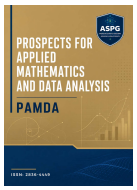
Prospects for Applied Mathematics and Data Analysis

♦ ISSN: 2836-4449 Vol: 04, No. 02, 2024 ♦ PP. 09–14

♦ Article DOI: https://doi.org/10.54216/PAMDA.040202



FULL LENGTH ARTICLE

Hessian Matrix for Testing the Convexity and Concavity of the Objective Function in Nonlinear Programming and Neutrosophic Nonlinear Programming Problems

Maissam Jdid[1,2,*]

[1] Faculty of Science, Damascus University, Damascus, Syria

[2] Department of Requirements, International University for Science and Technology, Ghabageb, Syrian Arab Republic

Email: maissam.jdid66@damascusuniversity.edu.sy

Received: September 05, 2024 Revised: November 04, 2024 Accepted: December 24, 2024 ⋆ Corresponding author

ABSTRACT

Mathematical examples rely on constructing mathematical models consisting of an objective function and constraints. These models may be linear, nonlinear, or otherwise. The objective function is either a maximization function or a minimization function for a given quantity. Nonlinear programming constitutes an important and fundamental part of operations research and is more comprehensive than linear programming. Therefore, researchers have focused on presenting studies that help find the optimal solution to these problems. Most of these studies have focused on the importance of knowing the type of objective function—whether it is convex or concave—because this knowledge helps determine the type of maximum value we obtain when studying a nonlinear programming problem. The Hessian matrix was used for this purpose. In this research, we will present the most important concepts that can be used when determining the type of maximum value for a nonlinear programming problem, as mentioned in some classic references. We will then reformulate them using the concepts of neutrosophic logic.

Keywords: Operations research ▪ Nonlinear models ▪ Neutrosophic logic ▪ Neutrosophic nonlinear models ▪ Concavity of functions ▪ Convexity of functions ▪ Hessian matrix

1. INTRODUCTION

We call mathematical programming the science that investigates determining the maximum or minimum value of a specific function, called the objective function, which depends on a number of variables that may be interconnected or independent of each other, known as constraints. It is one of the methods of operations research, a science that emerged during World War II due to the presence of many complex strategic issues that require optimal solutions. This science has gained a prominent position among the mathematical methods used in planning and managing various economic and military activities because it helps specialists develop ideal plans in terms of costs, production, financing, distribution, profit, or investment of human resources and energies. Its essence is to build a practical scientific model for a specific system that includes identifying the influencing factors and then obtaining the optimal solution for this model.

The mathematical model is a nonlinear model if any component of the objective function or constraints is a nonlinear expression, and the nonlinear expressions may be in both. The region of possible solutions for a nonlinear mathematical model is the set of vectors whose components satisfy all constraints, while the optimal solution is the vector that satisfies all constraints and where the function reaches an optimal value (maximum or minimum). The efforts of researchers in the field of nonlinear programming have focused

© 2024 The Author(s). Published by ASPG. This article is licensed under CC BY 4.0.

on presenting many concepts and methods that help in obtaining the optimal solution. In this research, we present some of these concepts based on what was mentioned in references [1, 2, 3, 4]. Then we will reformulate these concepts using the new vision presented by neutrosophic logic, the logic that is the latest achievement of logical thought in its persistent quest to comprehend reality and expresses the uprising of the mind when it collides with the unreasonableness of reality or with a false rationality that covers fossilized systems that seek stability in a world in which the pace of development and change is accelerating. To view the stages of development of this science from its inception until today and the most important research and studies that have been published using the concepts of this logic, which have included most fields of science, see [5].

2. DISCUSSION

The mathematical model consists of decision variables, an objective function, and constraints. The region of possible solutions for a nonlinear mathematical model is the set of vectors whose components satisfy all constraints. The optimal solution is the vector that satisfies all constraints and at which the function reaches a maximum value (maximum or minimum).

Previous Studies

Unconstrained Problems [4]

Solving unconstrained nonlinear programming problems means finding the general minimum or maximum value of a real function f that follows n real variables $x_1, x_2, \ldots, x_n$, which takes values from $-\infty$ to $+\infty$, meaning there is no constraint on the vector $x \in \mathbb{R}^n$. We are interested in this type of problem because the optimal conditions for solving constrained problems are merely a logical extension of the conditions for solving unconstrained problems.

