Hessian matrix for testing the convexity and concavity of the objective function in nonlinear programming and neutrosophic nonlinear programming problems

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Abstract

Mathematical examples rely on constructing mathematical models consisting of an objective function and constraints. These models may be linear, nonlinear, or otherwise. The objective function is either a maximization function or a minimization function for a given quantity. Nonlinear programming constitutes an important and fundamental part of operations research and is more comprehensive than linear programming. Therefore, researchers have focused on presenting studies that help find the optimal solution to these problems. Most of these studies have focused on the importance of knowing the type of objective function—whether it is convex or concave—because this knowledge helps determine the type of maximum value we obtain when studying a nonlinear programming problem. The Hessian matrix was used for this purpose. In this research, we will present the most important concepts that can be used when determining the type of maximum value for a nonlinear programming problem, as mentioned in some classic references. We will then reformulate them using the concepts of neutrosophic logic.

Keywords: Operations research; Nonlinear models; Neutrosophic logic; Neutrosophic nonlinear models; Concavity of a functions; Convexity of functions; Hessian matrix

1. Introduction

We call mathematical programming the science that investigates determining the maximum or minimum value of a specific function, called the objective function, which depends on a number of variables that may be interconnected or independent of each other, known as constraints. It is one of the methods of operations research, a science that emerged during World War II due to the presence of many complex strategic issues that require optimal solutions. This science has gained a prominent position among the mathematical methods used in planning and managing various economic and military activities because it helps specialists develop ideal plans in terms of costs, production, financing, distribution, profit, or investment of human resources and energies. Its essence is to build a practical scientific model for a specific system that includes identifying the influencing factors and then obtaining the optimal solution for this model. The mathematical model is a nonlinear model if any component of the objective function or constraints is a nonlinear expression, and the nonlinear expressions may be in both. The region of possible solutions for a nonlinear mathematical model is the set of vectors whose components satisfy all constraints, while the optimal solution is the vector that satisfies all constraints and where the function reaches an optimal value (maximum or minimum). The efforts of researchers in the field of nonlinear programming have focused on presenting many concepts and methods that help in obtaining the optimal solution. In this research, we present some of these concepts based on what was mentioned in references [1-4]. Then we will reformulate these concepts using the new vision presented by logic. Neutrosophic, the logic that is the latest achievement of logical

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thought in its persistent quest to comprehend reality and expresses the uprising of the mind when it collides with the unreasonableness of reality or with a false rationality that covers fossilized systems that seek stability in a world in which the pace of development and change is accelerating. To view the stages of development of this science from its inception until today and the most important research and studies that have been published using the concepts of this logic, which have included most fields of science, see [5].

2. Discussion

The mathematical model consists of decision variables, an objective function, and constraints. The region of possible solutions for a nonlinear mathematical model is the set of vectors whose components satisfy all constraints. The optimal solution is the vector that satisfies all constraints and at which the function reaches a maximum value (maximum or minimum).

Previous Studies:

- **Unconstrained Problems: [4]**

Solving unconstrained nonlinear programming problems means finding the general minimum or maximum value of a real function f that follows n real variables $x_1, x_2, \dots, x_n$, which takes values from $-\infty$ to $+\infty$, meaning there is no constraint on the vector $x \in R^n$. We are interested in this type of problem because the optimal conditions for solving constrained problems are merely a logical extension of the conditions for solving unconstrained problems.

Definition of an unconstrained nonlinear programming problem:

Let us have a function $f: R^n \longrightarrow R$ defined in the form $f(x) = f(x_1, x_2, \dots, x_n)$ for any vector $x \in R^n$:

$$Min f(x)$$

$$x \in R^n$$

We say that this problem is unconstrained. That is, we must look for a point $\bar{x} \in R^n$ such that:

$$f(\bar{x}) \leq f(x) \quad ; \forall x \in R^n \quad (1)$$

We say that the point $\bar{x}$ that satisfies (1) is a general minimum limit point.

If the point $\bar{x}$ satisfies the relation:

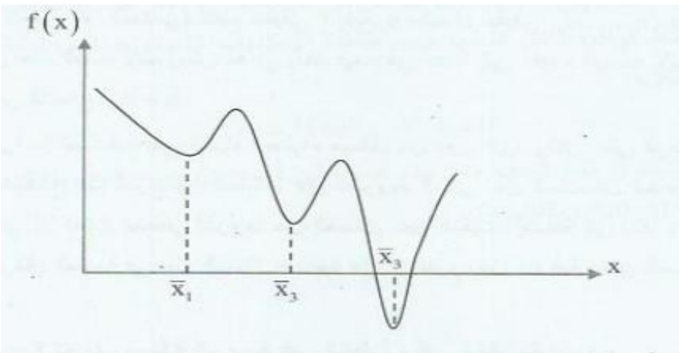
$$f(\bar{x}) < f(x) \quad ; \forall x \in R^n$$

we say that the point $\bar{x}$ is a unique general minimum limit point. Definition: If there is a neighborhood $N(x)$ around the point $\bar{x}$ such that the relation:

$$f(\bar{x}) < f(x) \quad ; \forall x \in N(x)$$

is satisfied only in the neighborhood, then $\bar{x}$ is said to be a local minimum limit.

