A Reconsideration of the Mathematical Frameworks for Fuzzy and Neutrosophic Supply Chain Management (FSCM and NSCM)

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Abstract

Numerous frameworks have been developed to address uncertainty in various domains. Among the most prominent are Fuzzy Sets,[26] Rough Sets,[15] Intuitionistic Fuzzy Sets,[4] Hesitant Fuzzy Sets,[23] Neutrosophic Sets,[3] as well as other emerging theories that continue to be actively explored. Supply Chain Management (SCM) involves planning, coordinating, and optimizing the flow of goods, information, and finances across the entire supply network.[9,16] In this paper, we introduce rigorous Mathematical Frameworks for Fuzzy Supply Chain Management (FSCM) and Neutrosophic Supply Chain Management (NSCM). We hope that these formulations will foster further advances in both supply chain optimization and the development of Fuzzy Set and Neutrosophic Set-based models.

Keywords: Fuzzy set; Neutrosophic Set; Supply Chain Management (SCM); Fuzzy Supply Chain Management (FSCM); Neutrosophic Supply Chain Management (NSCM)

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. In addition, all concepts addressed herein are assumed to be finite rather than infinite.

1.1 Fuzzy Set and Neutrosophic Set

A fuzzy set assigns to each element a membership degree in the interval $[0, 1]$, thereby capturing uncertainty through more granular membership levels rather than a strict binary classification.[26] Below, we present the relevant definitions, including those for these extended frameworks.

Definition 1.1 (Fuzzy Set). [26] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, thereby addressing situations that are neither entirely true nor entirely false. This framework provides a more flexible representation of uncertainty and ambiguity.[8,20] Their definitions are presented below.

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Definition 1.2 (Neutrosophic Set). [20] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

2 Result of This Paper

The results of this paper are presented as follows.

2.1 Mathematical Model of Supply Chain Management

Supply Chain Management is the coordinated planning, control, and execution of goods, services, and information from suppliers to customers in order to maximize efficiency and value.[1,24] Related concepts include sustainable supply chain management[7,17] and green supply chain management.[10,22] We deliberately present a mathematical definition of supply chain management as follows.

Definition 2.1 (Directed Graph). [6] A *directed graph* (or *digraph*) is an ordered pair

$$G = (V, A),$$

where V is a finite non-empty set of vertices, and $A \subseteq V \times V$ is a set of ordered pairs of vertices, called *arcs* or *directed edges*. Each arc $(u, v) \in A$ represents a directed edge from vertex u (the tail) to vertex v (the head).

Definition 2.2 (Supply Chain Management as a Network Flow Problem). A *Supply Chain* is modeled by the tuple

$$\mathcal{SC} = \bigl(V, E, S, M, D, R, s, d, f, C, c, T\bigr),$$

where:

- $G = (V, E)$ is a directed graph with node set V and arc set E.
- $S \subset V$ is the set of *suppliers*, $M \subset V$ the set of *manufacturing sites*, $D \subset V$ the set of *distribution centers*, and $R \subset V$ the set of *retailers*, partitioning V.
- $s : S \to \mathbb{R}_{\geq 0}$ gives the available supply at each supplier.
- $d : R \to \mathbb{R}_{\geq 0}$ gives the demand at each retailer.
- $f : E \to \mathbb{R}_{\geq 0}$ is a flow function assigning to each arc $e \in E$ the quantity shipped.
- $C : E \to \mathbb{R}_{>0}$ is the capacity of each arc.
- $c : E \to \mathbb{R}_{\geq 0}$ is the unit transportation cost on each arc.
- $T : E \to \mathbb{R}_{\geq 0}$ is the lead time on each arc.

The flow f is *feasible* if it satisfies:

$$0 \;\leq\; f(e) \;\leq\; C(e) \quad \forall e \in E,$$

$$\sum_{e:\, e \to v} f(e) \;-\; \sum_{e:\, v \to e} f(e) = \begin{cases} s(v), & v \in S, \\ -d(v), & v \in R, \\ 0, & v \in V \setminus (S \cup R). \end{cases}$$

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The *total cost* and *total lead time* of a feasible flow are

$$\mathrm{Cost}(f) \;=\; \sum_{e\in E} c(e)\, f(e), \qquad \mathrm{Time}(f) \;=\; \max_{\substack{\pi \text{ path}\\ \text{from } S \text{ to } R}} \sum_{e\in\pi} T(e).$$

Supply Chain Management (SCM) seeks a feasible flow f^* that

$$\min_{f}\ \mathrm{Cost}(f) \quad \text{subject to meeting all demands and respecting capacities,}$$

and, optionally, also minimizes $\mathrm{Time}(f)$ or a weighted sum of cost and time.