Definition of an unconstrained nonlinear programming problem: Let us have a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined in the form $f(x) = f(x_1, x_2, \ldots, x_n)$ for any vector $x \in \mathbb{R}^n$:

$$\operatorname*{Min}_{x \in \mathbb{R}^n} f(x).$$

We say that this problem is unconstrained. That is, we must look for a point $\bar{x} \in \mathbb{R}^n$ such that

$$f(\bar{x}) \leq f(x), \qquad \forall x \in \mathbb{R}^n. \tag{1}$$

We say that the point $\bar{x}$ that satisfies (1) is a general minimum limit point.

If the point $\bar{x}$ satisfies the relation

$$f(\bar{x}) < f(x), \qquad \forall x \in \mathbb{R}^n,$$

we say that the point $\bar{x}$ is a unique general minimum limit point. If there is a neighborhood $N(\bar{x})$ around the point $\bar{x}$ such that the relation

$$f(\bar{x}) < f(x), \qquad \forall x \in N(\bar{x}),$$

is satisfied only in the neighborhood, then $\bar{x}$ is said to be a local minimum limit. Refer to Figure 1.

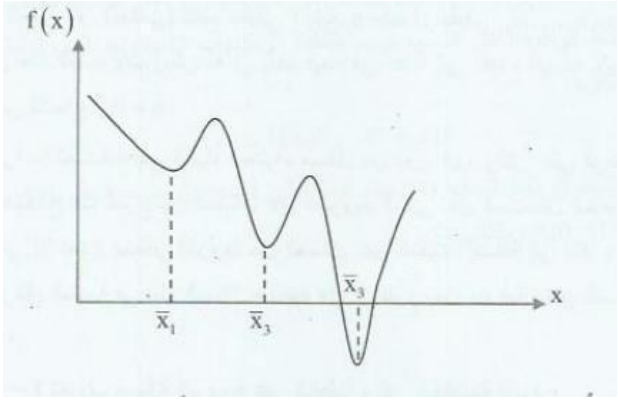


Figure 1. Illustration of local and general minimum limit points.

We note that

$$f(\bar{x}_1) < f(x), \qquad \forall x \in N(\bar{x}),$$

so $\bar{x}_1$ is a local minimum limit, while

$$f(\bar{x}_3) \leq f(x), \qquad \forall x \in \mathbb{R}^n,$$

and therefore $\bar{x}_3$ is a general minimum limit.

Necessary and Sufficient Conditions for General and Local Minimums [2, 4]

Assume that the function $f(x)$ is continuous and that its first partial derivatives $\partial f/\partial x_i$ and second partial derivatives $\partial^2 f/(\partial x_i \partial x_j)$ are also continuous for any point $x \in \mathbb{R}^n$; then we have the following.

Theorem. If the point $\bar{x} \in \mathbb{R}^n$ is a local minimum limit of the function $f(x)$, then:

a. $\nabla f(\bar{x}) = 0$.

b. The Hessian matrix of the function f,

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right],$$

is a positive near-perfect matrix.

We say that the point x^* that satisfies condition (a) of the previous theorem, i.e.,

$$\frac{\partial f}{\partial x_i}(x^*) = 0, \qquad i = 1, 2, \ldots, n,$$

is a stationary (critical) point.

Theorem. With the same assumptions as the previous theorem, we say that the point $\bar{x}$ is a local minimum limit of the function $f(x)$ over $\mathbb{R}^n$ if

$$\nabla f(\bar{x}) = 0$$

and the Hessian matrix

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

is a positive near-perfect matrix.

Note. A stationary point does not necessarily have to be a local minimum endpoint.

