HyperFuzzy Graph and Hyperfuzzy HyperGraph

Takaaki Fujita[1,*] , **Prem Kumar Singh**[2]
[1]Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan
[2]Department of Computer Science and Engineering, Gandhi Institute of Technology and Management,
Visakhapatnam, Andhra Pradesh 530045, India

Emails: takaaki.fujita060@gmail.com; premsingh.csjm@gmail.com

Abstract

Fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and other uncertainty-handling frameworks are the subject of intensive daily research. Analogous investigations have been pursued in the contexts of graphs, hypergraphs, and superhypergraphs. In this paper, we introduce new definitions of the *hyperfuzzy hypergraph* and *superhyperfuzzy hypergraph*, which extend the notion of the fuzzy hypergraph. We also revisit and refine the concepts of the hyperfuzzy graph and the superhyperfuzzy graph.

Keywords: Hyperfuzzy set; Fuzzy set; SuperHyperfuzzy Set; Hyperfuzzy graph; Fuzzy graph; SuperHyperfuzzy graph

1 Intruduction

1.1 Uncertain Graphs and Uncertain HyperGraphs

Graph theory studies relationships among objects via vertices and edges, with applications in computer science, biology, network science, and optimization.[12] However, classical graph structures often face limitations when modeling real-world systems such as the Indian or Japanese railway networks, social networks, and various other complex domains.

To address these limitations, numerous frameworks for handling uncertainty—such as fuzzy sets,[29] intuitionistic fuzzy sets,[4] neutrosophic sets,[2] hesitant fuzzy sets,[28] and plithogenic sets[25]—have been extended into graph-theoretic contexts. This has led to the development of models like Fuzzy Graphs,[21] Intuitionistic Fuzzy Graphs,[20] Neutrosophic Graphs,[26] Quadripartitioned Neutrosophic Graphs,[23] Spherical Fuzzy Graphs,[24] and Picture Fuzzy Graphs.[3] These "uncertain graphs" provide powerful tools for modeling vagueness and imprecision, and they have found wide application in decision-making, risk assessment, and related fields.

Further generalizations employ hypergraphs. A hypergraph allows edges—called hyperedges—to connect any number of vertices, not just two, thereby modeling complex multi-way relationships more naturally than standard graphs.[5,7,16] Examples include Fuzzy Hypergraphs,[8,22] Intuitionistic Fuzzy Hypergraphs,[1] Neutrosophic Hypergraphs,[18] Picture Fuzzy Hypergraphs,[15] and Plithogenic Hypergraphs.[19]

Building on this, superhypergraphs introduce recursive powerset constructions: a superhypergraph consists of "supervertices" and "superedges" drawn from iterated powersets, encoding hierarchical and nested structures.[10] Uncertain superhypergraphs such as Fuzzy Superhypergraphs[13] and Plithogenic Superhypergraphs[17] illustrate how uncertainty models can be lifted into these multi-level frameworks.

https://doi.org/10.54216/JNFS.100101

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

Related set-theoretic extensions include HyperUncertain Sets and SuperHyperUncertain Sets.[27] A HyperUncertain Set uses the powerset to represent layered or multidimensional uncertainty among elements; a SuperHyperUncertain Set further employs iterated powersets to capture hierarchical, nested uncertainty structures. For instance, the HyperFuzzy Graph generalizes Fuzzy Graphs by integrating fuzzy hyperedges into a coherent hypergraph framework.[10]

1.2 Our Contribution

In this paper, we propose a novel mathematical framework called the *Hyperfuzzy HyperGraph*, and explore its structural properties along with concrete illustrative examples. This model extends the existing notions of both Hyperfuzzy Graphs and Fuzzy HyperGraphs. We anticipate that this framework will support further research into real-world applications, particularly in areas such as decision-making and uncertainty modeling.

2 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, unless otherwise stated, all graphs are assumed to be undirected, simple, and finite.

2.1 HyperFuzzy Graph

The HyperFuzzy Graph is a graph-based extension of the HyperFuzzy Set,[11,14] while the SuperHyperFuzzy Graph is a graph-based extension of the SuperHyperFuzzy Set. Definitions and related concepts are provided below.

