1}(1+A_4^{-1}\delta A_4)^{-1}(A_3+\delta A_3)x_1(t)$$

3) $\lim_{t\to\infty}\|x_2(t)\| \le \frac{\|A_4^{-1}\|}{I-\|A_4^{-1}\delta A_4\|}(\|A_3\|+A_3)\lim_{t\to\infty}\|x_1(t)\| \to 0.$

Then the solution of bilinear time varying Differential -algebraic Systems $\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = p(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is asymptoticly stable

6. An Algorithm for organization the Stability of PBTVDAs

Recall the system (10-11)

Step1: Computing the space of consistent initial condition $\mathcal{W}_{E,A} = \left\{ w_0(t) = \begin{bmatrix} w_1(t_0) \\ w_2(t_0) \end{bmatrix} \in \mathcal{N}([A_3+\delta A_3 \;\; A_4 + \delta A_4]) \right\}$.

Step 2: Solve the differential time varying controlled system

$$\dot{w}_1 = (A_1(t)+\delta A_1(t))w_1(t) + (A_2+\delta A_2)u(t)w_1(t)$$

$$\Rightarrow w_1(t) = w_1(t_0)\emptyset_{A_1(t)+\delta A_1(t)}(t,t_0) + \int_{t_0}^{t}(A_2+\delta A_2)u(\tau)\,w_1(\tau)\emptyset_{A_1(t)+\delta A_1(t)}(t,\tau)d\tau$$

With $\emptyset_{A_1(t)+\delta A_1(t)}$ denotes the transition matrix of $\dot{w}_1 = (A_1(t)+\delta A_1(t))w_1(t)$.

Step 3: Using canonical standard from concepts, there exists non- singular time varying matrix $p(t)$, with $w_1(t) = p(t)^{-1}x_1(t)$, then

$$x_1 = p(t)\emptyset_{A_1(t)+\delta A_1(t)}(t,t_0)p^{-1}(t_0)x_1(t_0)$$

$$+p(t)\int_{t_0}^{t}(A_2+\delta A_2)u(\tau)\,p^{-1}(t_0)\,\emptyset_{A_1(t)+\delta A_1}(t,\tau)x_1(\tau)d\tau$$

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Step 4: Define generalized transition matrix $\mathcal{Y}(t,\tau) = p(t)\, \emptyset_{A_1(t)+\delta A_1(t)}(t,\tau)\, p^{-1}(\tau)$, this leads to $x_1 = \mathcal{Y}(t,t_0)\, x_1(t_0) + \int_{t_0}^{t} (A_2 + \delta A_2) u(\tau)\, \mathcal{Y}(t,\tau)\, x_1(\tau) d\tau$.

Step 5: Since $w_1 = p^{-1} x_1$ and by assumption of $u(t) = \frac{f(\,p^{-1}(t) x_1(t))}{\|p(t_0)\|^{-2} \|x_1(t_0)\|^2}$

one can get

$$\frac{\|x_1(t)\|}{\|\mathcal{Y}(t,t_0)\|\, \|x_1(t_0)\|} \le 1 + \int_{t_0}^{t} \|\mathcal{Y}(t_0,\tau)\|^{-2}\, \|A_2 + \delta A_2\|\ \|f(\tau)\| \|p(t)\|^{-2} \left[\frac{\|x_1(\tau)\|}{\|\mathcal{Y}(t_0,\tau)\| \|x_1(t_0)\|}\right]^3 d\tau$$

Step 6: Use Granwall lemma so there exist a positive integers

γ, α such that $t \ge t_0, 1 - 2\gamma^{-2}(\|A_2\| + a_2) \int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\, d\tau$ positive and bounded, then

$$\|x_1(t)\| \le \frac{\gamma e^{-2\alpha(t-t_0)}\, \|x_1(t_0)\|}{\sqrt{1-2\gamma^{-2}(\|A_2\|+a_2) \int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\, \|f(\tau)\| \|p(t)\|^{-2} d\tau}}$$

Step 7: For the algebraic system $\lim_{t\to\infty} \|x_2(t)\| \le \frac{\|A_4^{-1}\|}{I - \|A_4^{-1} \delta A_4\|} (\|A_3\| + A_3) \lim_{t\to\infty} \|x_1(t)\| \to 0$

Step 8: The solution of BTVDAs $\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ is asymptoticly stable.

