



A Study of Some Neutrosophic Derivatives Problems Based On Newton's BDF and CDF Numerical Methods

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Abstract

This paper is dedicated to study for the first time the applications of neutrosophic BDF and CDF Newton's methods for finding the numerical solutions of some different problems related to the derivations from first and second order applied on neutrosophic-tabulated functions, where we apply those novel methods on some problems and list the solutions by using the numerical tables. In addition, we provide a theoretical discussion and description of these methods to be applicable on other numerical problems.

Keywords: Numerical table; Numerical function; Derivation; Newton's methods

1. Introduction

Numerical analysis is one of the most important branches of both applied and theoretical mathematics, as it uses algorithms and numerical techniques to find numerical solutions to many problems that may be difficult to find an exact solution for [1-2].

In the scientific literature, numerical analysis and neutrosophic numerical algorithms have been studied in many references [3-5], especially the methods of Euler and Newton and their applications in solving differential problems [6-8], where numerical tables have been found that simulate exact solutions to these problems in the field of neutrosophic numbers [9-10]. These new numerical approaches are useful for analyzing problems that depend on uncertainty [12], through numerical algorithms that are a generalization of traditional numerical algorithms [11, 13-14].

In this work, we will study for the first time the applications of neutrosophic BDF and CDF Newton's methods for finding the numerical solutions of some different problems related to the derivations from first and second order applied on neutrosophic-tabulated functions, where we apply those novel methods on some problems and list the solutions by using the numerical tables. We note that the numerical results and methods discussed in this work will be useful in studying higher-order numerical derivatives, such as the third, fourth, and fifth orders. Furthermore, the tabulated numerical results are applicable to various higher-order numerical problems.

2. Main Discussion

Neutrosophic Derivatives by Newton's BDF:

Neutrosophic Newton's Backward Difference Interpolation Formula is:

$$f(x + yI) = f_n + (r + qI)\Delta f_n + \frac{(r + qI)(r + qI + 1)}{2!}\Delta^2 f_n + \frac{(r + qI)(r + qI + 1)(r + qI + 2)}{3!}\Delta^3 f_n + \dots \dots \dots \quad (1)$$

Where $r + qI = \frac{x+yI-x_n-y_nI}{h}$ (2)

$$\frac{df}{d(r+qI)} = \left[\Delta f_n + \frac{2(r+qI)+1}{2}\Delta^2 f_n + \frac{3(r+qI)^2+6(r+qI)+2}{6}\Delta^3 f_n + \frac{4(r+qI)^3+18(r+qI)^2+22(r+qI)+6}{24}\Delta^4 f_n + \dots \right].$$

we have $\frac{d(r+qI)}{d(x+yI)} \approx \frac{1}{h}$, now:

$$\begin{aligned} \frac{df}{d(x+yI)} &= \frac{df}{d(r+qI)} \cdot \frac{d(r+qI)}{d(x+yI)} \\ &= \frac{1}{h} \left[\nabla f_n + \frac{2(r+qI)+1}{2}\Delta^2 f_n + \frac{3(r+qI)^2+6(r+qI)+2}{6}\Delta^3 f_n + \frac{4(r+qI)^3+18(r+qI)^2+22(r+qI)+6}{24}\Delta^4 f_n + \dots \right] \end{aligned} \quad (3)$$

$$\begin{aligned} \left(\frac{df}{d(x+yI)}\right)(x+yI) &\approx \left(\frac{df}{d(x+yI)}\right)(x+yI)_n \\ &= \frac{1}{h} \left[\Delta f_n + \frac{1}{2}\Delta^2 f_n + \frac{1}{3}\Delta^3 f_n + \frac{1}{4}\Delta^4 f_n + \dots \right] \end{aligned} \quad (4)$$

$$\left(\frac{d^2f}{d(x+yI)^2}\right)(x+yI) \approx \frac{1}{h^2} \left[\Delta^2 f_n + \Delta^3 f_n + \frac{11}{12}\Delta^4 f_n + \frac{5}{6}\Delta^5 f_n + \dots \right]$$