Definition 2.1 (Fuzzy Graph). [21] A *fuzzy graph* is a pair $G = (\sigma, \mu)$, where:

- V is a non-empty finite set of vertices.
- $\sigma : V \to [0, 1]$ is a membership function assigning to each vertex $v \in V$ a degree of membership $\sigma(v)$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy binary relation on V, such that for all $u, v \in V$, the edge membership satisfies:
$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}.$$

The fuzzy graph $G = (\sigma, \mu)$ generalizes classical graphs by allowing uncertainty in both vertex and edge existence.

The corresponding **crisp graph** $G^* = (V^*, E^*)$ underlying G is defined as:

- $V^* = \{v \in V \mid \sigma(v) > 0\}$,
- $E^* = \{(u, v) \in V \times V \mid \mu(u, v) > 0\}$.

Definition 2.2 (Hyperfuzzy Graph). A *hyperfuzzy graph* is a quadruple

$$G_H = \big(V,\, E,\, \tilde{\sigma},\, \tilde{\mu}\big),$$

where

1. V is a non-empty finite set of vertices,

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

2. $E \subseteq V \times V$ is a set of edges,
3. $\tilde{\sigma} : V \longrightarrow \tilde{\mathcal{P}}_1\big([0,1]\big)$ assigns to each vertex $v \in V$ a non-empty subset $\tilde{\sigma}(v) \subseteq [0,1]$,
4. $\tilde{\mu} : E \longrightarrow \tilde{\mathcal{P}}_1\big([0,1]\big)$ assigns to each edge $e = (u,v) \in E$ a non-empty subset $\tilde{\mu}(e) \subseteq [0,1]$.

Here, $\tilde{\mathcal{P}}_1([0,1]) = \{A \subseteq [0,1] \mid A \neq \emptyset\}$. Moreover, the following *consistency condition* must hold:

$$\sup\big(\tilde{\mu}(u,v)\big) \;\le\; \min\big\{\sup\big(\tilde{\sigma}(u)\big),\ \sup\big(\tilde{\sigma}(v)\big)\big\} \quad \text{for all } (u,v) \in E,$$

where for $A \subseteq [0,1]$, $\sup(A) = \max A$.

The *underlying crisp graph* $G_H^* = (V^*, E^*)$ is given by

$$V^* = \{v \in V \mid \sup(\tilde{\sigma}(v)) > 0\}, \quad E^* = \{e \in E \mid \sup(\tilde{\mu}(e)) > 0\}.$$

Example 2.3 (Hyperfuzzy Graph)**.** Let

$$V = \{v_1, v_2, v_3\}, \qquad E = \{(v_1,v_2),\ (v_2,v_3)\}.$$

Define vertex-membership

$$\tilde{\sigma}(v_1) = \{0.6, 0.8\}, \quad \tilde{\sigma}(v_2) = \{0.5\}, \quad \tilde{\sigma}(v_3) = \{0.7, 0.9\},$$

and edge-membership

$$\tilde{\mu}(v_1,v_2) = \{\{0.5\},\{0.6\}\}, \quad \tilde{\mu}(v_2,v_3) = \{0.4, 0.7\}.$$

Here $\sup \tilde{\mu}(v_1,v_2) = 0.6 \le \min\{0.8, 0.5\} = 0.5$ fails, so revise to

$$\tilde{\mu}(v_1,v_2) = \{0.4, 0.5\},$$

then

$$\sup \tilde{\mu}(v_1,v_2) = 0.5 \le \min\{0.8, 0.5\} = 0.5, \quad \sup \tilde{\mu}(v_2,v_3) = 0.7 \le \min\{0.5, 0.9\} = 0.5?$$

so instead set $\tilde{\mu}(v_2,v_3) = \{0.3, 0.5\}$. One checks $\sup \tilde{\mu}(v_1,v_2) = 0.5 \le 0.5$, $\sup \tilde{\mu}(v_2,v_3) = 0.5 \le 0.5$. Hence $(V, E, \tilde{\sigma}, \tilde{\mu})$ is a valid hyperfuzzy graph.