Refer to the following figure:



We note that:

$$f(\bar{x}_1) < f(x) \quad ; \forall x \in N(\bar{x})$$

$\bar{x}_1$ is a local minimum limit.

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$$f(\bar{x}_3) \leq f(x) \quad ; \forall x \in R^n$$

$\bar{x}_3$ is a general minimum limit.

- **Necessary and sufficient conditions for the existence of general and local minimums**: $[\mathbf{2}, \mathbf{4}]$

Assume that the function $f(x)$ is continuous and that its first partial derivatives $\frac{\partial f}{\partial x_n}$ and second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are also continuous for any point $x \in R^n$, then we have:

Theorem: If the point $\bar{x} \in R^n$ is a local minimum limit of the function $f(x)$, then:

a. $\nabla f(\bar{x}) = 0$
b. The Hessian matrix of the function f:

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

Is a positive near-perfect matrix.

We say that the point x^* that satisfies condition (a) of the previous theorem, i.e., $\frac{\partial f}{\partial x_i}(x^*)$; $i = 1,2, \dots, n$, is a stationaries (critical) point.

Theorem: With the same assumptions as the previous theorem, we say that the point x is a local minimum limit of the function $f(x)$ over R^n if:

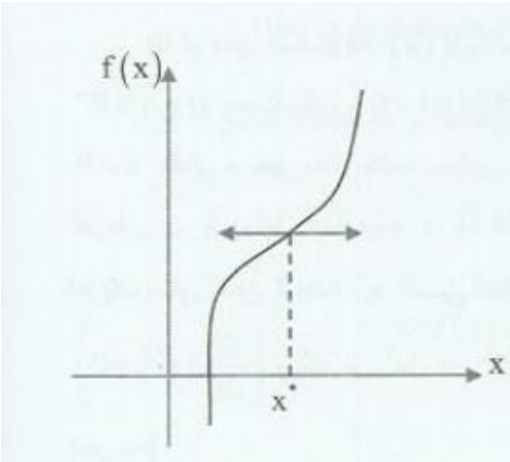
$$\nabla f(\bar{x}) = 0$$

The Hessian matrix of the function f:

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

Is a positive near-perfect matrix.

Note: A stationary point does not necessarily have to be a local minimum endpoint.



The figure shows a stationary point x^*, but it is not a local limit. We can also see from this figure that the sufficient conditions for the existence of a local minimum limit are not met. Because the function f accepts an inflection point x^*, the Hessian matrix is not perfectly positive. Rather, it is almost perfectly positive.

$$\frac{\partial^2 f}{\partial x^2}(x^*) = 0$$

- **Convex Functions:** $[\mathbf{2}, \mathbf{4}]$

When the function f is convex and defined on R^n, we can obtain necessary and sufficient conditions for a point to be a general minimum limit point. This can be summarized in the following theorem:

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Theorem: If f is a convex, continuously differentiable function, then the necessary and sufficient condition for the point x to be a general minimum limit point of the function f on R^n is that $\nabla f(\bar{x}) = 0$.

Necessary and sufficient conditions for the existence of general and local maximum limits:

Similarly, to those for general and local minimum limits, we summarize the necessary and sufficient conditions for the existence of maximum limits with the following theorems:

Theorem: If the function $f(x)$ is continuous and its first partial derivatives, $\frac{\partial f}{\partial x_i}$ and the second, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, are also continuous for any point $x \in R^n$, then we have:

If the point $\bar{x}$is a maximum limit (general or local) of the function f, then:

$$\nabla f(\bar{x}) = 0$$

The Hessian matrix of the function f:

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 \mathrm{f}}{\partial x_i \partial x_j}\ (\bar{x})\right]$$

is a near-perfect negative matrix.

Theorem: Using the same assumptions as the previous theorem, we say the point $\bar{x}$ is a local maximum limit point of the function $f(x)$ on R^n if: $\nabla f(\bar{x}) = 0$

The Hessian matrix of the function f $\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$

is a positive perfect matrix.