Similarly, $\left(\frac{d^3f}{d(x+yI)^3}\right)(x+yI) \approx \frac{1}{h^3} \left[\Delta^3 f_n - \frac{3}{2}\Delta^4 f_n + \dots \right]$ (5)

• **Neutrosophic Derivatives based Central Difference Formula CDF**

$$f = f_0 + \frac{r+qI}{2}(\Delta f_0 + \Delta f) + \frac{(r+qI)^2}{2}\Delta^2 f_{-1} + \frac{(r+qI)^3 - (r+qI)}{12}(\Delta^3 f_{-1} + \Delta^3 f_{-2}) + \frac{(r+qI)^4 - (r+qI)^2}{24}\Delta^4 f_{-2} + \dots \dots \quad (6)$$

Where $r + qI = \frac{x+yI-(x+yI)_0}{h}$

$$\begin{aligned} \frac{df}{d(x+yI)} &= \frac{df}{d(r+qI)} \cdot \frac{d(r+qI)}{d(x+yI)} = \frac{1}{h} \cdot \frac{df}{d(r+qI)} \\ &= \frac{1}{h} \left[\frac{1}{2}(\Delta f_0 + \Delta f_{-1}) + \Delta^2 f_{-1} + \frac{3(r+qI)^2 - 1}{12}(\Delta^3 f_{-1} + \Delta^3 f_{-2}) + \frac{(2(r+qI)^3 - (r+qI))}{12}\Delta^4 f_{-2} + \frac{5(r+qI)^4 - 15(r+qI)^2 - 4}{240}(\Delta^5 f_{-2} + \Delta^5 f_{-3}) + \dots \right] \end{aligned}$$

$$\left(\frac{d^2f}{d(x+yI)^2}\right)(x+yI) \approx \frac{1}{h^2} \left[\Delta^2 f_{-1} - \frac{1}{12}\Delta^4 f_{-2} + \frac{1}{90}\Delta^6 f_{-3} - \dots \right]$$

$$\left(\frac{d^3f}{d(x+yI)^3}\right)(x+yI) \approx \frac{1}{h^3} \left[\frac{1}{2}(\Delta^3 f_{-1} + \Delta^3 f_{-2}) + \dots \right]$$

• **Max and Min of a neutrosophic Function**

Given a set of neutrosophic data $(x_i + I, y_i + I), i = 0, 1, 2, \dots, n$, we can get the interpolating polynomial of degree n. Now we wish to estimate the value of x at which the curve is maximum or minimum.

Consider the neutrosophic Newton's forward difference interpolation formula:

$$f \approx f_0 + (r + qI)\Delta f_0 + \frac{(r+qI)(r+qI+1)}{2!}\Delta^2 f_0 + \frac{(r+qI)(r+qI-1)(r+qI-3)}{3!}\Delta^3 f_0 + \dots \dots \dots$$

$$\frac{df}{d(r+qI)} = \left[\Delta f_n + \frac{2(r+qI)-1}{2} \Delta^2 f_n + \frac{3(r+qI)^2 - 6(r+qI) + 2}{6} \Delta^3 f_n + \dots \dots \dots \right] \tag{7}$$

$$\left[\Delta f_n + \frac{2(r+qI)-1}{2} \Delta^2 f_n + \frac{3(r+qI)^2 - 6(r+qI) + 2}{6} \Delta^3 f_n + \dots \dots \dots \right] = 0$$

$$\left(\frac{1}{2} \Delta^3 f_0\right) p^2 + (\Delta^2 f_0 - \Delta^3 f_0) 0 + \left(\Delta f_0 + \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0\right) = 0 \tag{8}$$

Substituting the values of $\Delta f_0, \Delta^2 f_0, \frac{1}{3} \Delta^3 f_0$, from the difference table, we solve the equation (8) for p.