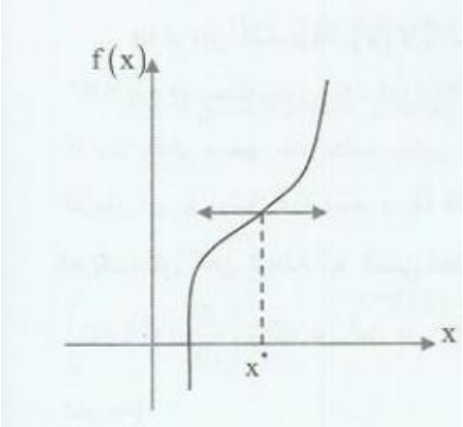


Figure 2. A stationary inflection point that is not a local limit.

Figure 2 shows a stationary point x^*, but it is not a local limit. We can also see from this figure that the sufficient conditions for the existence of a local minimum limit are not met. Because the function f accepts an inflection point x^*, the Hessian matrix is not perfectly positive. Rather, it is almost perfectly positive:

$$\frac{\partial^2 f}{\partial x^2}(x^*) = 0.$$

Convex Functions [2, 4]

When the function f is convex and defined on $\mathbb{R}^n$, we can obtain necessary and sufficient conditions for a point to be a general minimum limit point. This can be summarized in the following theorem.

Theorem. If f is a convex, continuously differentiable function, then the necessary and sufficient condition for the point $\bar{x}$ to be a general minimum limit point of the function f on $\mathbb{R}^n$ is that $\nabla f(\bar{x}) = 0$.

Necessary and Sufficient Conditions for General and Local Maximum Limits

Similarly to those for general and local minimum limits, we summarize the necessary and sufficient conditions for the existence of maximum limits with the following theorems.

Theorem. If the function $f(x)$ is continuous and its first partial derivatives $\partial f/\partial x_i$ and the second partial derivatives $\partial^2 f/(\partial x_i \partial x_j)$ are also continuous for any point $x \in \mathbb{R}^n$, then we have: if the point $\bar{x}$ is a maximum limit (general or local) of the function f, then

$$\nabla f(\bar{x}) = 0,$$

and the Hessian matrix of the function f,

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x})\right],$$

is a near-perfect negative matrix.

Theorem. Using the same assumptions as the previous theorem, we say the point $\bar{x}$ is a local maximum limit point of the function $f(x)$ on $\mathbb{R}^n$ if $\nabla f(\bar{x}) = 0$ and the Hessian matrix

$$\nabla^2 f(\bar{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

is a positive perfect matrix.

Result. Assume that $\nabla f(\bar{x}) = 0$.

- If $H(\bar{x})$ is perfectly positive, then $\bar{x}$ is a local minimum limit.
- If $H(\bar{x})$ is perfectly negative, then $\bar{x}$ is a local maximum limit.
- If $H(\bar{x})$ is almost perfectly positive (or almost perfectly negative), then $\bar{x}$ may or may not be a maximum (here we use limit relations).
- If $H(\bar{x})$ is neither perfectly negative nor perfectly positive, then $\bar{x}$ is neither a minimum nor a maximum.

Neutrosophic Mathematical Model

In the problem of examples where the objective and constraints are in the form of neutrosophic mathematical functions, the neutrosophic mathematical model is written in the following form:

$$Nf = Nf(x_1, x_2, \ldots, x_n) \longrightarrow (\text{Max}) \text{ or } (\text{Min}),$$

according to the restrictions

$$Ng_i(x_1, x_2, \ldots, x_n) \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} Nb_i, \qquad i = 1, 2, \ldots, m,$$

$$x_1, x_2, \ldots, x_n \geq 0.$$

In this model, such variables in the objective function and in constraints are neutrosophic values, as well as the second side of the relationships that represent constraints. This model is a nonlinear model if any component of the target function or constraints is a nonlinear statement, and the nonlinear statements may be in both.

Some Important Neutrosophic Definitions in Nonlinear Programming

The Quadratic Form of the Neutrosophic Target Function

A function with variables $f(x_1, x_2, \ldots, x_n)$ is called a quadratic form if the following is true:

$$Nf(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n}(Nq_{ij})x_i x_j = X^T(NQ)X.$$

Here $NQ_{n\times n} = [Nq_{ij}]$ is a neutrosophic matrix, meaning that all or some of its elements are neutrosophic values of the shape $Nq_{ij} = q_{ij} + \varepsilon_{ij}$, where ε_{ij} is the indeterminism of the elements of the matrix and any neighborhood may be close to q_{ij}. It shall be written in one of the forms $[\lambda_{1ij}, \lambda_{2ij}]$, $\{\lambda_{1ij}, \lambda_{2ij}\}$, or otherwise, and $X^T = (x_1, x_2, \ldots, x_n)$.