7. Practical Implementation of the Controlled Perturbed Bilinear Time Varying Differential -Algebraic Equations

Example 1:

consider the composite Perturbed bilinear time varying system:

$$\begin{bmatrix} \dot{w}_{11} \\ \dot{w}_{12} \\ \dot{w}_{13} \end{bmatrix} = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2+cost & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0.1sint & 0 \\ 0.1sint & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.01 & 0.02 & 0 \\ 0.02 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \right\} u(t) \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.01 & 0.02 & 0 \\ 0.02 & 0.01 & 0.03 \\ 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \end{bmatrix} + \left\{ \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} + \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix} \right\} u(t) \begin{bmatrix} w_{21} \\ w_{22} \\ w_{23} \end{bmatrix}$$

Where

$$\dot{w}_1 = (A_1(t) + \delta A_1(t)) w_1(t) + (A_2 + \delta A_2) u(t) w_1(t)$$
$$0 = (A_3 + \delta A_3) w_1(t) + (A_4 + \delta A_4) w_2(t)$$

With

$$A_1(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2+cost & 0 \\ 0 & 0 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \delta A_1(t) = \begin{bmatrix} 0 & 0.1sint & 0 \\ 0.1sint & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\delta A_2 = \begin{bmatrix} 0.01 & 0.02 & 0 \\ 0.02 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}, \delta A_3 = \begin{bmatrix} 0.01 & 0.02 & 0 \\ 0.02 & 0.01 & 0.03 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\delta A_4 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}, u(t) = \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix}, w_1(t) = \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \end{bmatrix}, w_2(t) = \begin{bmatrix} w_{21} \\ w_{22} \\ w_{23} \end{bmatrix}$$

Step 1: To determine the space of consistent initial condition

$\mathcal{W} = \left\{ w_0(t) = \begin{bmatrix} w_1(t_0) \\ w_2(t_0) \end{bmatrix} | w_{11}(0) = 0, w_{13}(0) = -3w_{12}(0), w_{21}(0) = -9w_{12}(0), w_{22}(0) = -0.04, w_{23}(0) = 0 \right\}$.

Step 2: Calculate the non-singular matrix $p(t) = \begin{bmatrix} sint & -cost & 0 \\ cost & sint & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and Define $x_1(t) = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = p(t) \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \end{bmatrix}$

Step 3: The transition matrix can be calculated as follows:

$$\emptyset_{A_1(t)+\delta A_1(t)} = \emptyset_{A_1(t)}(t, t_0) + \int_{t_0}^{t} \emptyset_{A_1(t)}(t, \tau)\delta A_1(t)\emptyset_{A_1(t)}(\tau, t_0)d\tau$$

With $\emptyset_{A_1(t)}(t, t_0) = \begin{bmatrix} e^{-(t-t_0)} & 0 & 0 \\ 0 & e^{-2(t-t_0)-(sint-sint_0)} & 0 \\ 0 & 0 & e^{-3(t-t_0)} \end{bmatrix}$ and the integral account the effect of the small perturbation $\delta A_1(t)$

Step 4: Using the transition matrix to define generalized transition matrix as:

$\mathcal{Y}(t, t_0) = p(t) \begin{bmatrix} \emptyset_{A_1(t)+\delta A_1(t)} & 0 \\ 0 & 0 \end{bmatrix} p^{-1}(t)$

Where $\emptyset_{A_1(t)+\delta A_1(t)}$ denotes the transition matrix of $\dot{w}_1 = (A_1(t) + \delta A_1(t))w_1(t)$

Step 5:

$x_1 = \mathcal{Y}(t, t_0)\, x_1(t_0) + \int_{t_0}^{t} (A_2 + \delta A_2)u(\tau)\, \mathcal{Y}(t, \tau)\, x_1(\tau)d\tau$

$$\frac{\|x_1(t)\|}{\|\mathcal{Y}(t, t_0)\|\, \|x_1(t_0)\|} \leq 1 + \int_{t_0}^{t} \|\mathcal{Y}(t_0, \tau)\|\, \|A_2 + \delta A_2\|\, \|f(\tau)\| \|p(t)\|^{-2} \|p(t_0)\|^2 \|x_1(t_0)\|^{-3} \|x_1(\tau)\|^3 d\tau$$

$$\frac{\|x_1(t)\|}{\|\mathcal{Y}(t, t_0)\|\, \|x_1(t_0)\|} \leq 1 + \int_{t_0}^{t} \|\mathcal{Y}(t_0, \tau)\|^{-2}\, \|A_2 + \delta A_2\|\, \|f(\tau)\| \|p(t)\|^{-2} \left[\frac{\|x_1(\tau)\|}{\|\mathcal{Y}(t_0, \tau)\| \|x_1(t_0)\|} \right]^3 d\tau$$