Problem 1 Given the following neutrosophic data find $f'(6+I)$

x + yI:	I	2+I	3+I	4+I	7+I	9+I
f:	4+I	26+I	58+I	112+I	466+I	922+I

$[f'(6+I) = 135+I]$

Problem 2: Find the first and second derivatives of the function at the point $x = 1.9 + I$.

x + yI:	1.5+I	2.0+I	2.7+I	3.1.0+I	3.8+I	4.0+I
f:	3.215+I	8.0+I	13.778+I	25.0+I	39.525+I	61.0+I

$[f'(6+I) = 4.75+I; f''(6+I) = 9+I]$

Problem 3: Given the following neutrosophic data find $f'(6+I)$

x + yI:	I	2+I	3+I	4+I	7+I	9+I
f:	4+I	26+I	58+I	112+I	466+I	922+I

$[f'(6+I) = 135+I]$

Solution: Given the following neutrosophic data:

x + yI	f(x + yI)	Δf(x + yI)	Δ²f(x + yI)	Δ³f(x + yI)	Δ⁴f(x + yI)
I	4+I				
		11 + I			
2+I	24+I		8 + I		
		33 + I		1 + I	
3+I	59+I		12 + I		I
		57 + I		1 + I	
4+I	118+I		17 + I		I
		119 + I		1 + I	
8+I	469+I		24 + I		
		237 + I			
10+I	930+I				

$$\begin{aligned}
 f(x + yI) &= f((x + yI)_0) + (x + yI - (x + yI)_0)f((x + yI)_0, (x + yI)_1) \\
 &\quad + (x + yI - (x + yI)_0)(x + yI - (x + yI)_1)f((x + yI)_0, (x + yI)_1, (x + yI)_2) + \dots \dots \dots \\
 &= 4 + I + (x + yI - 0) \times 11 + I + (x + yI - 0)(x + yI - 2) \times 7 + I \\
 &\quad + (x + yI - 0)(x + yI - 2)(x + yI - 3) \times 1 + \dots \dots \dots \\
 &= 4 + I + 11(x + yI) + 7((x + yI)^2 - 2(x + yI)) + (x + yI)((x + yI)^2 - 5(x + yI) + 6) = (x + yI)^3 + 2(x + yI)^2 + 3(x + yI) + 4 + I \text{ is the solution.}
 \end{aligned}$$

Problem 4: Find the value of the neutrosophic variable $x+yI$ for which f is maximum and find this value of f .

X+YI	1.3+I	1.4+I	1.5+I	1.6+I	1.7+I
f	0.9101+I	0.9622+I	0.9878+I	0.9965+I	0.9998+I

Solution:

X+YI	f	Δf	$\Delta^2 f$	$\Delta^3 f$
1.3+I	0.9101+I			
		0.0336+I		
1.4+I	0.9622+I		-0.0096+I	
		0.0223+I		-0.0001+I
1.5+I	0.9878+I		-0.0098+I	
		0.0113+I		I
1.6+I	0.9965+I		-0.0099+I	
		0.00134+I		
1.7+I	0.9998+I			

Here $h = 0.1+0.1I$.

For

$$x_0 = 1.3 + I, \text{ we have}$$

$$f_0 = 0.9101 + I,$$

$$\Delta f_0 = 0.0336 + I, \Delta^2 f_0 = -0.0096 + I \text{ and } \Delta^3 f_0 = -0.0001 + I.$$

By using the neutrosophic Newton’s Forward Difference Formula, we find

$$f = 0.9101 + I + (r + qI)(0.0336 + I) + \frac{(r+qI)(r+qI-1)}{2}(-0.0096 + I) \dots \dots \dots (1)$$

$$\therefore \frac{df}{d(r + qI)} \approx 0.0336 + I + \frac{2(r + qI) - 1}{2}(-0.0096 + I)$$