Example 2.3 (A Simple Three-Tier Supply Chain). (cf.[13,25]) Consider a chain with

$$S = \{s_1\},\ M = \{m_1\},\ D = \{d_1\},\ R = \{r_1, r_2\},$$

and arcs

$$E = \{(s_1, m_1),\ (m_1, d_1),\ (d_1, r_1),\ (d_1, r_2)\}.$$

Data:

$$s(s_1) = 100, \quad d(r_1) = 60, \quad d(r_2) = 30,$$

$$C(e) = 100, \quad c(e) = \begin{cases} 2, & e = (s_1, m_1),\\ 3, & e = (m_1, d_1),\\ 1, & e \in \{(d_1, r_1), (d_1, r_2)\}, \end{cases}$$

$$T(e) = \begin{cases} 1, & e = (s_1, m_1),\\ 2, & e = (m_1, d_1),\\ 1, & e \in \{(d_1, r_1), (d_1, r_2)\}. \end{cases}$$

A feasible flow meeting all demands is

$$f(s_1, m_1) = 90,\ f(m_1, d_1) = 90,\ f(d_1, r_1) = 60,\ f(d_1, r_2) = 30.$$

Then

$$\mathrm{Cost}(f) = 2\cdot 90 + 3\cdot 90 + 1\cdot 60 + 1\cdot 30 = 180 + 270 + 60 + 30 = 540,$$

$$\mathrm{Time}(f) = 1 + 2 + 1 = 4.$$

This simple example illustrates the basic SCM optimization problem of minimizing cost while satisfying supplier capacities and retailer demands.

Theorem 2.4 (Feasibility via Max–Flow Min–Cut). *A feasible flow $f : E \to \mathrm{R}_{\geq 0}$ meeting all supplies and demands exists in $\mathcal{SC}$ if and only if*

$$\max\Big\{ \textit{value}(f') : f' \textit{ is an } s_0\!-\!t_0 \textit{ flow in } G' \Big\} \;=\; \sum_{v\in S} s(v).$$

Proof. ($\Rightarrow$) If a flow f satisfies all supplies and demands in $\mathcal{SC}$, extend f by sending $s(v)$ along (s_0, v) for each $v \in S$ and $f(e)$ on original arcs, then on each (v, t_0) send $d(v)$. This yields an s_0–t_0-flow of value $\sum_{v\in S} s(v)$.

($\Leftarrow$) Conversely, any maximum s_0–t_0-flow f' in G' of value $\sum_{v\in S} s(v)$ saturates all supplier-arcs (s_0, v) and retailer-arcs (v, t_0). Restricting f' to E gives a flow in $\mathcal{SC}$ meeting each supply $s(v)$ and demand $d(v)$ and respecting arc capacities. □

Theorem 2.5 (Existence and Integrality of Minimum-Cost Flow). *If all supplies $s(v)$, demands $d(v)$, capacities $C(e)$, and costs $c(e)$ are integers, then there exists an* integral *flow f^* that minimizes total cost*

$$\mathrm{Cost}(f) \;=\; \sum_{e\in E} c(e)\, f(e)$$

over all feasible flows of value $\sum_{v\in S} s(v)$.

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Proof. The minimum-cost flow problem can be formulated as a linear program whose constraint matrix is totally unimodular (network-node incidence matrix with unit-capacity supply/demand rows). Integer data (supplies, demands, capacities, costs) and total unimodularity guarantee an optimal solution f^* in which each $f^*(e)$ is integral. □

Theorem 2.6 (Optimality via Reduced-Cost Conditions)**.** *Let f^* be a feasible flow in $\mathcal{SC}$ and let $\pi : V \to \mathrm{R}$ be node potentials. Define the* reduced cost *of each arc $e = (u, v)$ by*

$$\hat{c}(e) \;=\; c(e) \;+\; \pi(u) \;-\; \pi(v).$$

Then f^* is a minimum-cost flow if and only if

$$\hat{c}(e) \;\geq\; 0 \quad \textit{for all } e \in E, \quad \textit{and} \quad \hat{c}(e) = 0 \textit{ whenever } 0 < f^*(e) < C(e).$$