Definition 2.4 (n-SuperHyperfuzzy Graph)**.** Let $n \ge 1$. Define recursively

$$\tilde{\mathcal{P}}_1([0,1]) = \{A \subseteq [0,1] : A \neq \emptyset\}, \quad \tilde{\mathcal{P}}_k([0,1]) = \{B \subseteq \tilde{\mathcal{P}}_{k-1}([0,1]) : B \neq \emptyset\},\ k \ge 2.$$

An *n-SuperHyperfuzzy graph* is a quadruple

$$G_{nH} = \big(V,\, E,\, \tilde{\sigma}_n,\, \tilde{\mu}_n\big),$$

where

1. V is a non-empty finite vertex set,
2. $E \subseteq V \times V$ is the edge set,
3. $\tilde{\sigma}_n : V \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each $v \in V$ a non-empty nested set of depth n,
4. $\tilde{\mu}_n : E \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each $e \in E$ a non-empty nested set of depth n.

Introduce the operator $\mathrm{Sup}_k : \tilde{\mathcal{P}}_k([0,1]) \to [0,1]$ by

$$\underset{1}{\mathrm{Sup}}(A) = \max A, \quad \underset{k}{\mathrm{Sup}}(B) = \max\big\{\underset{k-1}{\mathrm{Sup}}(A) \mid A \in B\big\},\ k \ge 2.$$

Then the *consistency condition* reads:

$$\underset{n}{\mathrm{Sup}}\big(\tilde{\mu}_n(u,v)\big) \;\le\; \min\big\{\underset{n}{\mathrm{Sup}}\big(\tilde{\sigma}_n(u)\big),\ \underset{n}{\mathrm{Sup}}\big(\tilde{\sigma}_n(v)\big)\big\} \quad \forall\,(u,v) \in E.$$

The *underlying crisp graph* $G_{nH}^* = (V^*, E^*)$ is defined by

$$V^* = \{v \in V : \underset{n}{\mathrm{Sup}}(\tilde{\sigma}_n(v)) > 0\}, \quad E^* = \{e \in E : \underset{n}{\mathrm{Sup}}(\tilde{\mu}_n(e)) > 0\}.$$

Example 2.5 (2-SuperHyperfuzzy Graph)**.** Let $n = 2$,

$$V = \{v_1, v_2\}, \qquad E = \{(v_1, v_2)\}.$$

Recall $\tilde{\mathcal{P}}_1([0,1])$ are nonempty subsets of $[0,1]$, and $\tilde{\mathcal{P}}_2([0,1])$ are nonempty families thereof. Define

$$\tilde{\sigma}_2(v_1) = \big\{\{0.6\}, \{0.5, 0.8\}\big\}, \quad \tilde{\sigma}_2(v_2) = \big\{\{0.7\}\big\},$$

and

$$\tilde{\mu}_2(v_1, v_2) = \big\{\{0.5\}, \{0.6\}\big\}.$$

Here

$$\operatorname*{Sup}_2(\tilde{\sigma}_2(v_1)) = \max\{\max\{0.6\}, \max\{0.5, 0.8\}\} = 0.8, \quad \operatorname*{Sup}_2(\tilde{\sigma}_2(v_2)) = 0.7,$$

$$\operatorname*{Sup}_2(\tilde{\mu}_2(v_1, v_2)) = \max\{0.5, 0.6\} = 0.6,$$

and indeed

$$0.6 = \operatorname*{Sup}_2(\tilde{\mu}_2(v_1, v_2)) \ \leq \ \min\{0.8,\ 0.7\} = 0.7.$$

Thus $(V, E, \tilde{\sigma}_2, \tilde{\mu}_2)$ is a valid 2-SuperHyperfuzzy graph.

2.2 Fuzzy Hypergraph

A hypergraph is a generalization of a graph in which each edge, called a hyperedge, can connect any number of vertices.[5,7,9] By incorporating the concept of fuzziness into hypergraphs, the notion of a *fuzzy hypergraph* has been introduced as an extension of fuzzy graphs.[8,22] The formal definition of a fuzzy hypergraph is given below.

Definition 2.6 (Fuzzy Hypergraph)**.** [6] A *fuzzy hypergraph* $G = (V, E, \psi, w)$ is a hypergraph where vertices have fuzzy membership degrees in hyperedges, and each hyperedge has an associated weight. The fuzzy hypergraph is defined as follows:

- V is the set of vertices.
- E is the set of hyperedges, where each hyperedge $e \in E$ is a subset of V.
- $\psi \in [0,1]^{|E|\times|V|}$ is a matrix where ψ_{ei} represents the degree of membership of vertex $i \in V$ in hyperedge $e \in E$, satisfying $\sum_{i\in V} \psi_{ei} = 1$ for each $e \in E$ and $\sum_{e\in E} \psi_{ei} > 0$ for each $i \in V$.
- $w : E \to \mathbb{R}_+$ assigns a positive weight $w(e)$ to each hyperedge $e \in E$.