Since $\frac{\|x_1(t)\|}{\|\mathcal{Y}(t,t_0)\|\, \|x_1(t_0)\|} \leq \frac{1}{\sqrt{1-2\gamma^{-2}(\|A_2\|+a_2)\int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\, \|f(\tau)\| \|p(t)\|^{-2} d\tau}}$

Step 6 : By setting $\gamma = 1, \alpha = 2$, we get

$$\|x_1(t)\| \leq \frac{e^{-4t}\, \|x_1(0)\|}{\sqrt{1 - 2(5)\int_0^t e^{-4\tau}\, \|f(\tau)\| \|p(t)\|^{-2} d\tau}}$$

Hence $\lim_{t\to\infty} \|x_1(t)\| \to 0$.

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And the nonlinear control $u(t)$ verfy the following inequality

$$\|u(t)\| \leq f(\tau)\|p(t)\|^{-2} \frac{e^{-4t}}{1-2(5)\int_{t_0}^{t} e^{-4\tau}\ \|f(\tau)\|\|p(t)\|^{-2} d\tau}$$

Step 7:

$$\lim_{t\to\infty}\|x_2(t)\| \leq \frac{-1}{2}\lim_{t\to\infty}\|x_1(t)\| \to 0$$

Then the solution of bilinear time varying Differential -algebraic Systems $\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = p(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is asymptoticly stable as shown in the following figures 1, 2.

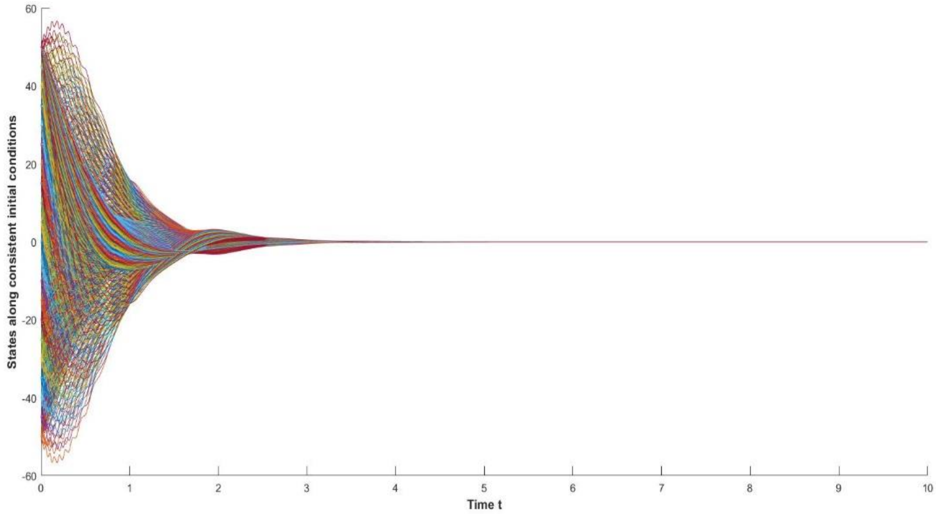

Figure 1. Asymptotically stability of the perturbed bilinear time varying differential- algebraic system along with consistent initial condition.

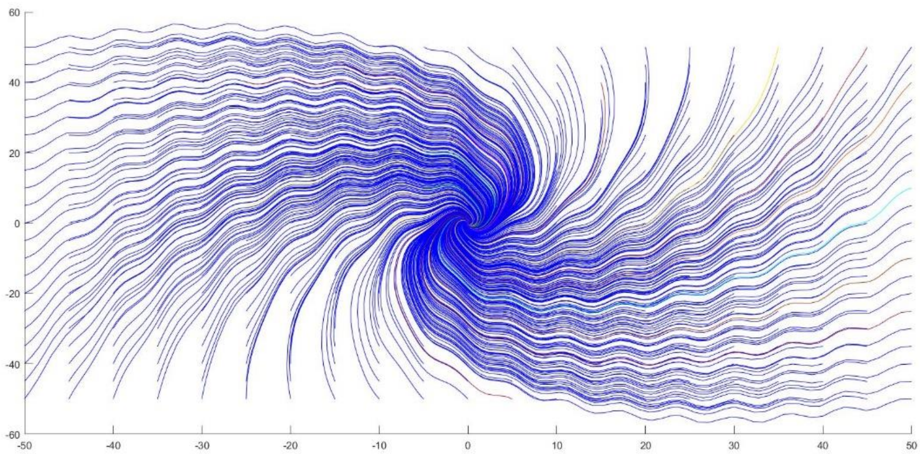

Figure 2. Representations the directional behaviour of the perturbed bilinear time varying differential- algebraic system.