For f to be maximum,

$$\begin{aligned}
 \frac{df}{d(r + qI)} \approx 0 &\Rightarrow \frac{df}{d(r + qI)} \approx 6.5122314 + I \\
 &\Rightarrow \boxed{p = 3.7282 + I}
 \end{aligned}$$

$$\text{Hence } x = x_0 + ph = 1.3 + I + (3.7282 + I)(0.1 + 0.1I)$$

Problem 5:

X+YI	0.6+I	0.65+I	0.7+I	0.75+I
f	0.6221+I	0.6155+I	0.6138+I	0.617+I

Solution:

X+YI	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0.6+I	0.6221+I			
		-0.0066+I		
0.65+I	0.6155+I		0.0049+I	
		-0.0017+I		I
0.7+I	0.6138+I		0.0049+I	
		0.0032+I		
0.75+I	0.617+I			

For $x_0 = 0.6 + I$, we have $f_0 = 0.6221 + I, \Delta f = -0.0066 + I, \Delta^2 f_0 = 0.0049 + I$ and $\Delta^3 f = I$.

$$f = 0.6221 + I + (r + qI)(-0.0066 + I) + \frac{(r + qI)(r + qI - 1)}{2}(0.0049 + I) \dots\dots\dots (1)$$

$$\frac{df}{d(r+qI)} \approx -0.0066 + I + \frac{2(r+qI)-1}{2}(0.0049 + I) \text{ For } f \text{ to be minimum,}$$

$$\begin{aligned} \frac{df}{d(r + qI)} \approx I &\Rightarrow \frac{df}{dr + qI} \approx -0.0066 + I + \frac{2(r + qI) - 1}{2}(0.0049 + I) = I \Rightarrow 2(0.0066 + I) \\ &= (2(r + qI) - 1)(0.0049 + I) \Rightarrow 2(r + qI) - 1 = \frac{2(0.0066 + I)}{0.0049 + I} \\ &\Rightarrow \boxed{r + qI = 1.8469 + I} \end{aligned}$$

Hence $x + yI = (x + yI)_0 + (r + qI)h = 0.6 + I + (1.8469 + I)(0.05 + I)$.

Problem 6: Compute the first and second neutrosophic derivatives for the following table of data at $x = -3 + I$ and $x = I$

X+yI	-3	-2	-1	0	1	2	3
f	-33	-12	-3	0	3	12	33

Solution:

x + yI	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
-3+I	-33+I						
		21+I					
-2+I	-12+I		-12+I				
		9 + I		6+I			
-1+I	-3+I		-6+I		I		
		3+I		6+I		I	

0+I	0+I		0+I		I		I
		3+I		6+I		I	
1+I	3+I		6+I		I		
		9+I		6+I			
2+I	12+I		12+I				
		21+I					
3+I	33+I						

Here $(x + yI)_0 = -3 + I, f_0 = -33 + I$ and $h = 1 + I \left(\frac{dy}{dx}\right) x = x_0 = \frac{1}{h} \left[\Delta f_0 - \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0 - \frac{1}{4} \Delta^4 f_0 + \frac{1}{5} \Delta^5 f_0 - \frac{1}{6} \Delta^6 f_0 + \dots \dots \right]$

$$\left(\frac{df}{dx + yI}\right) x + yI = (x + yI)_0 = \frac{1}{1} \left[21 - \frac{1}{2}(-12 + I) + \frac{1}{3}(6 + I) - \frac{1}{4}(0 + I) + \frac{1}{5}(0 + I) - \frac{1}{6}(0 + I) \right]$$

$$\left(\frac{df}{dx + yI}\right) x + yI = x + yI_0 = \frac{1}{1} [21 + 6 + 2 + I - 0 + 0 - 0] = 29 + I \Rightarrow \left(\frac{df}{dx + yI}\right) x + yI = (x + yI)_0 = 29 + I$$

$$\left(\frac{d^2f}{dx + yI^2}\right) x + yI = x + yI_0 = \frac{1}{h^2} \left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12} \Delta^4 f_0 - \frac{5}{6} \Delta^5 f_0 + \dots \dots \right]$$

$\left(\frac{d^2f}{dx + yI^2}\right) x + yI = x + yI_0 = \frac{1}{1^2} \left[(-12 + I) - 6 + \frac{11}{12}(0) - \frac{5}{6}(0) \right] \Rightarrow \left(\frac{d^2y}{dx + yI^2}\right) x + yI = x + yI_0 = -18 + I$ is solution.