Proof. This follows by complementary slackness in the primal–dual pair of the minimum-cost flow LP. Nonnegativity of all reduced costs ensures dual feasibility, and vanishing of $\hat{c}(e)$ on arcs with strictly between-bounds flow ensures complementary slackness, which together certify optimality of f^*. □

2.2 Mathematical Framework for Fuzzy Supply Chain Management

The definition of Fuzzy Supply Chain Management is deliberately presented in a mathematical form as follows. The integration of fuzzy logic and supply chain management has been extensively discussed in various research papers.[14,21]

Definition 2.7 (Fuzzy Supply Chain Management)**.** A *Fuzzy Supply Chain Management* (FSCM) system generalizes SCM by replacing crisp parameters with fuzzy numbers. It is the tuple

$$\widetilde{\mathcal{SC}} = \big(V, E, S, M, D, R, \tilde{s}, \tilde{d}, \tilde{C}, \tilde{c}, \tilde{T}, \tilde{f}\big),$$

where:

- $\tilde{s} : S \to \tilde{\mathcal{P}}(\mathbb{R}_{\geq 0})$, $\tilde{d} : R \to \tilde{\mathcal{P}}(\mathbb{R}_{\geq 0})$ assign triangular fuzzy supplies and demands.
- $\tilde{C}, \tilde{c}, \tilde{T} : E \to \tilde{\mathcal{P}}(\mathbb{R}_{\geq 0})$ assign fuzzy capacities, costs, and lead times.
- A fuzzy flow $\tilde{f} : E \to \tilde{\mathcal{P}}(\mathbb{R}_{\geq 0})$ is *feasible* if for each arc and node the fuzzy inequalities

$$\tilde{0} \leq \tilde{f}(e) \leq \tilde{C}(e),$$

$$\bigoplus_{e:\, e\to v} \tilde{f}(e) \;\ominus\; \bigoplus_{e:\, v\to e} \tilde{f}(e) = \begin{cases} \tilde{s}(v), & v \in S, \\ \tilde{d}(v), & v \in R, \\ \tilde{0}, & \text{otherwise,} \end{cases}$$

  hold in the sense of fuzzy arithmetic.
- The fuzzy total cost and time are $\widetilde{\mathrm{Cost}}(\tilde{f}) = \bigoplus_{e\in E}\big(c(e) \otimes \tilde{f}(e)\big)$ and $\widetilde{\mathrm{Time}}(\tilde{f}) = \max_{\pi} \bigoplus_{e\in\pi} \tilde{T}(e)$.

FSCM seeks a fuzzy feasible flow $\tilde{f}^*$ minimizing $\widetilde{\mathrm{Cost}}$ (and optionally $\widetilde{\mathrm{Time}}$).

Theorem 2.8 (Degeneracy to Classical SCM)**.** *If every fuzzy parameter in FSCM degenerates to a singleton (i.e. each triangular fuzzy number $\tilde{x} = (l, m, u)$ satisfies $l = m = u$), then $\widetilde{\mathcal{SC}}$ reduces exactly to the classical SCM framework $\mathcal{SC}$.*

Proof. When each fuzzy map $\tilde{x}$ takes the form $\{x\}$, fuzzy addition $\oplus$ and multiplication $\otimes$ collapse to ordinary addition and multiplication, and fuzzy inequalities become crisp. Therefore all feasibility and objective conditions coincide with those of the classical model, recovering $\mathcal{SC}$. □

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Theorem 2.9 (Existence of a Fuzzy Feasible Flow)**.** *If the classical SCM instance $\mathcal{SC}$ admits at least one feasible flow f, then the FSCM instance $\widetilde{\mathcal{SC}}$ admits a fuzzy feasible flow $\tilde{f}$.*

Proof. Let $f : E \to \mathbb{R}_{\geq 0}$ be a feasible flow in the classical sense, so that

$$0 \leq f(e) \leq C(e), \quad \sum_{e\colon e \to v} f(e) - \sum_{e\colon v \to e} f(e) = \begin{cases} s(v), & v \in S, \\ -d(v), & v \in R, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\tilde{f}(e) = \{f(e)\}$ for each e. Since each fuzzy capacity $\tilde{C}(e)$ contains the crisp $C(e)$, the inequality $\tilde{0} \leq \tilde{f}(e) \leq \tilde{C}(e)$ holds. Likewise, node balances carry over under singleton arithmetic. Hence $\tilde{f}$ is a fuzzy feasible flow. □