Result: Assume that $\nabla f(\bar{x}) = 0$

- If $H(\bar{x})$ is perfectly positive, then $\bar{x}$ is a local minimum limit.
- If $H(\bar{x})$ is perfectly negative, then $\bar{x}$ is a local maximum limit.
- If $H(\bar{x})$ is almost perfectly positive (or almost perfectly negative), then $\bar{x}$ may or may not be a maximum (here we use limit relations).
- If $H(\bar{x})$ is neither perfectly negative nor perfectly positive, then $\bar{x}$ is neither a minimum nor a maximum.

Neutrosophic Mathematical Model:

In the problem of examples where the objective and constraints are in the form of neutrosophic mathematical functions, then the neutrosophic mathematical model is written in the following form:

$$Nf = Nf(x_1, x_2, --, x_n) \rightarrow (Max)\,or\,(Min)$$

According to the following restrictions:

$$Ng_i(x_1, x_2, --, x_n) \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} Nb_i \quad ; i = 1,2, ---, m$$

$$x_1, x_2, --, x_n \geq 0$$

In this model, such variables in the objective function and in constraints are neutrosophic values as well as the second side of the relationships that represent constraints; this model is a nonlinear model if any component of the target function or constraints are nonlinear statements and the nonlinear statements may be in both.

Some Important Neutrosophic Definitions in Nonlinear Programming

- **The quadrature form of the neutrosophic target function:**

A function with variables $f(x_1, x_2, --, x_n)$ is called a quadratic form if the following is true:

$$Nf(x_1, x_2, --, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} (Nq_{ij}) x_i x_j = X^T (NQ)\, X$$

Where $NQ_{n\times n} = [Nq_{ij}]$is a neutrosophic matrix, meaning that all or some of its elements are neutrosophic values of the shape $Nq_{ij} = q_{ij} + \varepsilon_{ij}$ where$\boldsymbol{\varepsilon_{ij}}$ is the indeterminism of the elements of the matrix and any neighborhood may be close to q_{ij}

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It shall be written in one of the following forms: $[\boldsymbol{\lambda}_{1ij}, \boldsymbol{\lambda}_{2ij}]$ *or* $\{\boldsymbol{\lambda}_{1ij}, \boldsymbol{\lambda}_{2ij}\}$ or otherwise, and $X^T = (x_1, x_2, --, x_n)$

Here we can always assume without prejudice to the generality of the problem that NQ is symmetrical, we can replace the matrix NQ with the symmetrical matrix $\frac{(NQ+NQ^T)}{2}$ without changing the quadratic form.

- **Basic Minor Neutrosophic Matrices:**

If the matrix NQ has dimensions $n \times n$, then the basic minimum matrix of the rank k is a mini-matrix with $k \times k$ dimensions and we obtain it by neglecting any $(n-k)$ lines and the corresponding columns of the matrix

Note: The determinant of the basic minor neutrosophic matrix is called the basic determinant and for each square matrix of the rank, $n \times n$ there is $2^n - 1$ basic determinant.

- **Main Basic Minor Neutrosophic Matrices:**

The main basic minor neutrosophic matrices from rank k for matrix $n \times n$ are obtained from the matrix NQ by neglecting the last lines $n-k$ and its corresponding columns.

Example 1: Find the basic minor matrices and the main basic minor matrices from the following neutrosophic matrix:

$$NQ = \begin{bmatrix} [0,1] & [2,3] & [4,5] \\ [6,7] & [8,9] & [10,11] \\ [12,13] & [14,15] & [16,17] \end{bmatrix}$$

- **Basic minor matrices:**

Of the first rank: the elements of the main diameter

[0,1], [8,9] ,[16,17]

From the second rank: the following matrices:

$$\begin{bmatrix} [0,1] & [2,3] \\ [6,7] & [8,9] \end{bmatrix}, \begin{bmatrix} [6,7] & [10,11] \\ [12,13] & [16,17] \end{bmatrix}, \begin{bmatrix} [8,9] & [10,11] \\ [14,15] & [16,17] \end{bmatrix}$$

From the third rank: NQ

- **Main basic minor matrices:**

From the first rank, [0,1](we neglect the last two lines and the corresponding columns).

As for the main basic minor matrix of the second rank it is: $\begin{bmatrix} [0,1] & [2,3] \\ [6,7] & [8,9] \end{bmatrix}$

And the main basic minor matrix from the third rank is NQ.