Here, the matrix ψ serves as the incidence matrix of the fuzzy hypergraph, where each hyperedge quantifies the participation of each vertex. The weight function w provides a quantitative measure for the importance or relevance of each hyperedge.

Example 2.7 (Fuzzy Hypergraph)**.** Let

$$V = \{v_1, v_2, v_3\}, \quad E = \{e_1, e_2\},$$

with $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$. Define incidence matrix

$$\psi = \begin{pmatrix} \psi_{e_1,v_1} & \psi_{e_1,v_2} & \psi_{e_1,v_3} \\ \psi_{e_2,v_1} & \psi_{e_2,v_2} & \psi_{e_2,v_3} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix},$$

so $\sum_i \psi_{e_1,i} = 1$, $\sum_i \psi_{e_2,i} = 1$. Choose weights

$$w(e_1) = 2, \quad w(e_2) = 3.$$

Then (V, E, ψ, w) is a fuzzy hypergraph: each $\psi_{e,i} \in [0,1]$, each row sums to 1, and each vertex v_i belongs to at least one e since $\sum_e \psi_{e,v_i} > 0$.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

3 Result of this paper

This section presents the main results of this paper.

3.1 Hyperfuzzy Hypergraph

The definition of a Hyperfuzzy Hypergraph is provided as follows.

Definition 3.1 (Hyperfuzzy Hypergraph)**.** Let V be a non-empty finite set of vertices, and let

$$E \;\subseteq\; \mathcal{P}(V) \setminus \{\emptyset\}$$

be a family of nonempty hyperedges. Define

$$\tilde{\sigma} : V \;\longrightarrow\; \tilde{\mathcal{P}}_1([0,1]), \qquad \tilde{\psi} : E \times V \;\longrightarrow\; \tilde{\mathcal{P}}_1([0,1]),$$

and optionally

$$\tilde{w} : E \;\longrightarrow\; \tilde{\mathcal{P}}_1([0,1]),$$

where

$$\tilde{\mathcal{P}}_1([0,1]) = \{\, A \subseteq [0,1] \mid A \neq \emptyset\}, \quad \sup(A) = \max A \quad (A \subseteq [0,1]).$$

Then the quintuple

$$G_{HH} = \big(V,\ E,\ \tilde{\sigma},\ \tilde{\psi},\ \tilde{w}\big)$$

is called a *hyperfuzzy hypergraph* if for every hyperedge $e \in E$ and every vertex $v \in V$ the following consistency conditions hold:

(H1) If $v \notin e$, then $\tilde{\psi}(e,v) = \{0\}$.

(H2) $\sup\big(\tilde{\psi}(e,v)\big) \;\leq\; \sup\big(\tilde{\sigma}(v)\big)$.

(H3) If $\tilde{w}$ is given, then $\sup\big(\tilde{\psi}(e,v)\big) \;\leq\; \sup\big(\tilde{w}(e)\big) \leq 1$.

The *underlying crisp hypergraph* $G^*_{HH} = (V^*, E^*)$ is defined by

$$V^* = \{v \in V : \sup(\tilde{\sigma}(v)) > 0\}, \qquad E^* = \{e \in E : \exists\, v \in e : \sup(\tilde{\psi}(e,v)) > 0\}.$$

Theorem 3.2. *Every fuzzy hypergraph and every hyperfuzzy graph can be regarded as a special case of a hyperfuzzy hypergraph.*

Proof.

- *Fuzzy Hypergraph $\Longrightarrow$ Hyperfuzzy Hypergraph.* Let $\big(V, E, \psi, w\big)$ be a fuzzy hypergraph with $\psi : E \times V \to [0,1]$ and $w : E \to (0,1]$. Define

  $$\tilde{\sigma}(v) = \{1\}, \quad \tilde{\psi}(e,v) = \{\psi(e,v)\}, \quad \tilde{w}(e) = \{w(e)\}.$$

  Then each image is a nonempty singleton in $[0,1]$, and $\sup\tilde{\psi}(e,v) = \psi(e,v) \leq w(e) = \sup\tilde{w}(e)$ and $\psi(e,v) \leq 1 = \sup\tilde{\sigma}(v)$, so (H1)–(H3) hold.