Here we can always assume, without prejudice to the generality of the problem, that NQ is symmetrical. We can replace the matrix NQ with the symmetrical matrix

$$\frac{NQ + NQ^T}{2}$$

without changing the quadratic form.

Basic Minor Neutrosophic Matrices

If the matrix NQ has dimensions $n \times n$, then the basic minimum matrix of rank k is a mini-matrix with $k \times k$ dimensions, and we obtain it by neglecting any $(n-k)$ lines and the corresponding columns of the matrix.

Note. The determinant of the basic minor neutrosophic matrix is called the basic determinant, and for each square matrix of rank $n \times n$ there are $2^n - 1$ basic determinants.

Main Basic Minor Neutrosophic Matrices

The main basic minor neutrosophic matrices of rank k for a matrix $n \times n$ are obtained from the matrix NQ by neglecting the last $n-k$ lines and their corresponding columns.

Example 1. Find the basic minor matrices and the main basic minor matrices from the following neutrosophic matrix:

$$NQ = \begin{bmatrix} [0,1] & [2,3] & [4,5] \\ [6,7] & [8,9] & [10,11] \\ [12,13] & [14,15] & [16,17] \end{bmatrix}.$$

Basic minor matrices. Of the first rank, the elements of the main diameter are

$$[0,1],\ [8,9],\ [16,17].$$

From the second rank, the matrices are

$$\begin{bmatrix} [0,1] & [2,3] \\ [6,7] & [8,9] \end{bmatrix}, \quad \begin{bmatrix} [6,7] & [10,11] \\ [12,13] & [16,17] \end{bmatrix}, \quad \begin{bmatrix} [8,9] & [10,11] \\ [14,15] & [16,17] \end{bmatrix}.$$

From the third rank, the matrix is NQ.

Main basic minor matrices. From the first rank, the matrix is $[0,1]$ (we neglect the last two lines and the corresponding columns). The main basic minor matrix of the second rank is

$$\begin{bmatrix} [0,1] & [2,3] \\ [6,7] & [8,9] \end{bmatrix},$$

and the main basic minor matrix of the third rank is NQ.

Note. The number of major basic minor matrices in a matrix $n \times n$ is n.

Neutrosophic Function Gradation

For $Nf(x_1, x_2, \ldots, x_n)$, the relation defines the neutrosophic function gradient as

$$\nabla f(x_1, x_2, \ldots, x_n) = \left[\frac{\partial Nf}{\partial x_1}, \frac{\partial Nf}{\partial x_2}, \ldots, \frac{\partial Nf}{\partial x_n} \right].$$

Hessian Matrix of the Neutrosophic Function

For $Nf(x_1, x_2, \ldots, x_n)$, it is a square and symmetrical matrix of rank $n \times n$, denoted by $NH(x_1, x_2, \ldots, x_n)$, and defined by the following relation:

$$NH(x_1, x_2, \ldots, x_n) = \left[\frac{\partial^2 Nf}{\partial Nx_i \partial Nx_j} \right].$$

Positive Defined Neutrosophic Matrix

A symmetrical neutrosophic matrix NQ is positively defined if the following is true:

- All diagonal elements must be positive.
- The main basic determinants should be positive.

Using the quadratic form, we say that a matrix NQ is positive and defined if and only if

$$X^T (NQ) X > 0$$

for all X values.

Positive Quasi-Defined Neutrosophic Matrix

A symmetrical neutrosophic matrix NQ is a quasi-positive definition if the following is true:

- All diagonal elements must be non-negative.
- Basic determinants must be non-negative.

Using the quadratic form, we say that a matrix NQ is positive quasi-defined if and only if

$$X^T (NQ) X \geq 0$$

for all X values.

Note 1. To prove that a matrix NQ is negative (or negative quasi-defined), $-NQ$ must be positive (or positive quasi-defined).