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Example 2: Case Study under neutrosophic environment: battery model [7, 24]

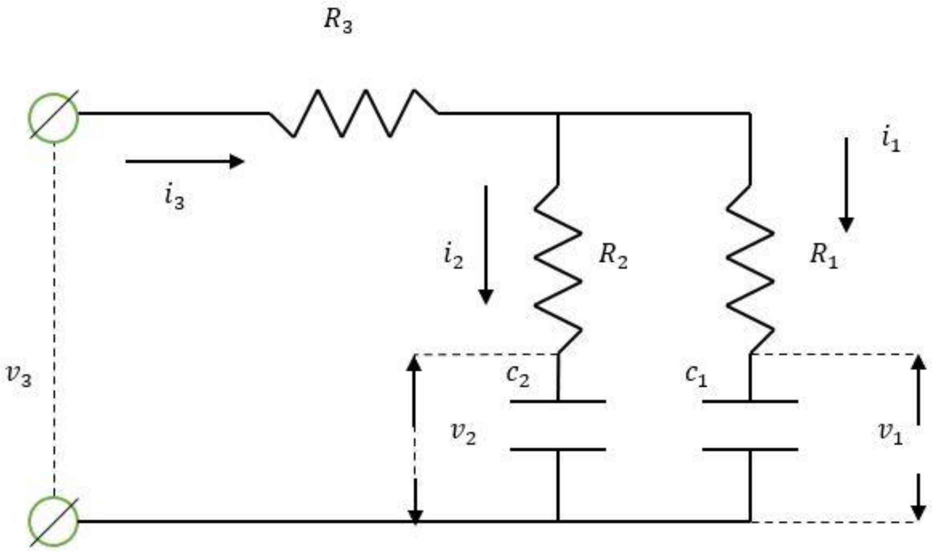

Figure 3. Circuit model of a Battery.

A resistance – capacitance equivalent circuit model is used to build a dynamic model of the battery, as shown in figure 3, where the variable $\mathcal{w}$ represents the state of charge. C_1 is the nominal capacity of the battery, v_1 represents the open circuit voltage, which is a function of state of charge, v_2 is the voltage across the polarized capacitor C_2. R_1 and R_2 represent the conduction resistance and the diffusion resistance, respectively. i_1 and i_2 are the currents of the two branches; R_3 is the terminal resistance, v_3 is the measurable terminal voltage, and i_3 is the charge and discharge current.

We will study it under a neutrosophic-represented environment, i.e. all coefficients are neutrosophic numbers with form a+bI.

From the definition of the state of charge, the dynamic relationship of state of charge is $\dot{\mathcal{w}} = \frac{i_1}{C_1}$. In addition, the dynamic equations of the battery are:

$$\dot{\mathcal{w}} = \frac{-v_1 + v_2}{C_1(R_1 + R_2)} + \frac{iR_2}{C_1(R_1 + R_2)}$$

$$\dot{v_3} = \frac{v_1 - v_2}{C_2(R_1 + R_2)} + \frac{iR_1}{C_2(R_1 + R_2)}$$

$$v_3 = \frac{R_2 v_2 + R_1 v_1}{R_1 + R_2} + \left(\frac{R_1 R_2}{R_1 + R_2} + R_3\right) i$$

Motivated by these considerations, this paper regards the function as an algebraic constraint between state variables, thereby modelling the system battery as a differential – algebraic system with $w = [\mathcal{w} \quad v_2 \quad v_1]^T$ is defined as state variable, and then the battery is modeled as follows:

$$\dot{\mathcal{w}} = -1.1\mathcal{w} + 0.25u_1\mathcal{w} + 0.25u_2v_2$$

$$\dot{v_2} = -e^{-t}v_2 + 1.01u_1\mathcal{w} + 1.01u_2v_2$$

$$0 = 2.1\mathcal{w} + 4.02v_1$$

Step 1: To determine the space of consistent initial condition

$$\mathcal{W} = \left\{ w_0(t + zI) = \begin{bmatrix} \mathcal{w}([t + zI]_0) \\ v_2([t + zI]_0) \\ v_1([t + zI]_0) \end{bmatrix} \middle| v_1(0) = \frac{-2.1}{4.02}\mathcal{w}, v_2(0) = 0 \right\}.$$