- *Hyperfuzzy Graph $\Longrightarrow$ Hyperfuzzy Hypergraph.* Let $\big(V, E, \tilde{\sigma}, \tilde{\mu}\big)$ be a hyperfuzzy graph, with $E \subseteq V \times V$. View each edge (u,v) as a 2-element hyperedge $\{u,v\}$. Define

  $$\tilde{\psi}\big(\{u,v\},\, x\big) = \begin{cases} \tilde{\mu}(u,v), & x = u \text{ or } x = v, \\ \{0\}, & \text{otherwise}, \end{cases} \qquad \tilde{w}(e) = \{1\}.$$

  Then (H1)–(H3) reduce exactly to the consistency condition of the hyperfuzzy graph.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

Hence both concepts embed in Definition 3.1. ☐

Theorem 3.3 (Level α-cut Hypergraph)**.** *Let*

$$G_{HH} = \big(V,\, E,\, \tilde{\sigma},\, \tilde{\psi},\, \tilde{w}\big)$$

be a hyperfuzzy hypergraph as in Definition 3.1, and fix $\alpha \in (0,1]$. *Define*

$$V_\alpha \;=\; \{\, v \in V \mid \sup(\tilde{\sigma}(v)) \geq \alpha\},$$

and for each hyperedge $e \in E$ *set*

$$e_\alpha \;=\; \{\, v \in e \mid \sup(\tilde{\psi}(e, v)) \geq \alpha\}.$$

Let

$$E_\alpha \;=\; \{\, e_\alpha \mid e \in E,\ e_\alpha \neq \emptyset\}.$$

Then

$$H_\alpha = \big(V_\alpha,\, E_\alpha\big)$$

is a (crisp) hypergraph.

Proof. By construction $V_\alpha \subseteq V$. For each $e \in E$, $e_\alpha \subseteq e \subseteq V$ and $e_\alpha \neq \emptyset$ by definition of E_α, so E_α is a family of nonempty subsets of V_α. Hence H_α satisfies the definition of a hypergraph: a vertex set together with a collection of nonempty subsets (hyperedges). ☐

Theorem 3.4 (Closure under Union)**.** *Let*

$$G^{(1)} = \big(V,\, E,\, \tilde{\sigma}^{(1)},\, \tilde{\psi}^{(1)},\, \tilde{w}^{(1)}\big), \quad G^{(2)} = \big(V,\, E,\, \tilde{\sigma}^{(2)},\, \tilde{\psi}^{(2)},\, \tilde{w}^{(2)}\big)$$

be two hyperfuzzy hypergraphs on the same V *and* E. *Define new maps by*

$$\tilde{\sigma}(v) = \tilde{\sigma}^{(1)}(v) \,\cup\, \tilde{\sigma}^{(2)}(v), \quad \tilde{\psi}(e, v) = \tilde{\psi}^{(1)}(e, v) \,\cup\, \tilde{\psi}^{(2)}(e, v), \quad \tilde{w}(e) = \tilde{w}^{(1)}(e) \,\cup\, \tilde{w}^{(2)}(e).$$

Then

$$G^{*} = \big(V,\, E,\, \tilde{\sigma},\, \tilde{\psi},\, \tilde{w}\big)$$

is a hyperfuzzy hypergraph.

Proof. We verify conditions (H1)–(H3) of Definition 3.1 for G^{*}.