Negative Defined and Quasi-Defined Neutrosophic Matrices

The defined neutrosophic matrix is negative if and only if $-NQ$ is positive and defined, or if the quadratic form satisfies

$$X^T (NQ) X < 0$$

for all X values.

The quasi-defined neutrosophic matrix is negative if and only if $-NQ$ is positive and defined, or if the quadratic form satisfies

$$X^T (NQ) X \leq 0.$$

Note 2. We say that a matrix Q is not defined if $X^T (NQ) X$ is positive for some values and negative for other values.

Note 3. The previous tests apply only to symmetrical matrices. If the matrix NQ is asymmetrical, we replace it with the matrix

$$\frac{NQ - NQ^T}{2}$$

and apply the tests.

Convex and Concave Functions

The function Nf is a convex function if the Hessian matrix of this function is positive or quasi-positive and defined for all variables. The function Nf is concave if the Hessian matrix of this function is negative or quasi-negative and defined for all variables.

Example 2. Show whether the function Nf is convex or

concave, where Nf is given by the relationship

$$Nf = [3,3.5]x_1^2 + 4x_2^2 + 3x_3^2 + 2x_1x_2 + 3x_1x_3 + x_2x_3 - 4x_1 + 5x_2 + 3x_3.$$

We find the partial derivatives of the function Nf with respect to the variables x_1, x_2, x_3:

$$\frac{\partial Nf}{\partial x_1} = 2[3,3.5]x_1 + 2x_2 + 3x_3 - 4,$$
$$\frac{\partial Nf}{\partial x_2} = 8x_2 + 2x_1 + x_3 + 5,$$
$$\frac{\partial Nf}{\partial x_3} = 6x_3 + 3x_1 + x_2 + 3.$$

Then we find the Hessian matrix of the function Nf:

$$H_{Nf}(x_1, x_2, \ldots, x_n) = \left[\frac{\partial^2 Nf}{\partial Nx_i \partial Nx_j}\right]$$

$$H_{Nf} = \begin{bmatrix} \dfrac{\partial^2 Nf}{\partial x_1^2} & \dfrac{\partial^2 Nf}{\partial x_1 \partial x_2} & \dfrac{\partial^2 Nf}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 Nf}{\partial x_2 \partial x_1} & \dfrac{\partial^2 Nf}{\partial x_2^2} & \dfrac{\partial^2 Nf}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 Nf}{\partial x_3 \partial x_1} & \dfrac{\partial^2 Nf}{\partial x_3 \partial x_2} & \dfrac{\partial^2 Nf}{\partial x_3^2} \end{bmatrix}.$$

Thus,

$$H_{Nf} = \begin{bmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}.$$

We note that H_{Nf} is a symmetrical matrix and all diagonal elements are positive. The basic minimum determinants are positive because the main determinant of the third rank is equal to

$$D_3 = \begin{vmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{vmatrix}$$
$$= [6,7](8 \times 6 - 1) - 2(2 \times 6 - 3) + 3(2 - 3 \times 8) = [198,245] > 0.$$

The main determinant of the second rank is

$$\begin{vmatrix} [6,7] & 2 \\ 2 & 8 \end{vmatrix} = [6,7] \times 8 - 4 = [48,56] - 4 = [44,52] > 0.$$

The main determinant of the first rank is $|[6,7]| > 0$. From the above, we can see that the Hessian matrix of the function Nf is positive; that is, the function Nf is a convex function.

The Maximum and Minimum Value of a Bound Nonlinear Function

From the previous study, we can determine whether the value of a target function in a nonlinear model is a minimum value or a maximum value as follows:

- The value is minimal if the target is a convex function and the set of constraints forms a convex zone.
- The value is maximal if the target is a concave function and the set of constraints forms a convex zone.

Current Study

From the information provided in previous studies, we conclude the following. To study unrestricted neutrosophic nonlinear programming problems, we calculate the partial derivatives of the objective function

$$\frac{\partial Nf}{\partial x_i}, \qquad i = 1,2,\ldots,n.$$

We solve the set of equations that we obtain from $\nabla Nf(\bar{x}) = 0$. The solution represents a limit point of the function. To determine its type, we find a Hessian matrix and determine its type using the following result.

Result. Assume that $\nabla Nf(\bar{x}) = 0$.