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Step 2: Calculate the non-singular matrix $p(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and Define $x_1(t+zI) = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = p(t+zI)\begin{bmatrix} w \\ v_2 \end{bmatrix}$

Step 3: The transition matrix can be calculated as follows:

$$\emptyset_{A_1(t+zI)+\delta A_1(t+zI)} = \emptyset_{A_1(t+zI)}(t+zI,[t+zI]_0) + \int_{[t+zI]_0}^{t+zI} \emptyset_{A_1(t+zI)}(t+zI,\tau)\delta A_1(t+zI)\emptyset_{A_1(t+zI)}(\tau,[t+zI]_0)d\tau$$

With $\emptyset_{A_1(t)}(t+zI,[t+zI]_0) = \begin{bmatrix} e^{-1.1[t+zI]} & 0 \\ 0 & e^{-[t+zI]} \end{bmatrix}$

Step 4: Using the transition matrix to define generalized transition matrix as:

$$\mathcal{Y}(t+zI,[t+zI]_0) = p(t+zI)\begin{bmatrix} \emptyset_{A_1(t+zI)+\delta A_1(t+zI)} & 0 \\ 0 & 0 \end{bmatrix} p^{-1}(t+zI)$$

Step 5:

$$x_1 = \mathcal{Y}(t+zI,[t+zI]_0)\, x_1([t+zI]_0) + \int_{[t+zI]_0}^{t+zI}(A_2+\delta A_2)u(\tau)\, \mathcal{Y}(t+zI,\tau)\, x_1(\tau)d\tau$$

$$\frac{\|x_1(t+zI)\|}{\|\mathcal{Y}(t+zI,[t+zI]_0)\|\;\|x_1([t+zI]_0)\|} \le 1 + \int_{[t+zI]_0}^{t+zI} \|\mathcal{Y}([t+zI]_0,\tau)\|\;\|A_2 + \delta A_2\|\;\|f(\tau)\|\|p(t+zI)\|^{-2}\|p([t+zI]_0)\|^2\|x_1([t+zI]_0)\|^{-3}\|x_1(\tau)\|^3 d\tau$$

$$\frac{\|x_1(t+zI)\|}{\|\mathcal{Y}(t+zI,[t+zI]_0)\|\;\|x_1([t+zI]_0)\|} \le 1 + \int_{[t+zI]_0}^{t+zI} \|\mathcal{Y}([t+zI]_0,\tau)\|^{-2}\;\|A_2 + \delta A_2\|\;\|f(\tau)\|\|p(t+zI)\|^{-2}\left[\frac{\|x_1(\tau)\|}{\|\mathcal{Y}([t+zI]_0,\tau)\|\|x_1([t+zI]_0)\|}\right]^3 d\tau$$

Since $\frac{\|x_1(t+zI)\|}{\|\mathcal{Y}(t+zI,[t+zI]_0)\|\;\|x_1([t+zI]_0)\|} \le \frac{1}{\sqrt{1-2\gamma^{-2}(\|A_2\|+a_2)\int_{t_0}^{t} e^{2\alpha(t_0-\tau)}\;\|f(\tau)\|\|p(t+zI)\|^{-2}d\tau}}$

Step 6: By setting $\gamma = 1, \alpha = 1$, we get

$$\|x_1(t+zI)\| \le \frac{e^{-2[t+zI]}\;\|x_1(0)\|}{\sqrt{1-2(4)\int_0^t e^{-2\tau}\;\|f(\tau)\|\|p(t+zI)\|^{-2}d\tau}}$$

Hence $\lim_{t\to\infty}\|x_1(t+zI)\| \to 0$.

And the nonlinear control $u(t)$ verfy the following inequality

$$\|u(t+zI)\| \le f(\tau)\|p(t+zI)\|^{-2}\frac{e^{-2[t+zI]}}{1-2(4)\int_{[t+zI]_0}^{t+zI} e^{-2\tau}\;\|f(\tau)\|\|p(t+zI)\|^{-2}d\tau}$$

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Step 7:

$$\lim_{t\to\infty}\|x_2(t+zI)\| \le \frac{-1}{3.04}\lim_{t\to\infty}\|x_1(t+zI)\| \to 0$$

Then the solution of bilinear time varying Differential -algebraic Systems $\begin{bmatrix} w \\ v_2 \\ v_1 \end{bmatrix} = p(t+zI)\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \end{bmatrix}$ is asymptoticly stable and perturbed factor and the current of the time varying of the model are given in figures 4, 5 respectively.