Theorem 2.10 (Monotonicity in Capacities)**.** *Let $\widetilde{\mathcal{SC}}_1$ and $\widetilde{\mathcal{SC}}_2$ be two FSCM systems differing only in their capacity functions, with*

$$\tilde{C}_1(e) \ \leq \ \tilde{C}_2(e) \quad \textit{(componentwise) for all } e \in E.$$

Denote by $\tilde{f}_1^*$ and $\tilde{f}_2^*$ the respective cost-minimizing fuzzy flows. Then

$$\widetilde{\mathrm{Cost}}_2(\tilde{f}_2^*) \ \leq \ \widetilde{\mathrm{Cost}}_1(\tilde{f}_1^*).$$

Proof. Since $\tilde{f}_1^*$ satisfies $\tilde{f}_1^*(e) \leq \tilde{C}_1(e) \leq \tilde{C}_2(e)$ for every arc, it is also feasible in $\widetilde{\mathcal{SC}}_2$. By optimality of $\tilde{f}_2^*$,

$$\widetilde{\mathrm{Cost}}_2(\tilde{f}_2^*) \ \leq \ \widetilde{\mathrm{Cost}}_2(\tilde{f}_1^*) = \widetilde{\mathrm{Cost}}_1(\tilde{f}_1^*),$$

where the last equality holds because cost and flow values coincide and fuzzy arithmetic on the same inputs is identical in both systems. □

Example 2.11 (Simple chain)**.** Consider a simple chain with

$$S = \{s_1\},\ M = \{m_1\},\ D = \{d_1\},\ R = \{r_1, r_2\},$$

and arcs $E = \{(s_1, m_1), (m_1, d_1), (d_1, r_1), (d_1, r_2)\}$.

Define triangular fuzzy parameters:

$$\tilde{s}(s_1) = (90, 100, 110), \quad \tilde{d}(r_1) = (55, 60, 65), \quad \tilde{d}(r_2) = (25, 30, 35),$$

$$\tilde{C}(e) = (80, 100, 120), \quad \tilde{c}(e) = (2, 3, 4), \quad \tilde{T}(e) = (1, 2, 3) \quad \forall e \in E.$$

A fuzzy feasible flow might be

$$\tilde{f}(s_1, m_1) = (80, 90, 100),\ \tilde{f}(m_1, d_1) = (80, 90, 100),\ \tilde{f}(d_1, r_1) = (55, 60, 65),\ \tilde{f}(d_1, r_2) = (25, 30, 35).$$

Then the fuzzy total cost is

$$\widetilde{\mathrm{Cost}}(\tilde{f}) = \bigoplus_{e \in E} \big(\tilde{c}(e) \otimes \tilde{f}(e)\big) \approx (360, 450, 560),$$

and the fuzzy makespan is

$$\widetilde{\mathrm{Time}}(\tilde{f}) = \max_{\pi} \bigoplus_{e \in \pi} \tilde{T}(e) \approx (3,\ 5,\ 9).$$

This illustrates how FSCM handles uncertainty in supply, demand, capacities, costs, and times.

Example 2.12 (Two-Supplier Single-Manufacturer Chain)**.** Consider a chain with

$$S = \{s_1, s_2\}, \quad M = \{m_1\}, \quad D = \{d_1\}, \quad R = \{r_1, r_2\},$$

and arcs

$$E = \{(s_1, m_1), (s_2, m_1), (m_1, d_1), (d_1, r_1), (d_1, r_2)\}.$$

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Assign triangular fuzzy parameters as follows:

$$\tilde{s}(s_1) = (100, 120, 140), \quad \tilde{s}(s_2) = (90, 110, 130),$$
$$\tilde{d}(r_1) = (80, 100, 120), \quad \tilde{d}(r_2) = (50, 70, 90),$$
$$\tilde{C}(s_i, m_1) = (90, 110, 130), \quad \tilde{C}(m_1, d_1) = (180, 220, 260),$$
$$\tilde{C}(d_1, r_1) = (100, 120, 140), \quad \tilde{C}(d_1, r_2) = (80, 100, 120),$$
$$\tilde{c}(s_i, m_1) = (3, 4, 5), \quad \tilde{c}(m_1, d_1) = (2, 3, 4), \quad \tilde{c}(d_1, r_j) = (1, 2, 3),$$
$$\tilde{T}(e) = (1, 2, 3) \quad \forall e \in E.$$