Note: The number of major basic minor matrices in the matrix $n \times n$ the value n

- **Neutrosophic function gradation:** $Nf(x_1, x_2, --, x_n)$

The relation defines the neutrosophic function gradient

$$\nabla f(x_1, x_2, --, x_n) = \left[\frac{\partial Nf}{\partial x_1}, \frac{\partial Nf}{\partial x_2}, ---, \frac{\partial Nf}{\partial x_n}\right]$$

- **Hessian matrix of the neutrosophic function** $Nf(x_1, x_2, --, x_n)$**:**

It is a square and symmetrical matrix of the rank $n \times n$, denoted by $NH(x_1, x_2, --, x_n)$ and defined by the following relation:

$$NH(x_1, x_2, --, x_n) = \left[\frac{\partial^2 Nf}{\partial Nx_i \partial Nx_j}\right]$$

- **Positive defined neutrosophic matrix:**

A symmetrical neutrosophic matrix NQ is positively defined if the following is true**:**

- All diagonal elements must be positive.
- The main basic determinants should be positive.

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- **Using the quadratic form, it is defined as follows:**

We say about a matrix NQ that it is positive and defined if and only if the quadratic form satisfies the following: $X^T(NQ)X > 0$ For all X values

- **Positive quasi-defined neutrosophic matrix:**

A symmetrical neutrosophic matrix is NQ a quasi-positive definition if the following is true**:**

- All diagonal elements must be non-negative.
- Basic determinants must be non-negative.

- **Using the quadratic form, it is defined as follows:**

We say about a matrix NQ that it is positive quasi-defined if and only if the quadratic form satisfies the following: $(NQ)X \geq 0X$ for all X^Tvalues.

Note 1: To prove that a matrix NQ is negative (or negative quasi-defined) (NQ -) must be positive (or positive quasi-defined).

- **Using the quadratic form, it is defined as follows:**

- **The defined neutrosophic matrix is negative:**

We say that a matrix NQ is defined and negative if and only if $(-NQ)$is positive and defined or if the quadratic form has satisfied$(NQ)X < 0X$ for all X^T values

- **The quasi-defined neutrosophic matrix is negative:**

We say about a matrix, NQ that it is quasi-negative and defined if and only if $(-NQ)$is positive and defined or if it satisfied the quadratic form: $X^T(NQ)X \leq 0$

Note 2: We say about Q matrix that it is not defined if $X^T(NQ)X$ was positive for values and negative for other values.

Note 3: The previous tests apply only to symmetrical matrices (and if the matrixNQ is asymmetrical, we replace it with the matrix $\frac{NQ-NQ^T}{2}$and apply the tests.

- **Convex function:** The function Nf is a convex function if the Hessian matrix of this function is positive or quasi-positive and defined for all variables.

- **Concave Function:** The function Nf is concave if the Hessian matrix of this function has negative or quasi-negative and defined for all variables.

Example2: Show whether the functionNf is convex or concave, where $Nf$is given by the following relationship

$$Nf = [3,3.5]x_1^2 + 4x_2^2 + 3x_3^2 + 2x_1x_2 + 3x_1x_3 + x_2x_3 - 4x_1 + 5x_2 + 3x_3$$

We find the partial derivatives of the function s$Nf$with respect to the variable x_1 , x_2 , x_3

$$\frac{\partial Nf}{\partial x_1} = 2[3,3.5]x_1 + 2x_2 + 3x_3 - 4$$

$$\frac{\partial Nf}{\partial x_2} = 8x_2 + 2x_1 + x_3 + 5$$

$$\frac{\partial Nf}{\partial x_3} = 6x_3 + 3x_1 + x_2 + 3$$

Then we find the Hessian matrix of the function Nf

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$$H_{Nf}(x_1, x_2, --, x_n) = \left[\frac{\partial^2 Nf}{\partial Nx_i \partial Nx_j}\right] = \begin{bmatrix} \frac{\partial^2 Nf}{\partial x_1^2} & \frac{\partial^2 Nf}{\partial x_1 \partial x_2} & \frac{\partial^2 Nf}{\partial x_1 \partial x_3} \\ \frac{\partial^2 Nf}{\partial x_2 \partial x_1} & \frac{\partial^2 Nf}{\partial x_2^2} & \frac{\partial^2 Nf}{\partial x_2 \partial x_3} \\ \frac{\partial^2 Nf}{\partial x_3 \partial x_1} & \frac{\partial^2 Nf}{\partial x_3 \partial x_2} & \frac{\partial^2 Nf}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

And from it

$$H_{Nf} = \begin{bmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

We note that H_{Nf} is symmetrical matrix and all diagonal elements are positive and the basic minimum determinants are positive because the main determinant of the third rank is equal to

$$\begin{vmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{vmatrix} = [6,7](8 \times 6 - 1) - 2(2 \times 6 - 3) + 3(2 - 3 \times 8) = [198,245] > 0$$

The main determinant of the second rank

$$\begin{vmatrix} [6,7] & 2 \\ 2 & 8 \end{vmatrix} = [6,7] \times 8 - 4 =$$

$$[48,56] - 4 = [44,52] > 0$$

The main determinant of the first rank $|[6,7]| > 0$

From the above, we can see that the Hessian matrix of Nf the function is positive, that is, the function Nf is a convex function.