Problem 7: Compute the first and second neutrosophic derivatives at $x = 1.1 + I$

X+yI	1.0+I	1.2+I	1.4+I	1.6+I	1.8+I	2.0+I
f	I	0.128+I	0.544+I	1.296+I	2.432+I	4.0+I

Solution: the difference table is as below:

X+yI	f	Δf	Δ²f	Δ³f	Δ⁴f	Δ⁵f
1.0+I	I					
		0.128+I				
1.2+I	0.128		0.288+I			
		0.416 + I		0.048+I		
1.4+I	0.544+I		0.336+I		I	
		0.752+I		0.048+I		I
1.6+I	1.296+I		0.384+I		I	
		1.136+I		0.048+I		
1.8+I	2.432+I		0.432+I			
		1.568+I				
2.0+I	4+I					

Here $(x + yI)_0 = 1 + I$, $f_0 = 0$ and $h = 0.2 + I$.

$$\left(\frac{df}{dx + yI}\right) x + yI = x + yI_0 = \frac{1}{h} \left[\Delta f_0 - \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0 - \frac{1}{4} \Delta^4 f_0 + \frac{1}{5} \Delta^5 f_0 - \dots \dots \right]$$

$$\left(\frac{df}{dx + yI}\right) x + yI = x + yI_0 = \frac{1}{0.2} \left[0.128 + I - \frac{1}{2} (0.288 + I) + \frac{1}{3} (0.048 + I) - \frac{1}{4} (0 + I) + \frac{1}{5} (0 + I) \right]$$

$$\left(\frac{df}{dx + yI}\right) x + yI = x + yI_0 = \frac{1}{0.2} [0.128 - 0.144 + 0.016 - 0 + 0 + I] = I \Rightarrow \left(\frac{df}{dx + yI}\right) x + yI = 1.1 + I \\ = 0.128/2 + I \approx 0.64 + I$$

$$\left(\frac{d^2f}{dx + yI^2}\right) x + yI = x + yI_0 = \frac{1}{h^2} \left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12} \Delta^4 f_0 - \frac{5}{6} \Delta^5 f_0 + \dots \dots \right]$$

$$\therefore \left(\frac{d^2f}{dx + yI^2}\right) x + yI = x + yI_0 = \frac{1}{0.2^2} \left[(0.288 + I) - 0.048 + I + \frac{11}{12} (0 + I) - \frac{5}{6} (0 + I) \right]$$

$$\left(\frac{d^2f}{dx + yI^2}\right) x + yI = x + yI_0 = \frac{1}{0.04} [0.240 + I] \approx 6 + I \Rightarrow \left(\frac{d^2f}{dx + yI^2}\right) x = 1.1 + I = 6 \times 1.1 + I = 6.6 + I .$$

3. Conclusion

In this paper, we studied for the first time the applications of neutrosophic BDF and CDF Newton's methods for finding the numerical solutions of some different problems related to the derivations from first and second order applied on neutrosophic-tabulated functions, where we applied those novel methods on some problems and list the solutions by using the numerical tables. In addition, we provided a theoretical discussion and description of these methods to be applicable on other numerical problems. We note that the numerical results and methods discussed in this work will be useful in studying higher-order numerical derivatives, such as the third, fourth, and fifth orders. Furthermore, the tabulated numerical results are applicable to various higher-order numerical problems.

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