- If $H_N(\bar{x})$ is perfectly positive, this means that $\bar{x}$ is a local minimum limit.
- If $H_N(\bar{x})$ is perfectly negative, this means that $\bar{x}$ is a local maximum limit.
- If $H_N(\bar{x})$ is almost perfectly positive (or almost perfectly negative), then $\bar{x}$ may or may not be a maximum value (here we use limit relations).
- If $H_N(\bar{x})$ is neither perfectly negative nor perfectly positive, then $\bar{x}$ is neither a minimum nor a maximum value.

We illustrate this with the following example.

Example 3. Find the maximum values of the following neutrosophic function, and then determine its type:

$$Nf = [3,3.5]x_1^2 + 4x_2^2 + 3x_3^2 + 2x_1x_2 + 3x_1x_3 + x_2x_3 - 4x_1 + 5x_2 + 3x_3.$$

Solution. We found that the partial derivatives are as follows (Example 2):

$$\frac{\partial Nf}{\partial x_1} = 2[3,3.5]x_1 + 2x_2 + 3x_3 - 4,$$
$$\frac{\partial Nf}{\partial x_2} = 8x_2 + 2x_1 + x_3 + 5,$$
$$\frac{\partial Nf}{\partial x_3} = 6x_3 + 3x_1 + x_2 + 3.$$

We solve the following set of equations:

$$2[3,3.5]x_1 + 2x_2 + 3x_3 - 4 = 0,$$
$$8x_2 + 2x_1 + x_3 + 5 = 0,$$
$$6x_3 + 3x_1 + x_2 + 3 = 0.$$

We get $\bar{x}$. To determine whether it is a maximum or minimum value, we determine the type of the objective function by testing the Hessian matrix as follows:

$$H_{Nf}(x_1, x_2, \ldots, x_n) = \left[\frac{\partial^2 Nf}{\partial Nx_i \partial Nx_j}\right]$$

$$H_{Nf} = \begin{bmatrix} \frac{\partial^2 Nf}{\partial x_1^2} & \frac{\partial^2 Nf}{\partial x_1 \partial x_2} & \frac{\partial^2 Nf}{\partial x_1 \partial x_3} \\ \frac{\partial^2 Nf}{\partial x_2 \partial x_1} & \frac{\partial^2 Nf}{\partial x_2^2} & \frac{\partial^2 Nf}{\partial x_2 \partial x_3} \\ \frac{\partial^2 Nf}{\partial x_3 \partial x_1} & \frac{\partial^2 Nf}{\partial x_3 \partial x_2} & \frac{\partial^2 Nf}{\partial x_3^2} \end{bmatrix}.$$

From this,

$$H_{Nf} = \begin{bmatrix} [6,7] & 2 & 3 \\ 2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}.$$

We found previously (Example 2) that this matrix is positively defined, meaning that the function Nf is a convex function.

3. CONCLUSION AND RESULTS

In this research, we presented a study of neutrosophic nonlinear dependencies, and what is mentioned in this research can be used when searching for optimal values for neutrosophic nonlinear models, as there are many practical problems that lead to nonlinear models and we need to obtain optimal solutions to them that take into account all the conditions and fluctuations that face us during the workflow. This is provided to us by the science of neutrosophy through the neutrosophic values that we can use to build neutrosophic nonlinear models, specified through the realistic study of the issues under study.

REFERENCES

[1] G. V. Reklaitis, A. Ravindran, and K. M. Ragsdell, *Engineering Optimization: Methods and Applications*. New York, NY, USA: Wiley-Interscience, 1983.

[2] J. S. Bakaja, W. Mualla *et al.*, *Operations Research Book [Translated into Arabic]*. Damascus, Syria: The Arab Center for Arabization, Translation, Authoring and Publishing, 1998.

[3] D. G. Luenberger and Y. Ye, *Linear and Nonlinear Programming*. New York, NY, USA: Springer Science+Business Media, 2015.

[4] M. D. Al Hamid, *Mathematical Programming*. Aleppo, Syria: Aleppo University, 2010.

[5] F. Smarandache and M. Jdid, "Research in the field of neutrosophic operations research," 2023.