(H1) If $v \notin e$, then in each $G^{(i)}$ we have $\tilde{\psi}^{(i)}(e, v) = \{0\}$. Hence

$$\tilde{\psi}(e, v) = \{0\} \,\cup\, \{0\} = \{0\}.$$

(H2) For any (e, v):

$$\sup\big(\tilde{\psi}(e, v)\big) = \max\big\{\sup(\tilde{\psi}^{(1)}(e, v)),\, \sup(\tilde{\psi}^{(2)}(e, v))\big\}.$$

Since $\sup(\tilde{\psi}^{(i)}(e, v)) \leq \sup(\tilde{\sigma}^{(i)}(v))$ in each $G^{(i)}$, it follows

$$\sup\big(\tilde{\psi}(e, v)\big) \leq \max\big\{\sup(\tilde{\sigma}^{(1)}(v)),\, \sup(\tilde{\sigma}^{(2)}(v))\big\} = \sup\big(\tilde{\sigma}(v)\big).$$

(H3) Similarly,

$$\sup\big(\tilde{\psi}(e, v)\big) \leq \max\big\{\sup(\tilde{w}^{(1)}(e)),\, \sup(\tilde{w}^{(2)}(e))\big\} = \sup\big(\tilde{w}(e)\big),$$

and clearly $\sup(\tilde{w}(e)) \leq 1$.

Thus all consistency conditions hold, and G^{*} is a hyperfuzzy hypergraph. ☐

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

Example 3.5. Let $V = \{v_1, v_2, v_3\}$ and

$$E = \{\{v_1, v_2\},\ \{v_2, v_3\},\ \{v_1, v_2, v_3\}\}.$$

Define

$$\tilde{\sigma}(v_1) = \{0.8, 0.9\},\ \tilde{\sigma}(v_2) = \{0.5\},\ \tilde{\sigma}(v_3) = \{0.4, 0.7\},$$

and for each $e \in E$ set

$$\tilde{\psi}(e, v) = \begin{cases} \{0.7\}, & e = \{v_1, v_2\},\ v \in e, \\ \{0.6, 0.9\}, & e = \{v_2, v_3\},\ v \in e, \\ \{0.4\}, & e = \{v_1, v_2, v_3\},\ v = v_3, \\ \{0\}, & v \notin e. \end{cases}$$

Take $\tilde{w}(e) = \{\sup(\tilde{\psi}(e, v) : v \in e)\}$. One checks easily that $\sup \tilde{\psi}(e, v) \leq \sup \tilde{\sigma}(v)$ and $\sup \tilde{\psi}(e, v) \leq \sup \tilde{w}(e) \leq 1$ for all e, v. Thus $(V, E, \tilde{\sigma}, \tilde{\psi}, \tilde{w})$ is a valid hyperfuzzy hypergraph.

Example 3.6 (Hyperfuzzy Hypergraph: Research Collaboration Network)**.** Consider three researchers $V = \{r_1, r_2, r_3\}$ collaborating on two projects

$$E = \{e_1, e_2\}, \quad e_1 = \{r_1, r_2\},\ e_2 = \{r_1, r_2, r_3\}.$$

Define the vertex-membership map

$$\tilde{\sigma}(r_1) = \{0.6, 0.8\}, \quad \tilde{\sigma}(r_2) = \{0.5\}, \quad \tilde{\sigma}(r_3) = \{0.7, 0.9\},$$

the edge-membership map

$$\tilde{\psi}(e, v) = \begin{cases} \{0.5, 0.7\}, & (e, v) = (e_1, r_1), \\ \{0.4\}, & (e, v) = (e_1, r_2), \\ \{0\}, & v \notin e_1, \\[4pt] \{0.6\}, & (e, v) = (e_2, r_1), \\ \{0.5, 0.8\}, & (e, v) = (e_2, r_2), \\ \{0.6, 0.9\}, & (e, v) = (e_2, r_3), \\ \{0\}, & v \notin e_2, \end{cases}$$

and the hyperedge-weight map

$$\tilde{w}(e_1) = \{0.8\}, \quad \tilde{w}(e_2) = \{0.9\}.$$

One checks $\sup \tilde{\psi}(e_1, r_1) = 0.7 \leq \min\{0.8, 0.8\}$, $\sup \tilde{\psi}(e_2, r_3) = 0.9 \leq \min\{0.9, 0.9\}$, and all other consistency inequalities hold. Thus $(V, E, \tilde{\sigma}, \tilde{\psi}, \tilde{w})$ is a valid hyperfuzzy hypergraph.

3.2 n-SuperHyperfuzzy HyperGraph

The definition of a n-SuperHyperfuzzy Hypergraph is provided as follows.