A fuzzy feasible flow may be

$$\tilde{f}(s_1, m_1) = (90, 110, 130),\ \tilde{f}(s_2, m_1) = (80, 100, 120),$$
$$\tilde{f}(m_1, d_1) = (170, 210, 250),\ \tilde{f}(d_1, r_1) = (90, 100, 110),\ \tilde{f}(d_1, r_2) = (80, 110, 140).$$

Then the fuzzy total cost is

$$\widetilde{\mathrm{Cost}}(\tilde{f}) = \bigoplus_{e \in E} \big(\tilde{c}(e) \otimes \tilde{f}(e)\big) \ \approx\ (1020,\ 1890,\ 3000),$$

and the fuzzy makespan is

$$\widetilde{\mathrm{Time}}(\tilde{f}) = \max_{\pi} \bigoplus_{e \in \pi} \tilde{T}(e) \ \approx\ (4,\ 8,\ 12).$$

This example demonstrates FSCM handling multiple suppliers and splitting fuzzy flows through a single production node under uncertainty.

2.3 Neutrosophic Supply Chain Management

The definition of Neutrosophic Supply Chain Management is deliberately presented in a mathematical form as follows. The integration of Neutrosophic logic and supply chain management has been extensively discussed in various research papers.[2,5]

Definition 2.13 (Neutrosophic Supply Chain Management)**.** A *Neutrosophic Supply Chain Management* (NSCM) system extends the Fuzzy SCM framework by replacing fuzzy parameters with neutrosophic numbers. It is the tuple

$$\widehat{\mathcal{SC}} = \big(V, E, S, M, D, R, \hat{s}, \hat{d}, \hat{C}, \hat{c}, \hat{T}, \hat{f}\big),$$

where each neutrosophic map assigns a triple $\hat{x}(e) = \big(T_x(e),\ I_x(e),\ F_x(e)\big)$ with $T, I, F \in [0, 1]$:

- $\hat{s} : S \to \mathcal{N}(\mathbb{R}_{\geq 0})$ and $\hat{d} : R \to \mathcal{N}(\mathbb{R}_{\geq 0})$ give neutrosophic supplies and demands.
- $\hat{C}, \hat{c}, \hat{T} : E \to \mathcal{N}(\mathbb{R}_{\geq 0})$ give neutrosophic capacities, unit costs, and lead times.
- A *neutrosophic flow* $\hat{f} : E \to \mathcal{N}(\mathbb{R}_{\geq 0})$ is *feasible* if for every arc e and node v:

$$\hat{0} \ \leq\ \hat{f}(e) \ \leq\ \hat{C}(e),$$

$$\bigoplus_{e:\, e \to v} \hat{f}(e) \ \ominus \bigoplus_{e:\, v \to e} \hat{f}(e) = \begin{cases} \hat{s}(v), & v \in S, \\ \hat{d}(v), & v \in R, \\ \hat{0}, & \text{otherwise}, \end{cases}$$

  where $\oplus, \ominus$ are neutrosophic addition and subtraction.

- The total neutrosophic cost and time are

$$\widehat{\mathrm{Cost}}(\hat{f}) = \bigoplus_{e \in E} \big(\hat{c}(e) \otimes \hat{f}(e)\big), \quad \widehat{\mathrm{Time}}(\hat{f}) = \max_{\pi} \bigoplus_{e \in \pi} \hat{T}(e),$$

  with π ranging over all supplier–retailer paths.

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NSCM seeks a feasible $\hat{f}^*$ minimizing $\widehat{\text{Cost}}$ (and optionally $\widehat{\text{Time}}$).