➢ **The maximum value and the minimum value of a bound nonlinear function:**

From the previous study, we can determine whether the value of a target function in a nonlinear model is a minimum value or a maximum value as follows:

- The value is minimal if the target is a convex function and the set of constraints forms a convex zone.
- The value is maximal if the target is a concave function and the set of constraints forms a convex zone.

Current Study:

From the information provided in previous studies, we conclude the following:

To study unrestricted neutrosophic nonlinear programming problems:

We calculate the partial derivatives of the objective function $\frac{\partial Nf}{\partial x_i}$ $; i = 1.2 \dots, n$

We solve the set of equations that we obtain from $\nabla Nf(\bar{x}) = 0$. The solution represents a limit point of the function.

To determine its type, we find a Hessian matrix. We determine its type using the following formula:

Result: Assume that $\nabla Nf(\bar{x}) = 0$

- If $H_N(\bar{x})$is perfectly positive, this means that $\bar{x}$ is a local minimum limit.
- If $H_N(\bar{x})$ is perfectly negative, this means that $\bar{x}$ is a local maximum limit.
- If $H_N(\bar{x})$ is almost perfectly positive (or almost perfectly negative), then $\bar{x}$ may or may not be a maximum value (here we use limit relations).
- If $H_N(\bar{x})$ is neither perfectly negative nor perfectly positive, then $\bar{x}$ is neither a minimum nor a maximum value.

We illustrate this with the following example:

Example 3:

Find the maximum values? Of the following neutrosophic function:

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$$Nf = [3,3.5]x_1^2 + 4x_2^2 + 3x_3^2 + 2x_1x_2 + 3x_1x_3 + x_2x_3 - 4x_1 + 5x_2 + 3x_3$$

Then determine its type.

Solution:

- We found that the partial derivatives are as follows: (Example 2)

$$\frac{\partial Nf}{\partial x_1} = 2[3,3.5]x_1 + 2x_2 + 3x_3 - 4$$

$$\frac{\partial Nf}{\partial x_2} = 8x_2 + 2x_1 + x_3 + 5$$

$$\frac{\partial Nf}{\partial x_3} = 6x_3 + 3x_1 + x_2 + 3$$

- We solve the following set of equations:

$$2[3,3.5]x_1 + 2x_2 + 3x_3 - 4 = 0$$

$$8x_2 + 2x_1 + x_3 + 5 = 0$$

$$6x_3 + 3x_1 + x_2 + 3 = 0$$

We get $\bar{x}$. To determine whether it is a maximum or minimum value, we determine the type of the objective function by testing the Hessian matrix as follows:

- Then we find the Hessian matrix for the function Nf

$$H_{Nf}(x_1, x_2, --, x_n) = \left[\frac{\partial^2 Nf}{\partial Nx_i \partial Nx_j}\right] = \begin{bmatrix} \frac{\partial^2 Nf}{\partial x_1^2} & \frac{\partial^2 Nf}{\partial x_1 \partial x_2} & \frac{\partial^2 Nf}{\partial x_1 \partial x_3} \\ \frac{\partial^2 Nf}{\partial x_2 \partial x_1} & \frac{\partial^2 Nf}{\partial x_2^2} & \frac{\partial^2 Nf}{\partial x_2 \partial x_3} \\ \frac{\partial^2 Nf}{\partial x_3 \partial x_1} & \frac{\partial^2 Nf}{\partial x_3 \partial x_2} & \frac{\partial^2 Nf}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

From this,

$$H_{Nf} = \begin{bmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

- We found previously (Example 2) that this matrix is positively defined, meaning that the function Nf is a convex function.

3. Conclusion and Results

In this research, we presented a study of neutrosophic nonlinear dependencies, and what is mentioned in this research can be used when searching for optimal values for neutrosophic nonlinear models, as there are many practical problems that lead to nonlinear models and we need to obtain optimal solutions to them that take into account all the conditions and fluctuations that face us during the workflow, which is provided to us by the science of neutrosophy through the neutrosophic values that we can use to build neutrosophic nonlinear models and the specified through the realistic study of the issues under study.

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