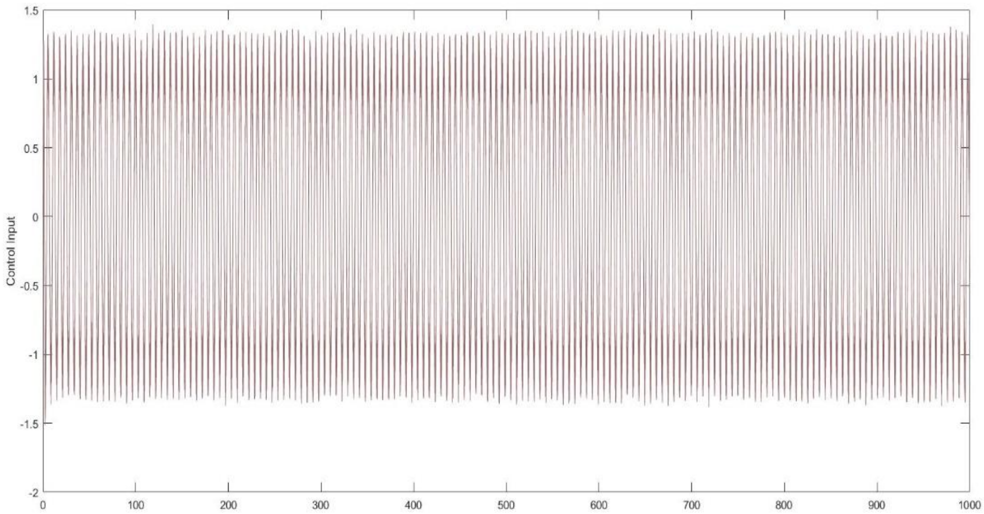

Figure 4. Full graph of the perturbed factor in the battery model.

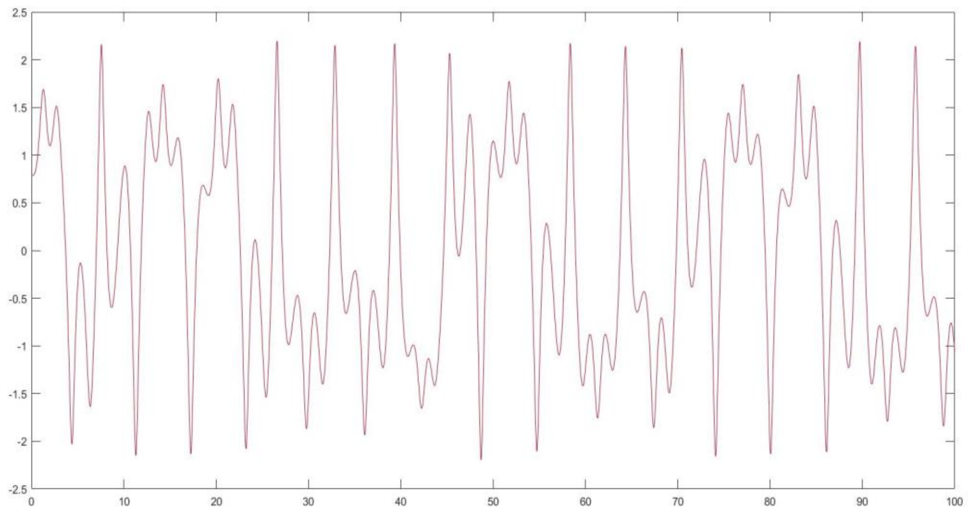

Figure 5. Zoomed graph of the current of the time varying bilinear differential -algebraic battery system.

8. Conclusion

This paper presents novel sufficient conditions for the designing generalized transition matrix of controlled bilinear time varying differential – algebraic equations with the presence of unknown inputs. These conditions are presented in the form of algebraic constrains on the rank of time varying matrices of the given model.

An algorithm is proposed to summarize the design procedure, the order of the stabilization is given in term of system matrices, and examples of practical models are implemented to confirm the theoretical results.

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9. Future works

1-The stability of more general semi-explicit bilinear time varying differential – algebraic equations under the perturbed conditions can be defined in completely analogous manner.

2-Calculations with respect to the non-bilinear systems are not much more complicated than the bilinear one.

We therefore feel that our approach could be made quite effective from more complicated time varying differential – algebraic equations

Data Availability: The data used in the article are generated by octave and can be made available upon request.

Conflicts of Interest: I declare no interest conflicts in publishing this work.

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