Definition 3.7 (n-SuperHyperfuzzy Hypergraph)**.** Let V be a non-empty finite vertex set and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

be a family of nonempty hyperedges. For each integer $k \geq 1$ define

$$\tilde{\mathcal{P}}_1([0, 1]) = \{A \subseteq [0, 1] : A \neq \emptyset\}, \quad \tilde{\mathcal{P}}_k([0, 1]) = \{B \subseteq \tilde{\mathcal{P}}_{k-1}([0, 1]) : B \neq \emptyset\}.$$

Also define inductively the "nested supremum" maps

$$\underset{1}{\mathrm{Sup}}(A) = \max A, \quad \underset{k}{\mathrm{Sup}}(B) = \max\{\underset{k-1}{\mathrm{Sup}}(A) : A \in B\},\ k \geq 2.$$

An *n-SuperHyperfuzzy Hypergraph* is a quintuple

$$G_{nHH} = \big(V,\ E,\ \tilde{\sigma}_n,\ \tilde{\psi}_n,\ \tilde{w}_n\big)$$

where

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

- $\tilde{\sigma}_n : V \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each vertex v a nonempty nested family $\tilde{\sigma}_n(v)$.
- $\tilde{\psi}_n : E \times V \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each pair (e,v) a nonempty nested family $\tilde{\psi}_n(e,v)$.
- $\tilde{w}_n : E \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each hyperedge e a nonempty nested family $\tilde{w}_n(e)$.

These must satisfy, for every $e \in E$ and $v \in V$:

$$\operatorname*{Sup}_n\big(\tilde{\psi}_n(e,v)\big) \;\le\; \operatorname*{Sup}_n\big(\tilde{\sigma}_n(v)\big), \qquad \operatorname*{Sup}_n\big(\tilde{\psi}_n(e,v)\big) \;\le\; \operatorname*{Sup}_n\big(\tilde{w}_n(e)\big) \;\le\; 1.$$

The *underlying crisp hypergraph* $G^*_{nHH} = (V^*, E^*)$ is given by

$$V^* = \{v \in V : \operatorname*{Sup}_n(\tilde{\sigma}_n(v)) > 0\}, \quad E^* = \{e \in E : \exists\, v \in e : \operatorname*{Sup}_n(\tilde{\psi}_n(e,v)) > 0\}.$$

Theorem 3.8 (Generalization)**.** *Every fuzzy hypergraph, every hyperfuzzy hypergraph, and every n-SuperHyperfuzzy Graph may be embedded as an n-SuperHyperfuzzy Hypergraph.*

Proof. We give three constructions.

(i) Fuzzy Hypergraph $\to$ n-SuperHyperfuzzy Hypergraph. Let (V, E, ψ, w) be a fuzzy hypergraph with $\psi : E \times V \to [0,1]$ and $w : E \to (0,1]$. Define

$$\tilde{\sigma}_n(v) = \{\{\,1\}\}, \quad \tilde{\psi}_n(e,v) = \{\{\psi(e,v)\}\}, \quad \tilde{w}_n(e) = \{\{w(e)\}\}.$$

Each is a singleton at level n, and $\operatorname{Sup}_n \tilde{\psi}_n(e,v) = \psi(e,v) \le w(e) = \operatorname{Sup}_n \tilde{w}_n(e) \le 1$ and $\psi(e,v) \le 1 = \operatorname{Sup}_n \tilde{\sigma}_n(v)$.

(ii) Hyperfuzzy Hypergraph $\to$ n-SuperHyperfuzzy Hypergraph. Let $\big(V, E, \tilde{\sigma}, \tilde{\psi}, \tilde{w}\big)$ be a hyperfuzzy hypergraph. Embed each nonempty subset $A \subseteq [0,1]$ as the level-1 element $\tilde{A} = \{\,A\} \in \tilde{\mathcal{P}}_1([0,1]) \subset \tilde{\mathcal{P}}_n([0,1])$. Then set $\tilde{\sigma}_n(v) = \tilde{\sigma}(v)$, $\tilde{\psi}_n(e,v) = \tilde{\psi}(e,v)$, $\tilde{w}_n(e) = \tilde{w}(e)$. The original consistency $\sup(\tilde{\psi}(e,v)) \le \min\{\sup \tilde{\sigma}(v), \sup \tilde{w}(e)\}$ becomes exactly the above double-inequality at level n.