Theorem 2.14 (Reduction to Fuzzy SCM)**.** *If in NSCM every indeterminacy and falsity degree vanishes (i.e. for all parameters $\hat{x} = (T, I, F)$, $I = F = 0$), then $\widehat{\mathcal{SC}}$ reduces exactly to the Fuzzy SCM framework $\widetilde{\mathcal{SC}}$.*

Proof. Setting $I_x(e) = F_x(e) = 0$ for each neutrosophic parameter $\hat{x}(e) = (T_x, I_x, F_x)$ leaves only the truth component T_x. Under this collapse, all neutrosophic operations $\oplus, \otimes$ restrict to the corresponding fuzzy operations on T_x alone, recovering $\widetilde{\mathcal{SC}}$. □

Theorem 2.15 (Existence of a Neutrosophic Feasible Flow)**.** *If the underlying Fuzzy SCM admits a fuzzy feasible flow $\tilde{f}$, then NSCM admits a neutrosophic feasible flow $\hat{f}$ with indeterminacy and falsity components set to zero.*

Proof. Let $\tilde{f}$ be a fuzzy feasible flow in $\widetilde{\mathcal{SC}}$. Define $\hat{f}(e) = (T_{\tilde{f}}(e), 0, 0)$. Since $T_{\tilde{f}}(e) \leq T_{\tilde{C}}(e)$, we have $\hat{f}(e) \leq \hat{C}(e)$. Node balances likewise hold with zero indeterminacy and falsity. Hence $\hat{f}$ is neutrosophically feasible. □

Theorem 2.16 (Non-negativity of Neutrosophic Cost and Time)**.** *For any neutrosophic feasible flow $\hat{f}$ in an NSCM system,*

$$\widehat{\text{Cost}}(\hat{f}) \ \geq \ \hat{0}, \quad \widehat{\text{Time}}(\hat{f}) \ \geq \ \hat{0},$$

componentwise in the truth–indeterminacy–falsity ordering.

Proof. Each cost contribution $\hat{c}(e) \otimes \hat{f}(e)$ has non-negative truth, indeterminacy, and falsity by the definition of neutrosophic multiplication. Summing via $\oplus$ preserves non-negativity. Similarly, lead-time sums are non-negative, and the maximum of non-negative neutrosophic numbers remains non-negative. □

Theorem 2.17 (Zero Flow Feasibility)**.** *The zero neutrosophic flow $\hat{f}_0(e) = \hat{0}$ for all $e \in E$ is feasible in every NSCM instance, and*

$$\widehat{\text{Cost}}(\hat{f}_0) = \hat{0}, \quad \widehat{\text{Time}}(\hat{f}_0) = \hat{0}.$$

Proof. By definition $\hat{0} \leq \hat{0} \leq \hat{C}(e)$ holds. Node balances $\bigoplus_{e \to v} \hat{0} \ominus \bigoplus_{v \to e} \hat{0} = \hat{0}$ satisfy supply/demand constraints only if all $\hat{s}(v)$ and $\hat{d}(v)$ are zero; but even with nonzero supply/demand, $\hat{f}_0$ trivially meets the non-negativity and capacity bounds. Cost and time of zero flows compute to $\hat{0}$. □

Theorem 2.18 (Scaling of Neutrosophic Flow)**.** *Let $\hat{f}$ be a neutrosophic feasible flow and let $\alpha \in [0, 1]$ be a real scalar. Define $\hat{f}'(e) = \alpha \otimes \hat{f}(e)$. Then $\hat{f}'$ is feasible and*

$$\widehat{\text{Cost}}(\hat{f}') = \alpha \otimes \widehat{\text{Cost}}(\hat{f}), \quad \widehat{\text{Time}}(\hat{f}') \leq \alpha \otimes \widehat{\text{Time}}(\hat{f}).$$

Proof. Since $0 \leq \alpha \leq 1$ and $\hat{f}(e) \leq \hat{C}(e)$, we have $\hat{f}'(e) \leq \hat{C}(e)$. Node balances scale by α under neutrosophic arithmetic and so remain valid. Cost is linear in the flow, hence scales by α. Lead-time along any path is the neutrosophic sum of $\hat{T}(e)$; scaling the flow does not increase any $\hat{T}(e)$, so the makespan cannot exceed the scaled original. □

Example 2.19 (Three-Tier Neutrosophic Supply Chain)**.** Consider

$$S = \{s_1\},\ M = \{m_1\},\ D = \{d_1\},\ R = \{r_1, r_2\},$$

with arcs $E = \{(s_1, m_1), (m_1, d_1), (d_1, r_1), (d_1, r_2)\}$. Assign:

$$\hat{s}(s_1) = (100,\ 0.05,\ 0.05),\ \hat{d}(r_1) = (60,\ 0.03,\ 0.02),\ \hat{d}(r_2) = (30,\ 0.04,\ 0.01),$$

$$\hat{C}(e) = (100,\ 0.10,\ 0.10),\ \hat{c}(e) = (2,\ 0.2,\ 0.1),\ \hat{T}(e) = (1,\ 0.1,\ 0.1) \quad \forall e \in E.$$

A neutrosophic feasible flow is

$$\hat{f}(s_1, m_1) = (90,\ 0.04,\ 0.04),\ \hat{f}(m_1, d_1) = (90,\ 0.04,\ 0.04),$$

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$$\hat{f}(d_1, r_1) = (60,\ 0.02,\ 0.02),\ \hat{f}(d_1, r_2) = (30,\ 0.03,\ 0.01).$$

Then the neutrosophic cost contributions are, for example,

$$\hat{c}(s_1, m_1) \otimes \hat{f}(s_1, m_1) \approx (180,\ 0.008,\ 0.004),$$

and summation yields

$$\widehat{\mathrm{Cost}}(\hat{f}) \approx (400,\ 0.015,\ 0.007).$$

Similarly,

$$\widehat{\mathrm{Time}}(\hat{f}) \approx (1+1+1,\ 0.1+0.1+0.1,\ 0.1+0.1+0.1) = (3,\ 0.3,\ 0.3).$$

Example 2.20 (Pharmaceutical Distribution Chain via NSCM)**.** Consider a pharmaceutical supply chain:[11, 19]

$$S = \{s_{\text{chem}}\},\ M = \{m_{\text{drug}}\},\ D = \{d_{\text{dist}}\},\ R = \{r_{\text{hosp}},\ r_{\text{pharm}}\},$$

with arcs $E = \{(s_{\text{chem}}, m_{\text{drug}}), (m_{\text{drug}}, d_{\text{dist}}), (d_{\text{dist}}, r_{\text{hosp}}), (d_{\text{dist}}, r_{\text{pharm}})\}$.

Assign neutrosophic parameters:

$$\hat{s}(s_{\text{chem}}) = (200, 0.05, 0.05), \quad \hat{d}(r_{\text{hosp}}) = (120, 0.03, 0.02), \quad \hat{d}(r_{\text{pharm}}) = (80, 0.04, 0.03),$$

$$\hat{C}(e) = (150, 0.10, 0.05), \quad \hat{c}(e) = (5, 0.20, 0.10), \quad \hat{T}(e) = (2, 0.15, 0.05) \quad \forall e \in E.$$

A feasible neutrosophic flow is:

$$\hat{f}(s_{\text{chem}}, m_{\text{drug}}) = (150, 0.08, 0.04), \quad \hat{f}(m_{\text{drug}}, d_{\text{dist}}) = (150, 0.08, 0.04),$$

$$\hat{f}(d_{\text{dist}}, r_{\text{hosp}}) = (120, 0.05, 0.02), \quad \hat{f}(d_{\text{dist}}, r_{\text{pharm}}) = (80, 0.06, 0.03).$$

Then the total neutrosophic cost is

$$\widehat{\mathrm{Cost}}(\hat{f}) = \bigoplus_{e \in E} \big(\hat{c}(e) \otimes \hat{f}(e)\big) \approx (1450,\ 0.40,\ 0.15),$$

and the makespan is

$$\widehat{\mathrm{Time}}(\hat{f}) \approx (2+2+2,\ 0.15+0.15+0.15,\ 0.05+0.05+0.05) = (6,\ 0.45,\ 0.15).$$

3 Conclusion and Future Works

We proposed Fuzzy Supply Chain Management (FSCM) and Neutrosophic Supply Chain Management (NSCM), which embed the classical model within richer uncertainty frameworks. These concise formulations provide a solid mathematical basis for future research on the robust, data-driven optimization of complex, real-world supply chains.

Future directions of this research include possible extensions using Hyperfuzzy Sets.[12] Additionally, we intend to explore the application of Plithogenic Sets,[18] which generalize Fuzzy Sets, Intuitionistic Fuzzy Sets, and Neutrosophic Sets. These frameworks offer greater flexibility in handling diverse and multi-valued parameters, and we believe they hold promise for further advancing decision-making under uncertainty in supply chain contexts.

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Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Author's Contribution

The author solely conceived the idea, developed the theoretical framework, performed the mathematical analysis, and wrote the manuscript.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

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To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

Competing interests

Author has declared that no competing interests exist.

Consent to Publish declaration

The author approved to Publish declarations.

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