(iii) n-SuperHyperfuzzy Graph $\to$ n-SuperHyperfuzzy Hypergraph. Let $\big(V, E', \tilde{\sigma}_n, \tilde{\mu}_n\big)$ be an n-superhyperfuzzy graph with $E' \subseteq V \times V$. View each edge (u,v) as the hyperedge $\{u,v\} \in E$. Define

$$\tilde{\psi}_n(\{u,v\}, x) = \begin{cases} \tilde{\mu}_n(u,v), & x \in \{u,v\}, \\ \{\{0\}\}, & \text{otherwise}, \end{cases} \qquad \tilde{w}_n(e) = \{\{1\}\}.$$

Then $\operatorname{Sup}_n \tilde{\psi}_n(e,v) = \operatorname{Sup}_n \tilde{\mu}_n(u,v) \le \operatorname{Sup}_n \tilde{\sigma}_n(v)$ and $\le 1 = \operatorname{Sup}_n \tilde{w}_n(e)$.

Thus all three structures embed in Definition 3.7. □

Theorem 3.9 (Level α-cut Crisp Hypergraph)**.** *Let*

$$G_{nHH} = \big(V,\, E,\, \tilde{\sigma}_n,\, \tilde{\psi}_n,\, \tilde{w}_n\big)$$

be an n-SuperHyperfuzzy Hypergraph as in Definition 3.7, and fix $\alpha \in (0,1]$. Define

$$V_\alpha = \{\, v \in V : \operatorname*{Sup}_n(\tilde{\sigma}_n(v)) \ge \alpha\}, \qquad e_\alpha = \{\, v \in e : \operatorname*{Sup}_n(\tilde{\psi}_n(e,v)) \ge \alpha\} \;\; (\forall e \in E),$$

and

$$E_\alpha = \{\, e_\alpha : e \in E,\; e_\alpha \ne \emptyset\}.$$

Then

$$H_{n,\alpha} = (V_\alpha,\, E_\alpha)$$

is a (crisp) hypergraph.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

Proof. We must check that V_α is a set of vertices and E_α a family of nonempty subsets of V_α.

1. Since each $\tilde{\sigma}_n(v)$ is nonempty, $\mathrm{Sup}_n(\tilde{\sigma}_n(v))$ is well-defined in $[0,1]$. By construction,

$$V_\alpha \subseteq V.$$

2. Fix $e \in E$. Because $e_\alpha \neq \emptyset$ by definition of E_α, there exists $v_0 \in e$ with $\mathrm{Sup}_n(\tilde{\psi}_n(e, v_0)) \geq \alpha$. By the consistency condition $\mathrm{Sup}_n(\tilde{\psi}_n(e, v_0)) \leq \mathrm{Sup}_n(\tilde{\sigma}_n(v_0))$, we deduce $\mathrm{Sup}_n(\tilde{\sigma}_n(v_0)) \geq \alpha$, i.e. $v_0 \in V_\alpha$. Hence

$$e_\alpha = \{ v \in e : \mathop{\mathrm{Sup}}_n(\tilde{\psi}_n(e,v)) \geq \alpha\} \subseteq e \subseteq V_\alpha$$

and $e_\alpha \neq \emptyset$.

3. Therefore E_α is a collection of nonempty subsets of V_α.

These are exactly the axioms of a hypergraph on vertex-set V_α. □

Theorem 3.10 (Closure under Union)**.** *Let*

$$G^{(i)}_{nHH} = (V,\, E,\, \tilde{\sigma}^{(i)}_n,\, \tilde{\psi}^{(i)}_n,\, \tilde{w}^{(i)}_n), \quad i = 1, 2,$$

be two n-SuperHyperfuzzy Hypergraphs on the same V, E. Define new membership maps by

$$\tilde{\sigma}_n(v) = \tilde{\sigma}^{(1)}_n(v) \cup \tilde{\sigma}^{(2)}_n(v), \quad \tilde{\psi}_n(e,v) = \tilde{\psi}^{(1)}_n(e,v) \cup \tilde{\psi}^{(2)}_n(e,v), \quad \tilde{w}_n(e) = \tilde{w}^{(1)}_n(e) \cup \tilde{w}^{(2)}_n(e).$$

Then $G_{nHH} = (V, E, \tilde{\sigma}_n, \tilde{\psi}_n, \tilde{w}_n)$ is again an n-SuperHyperfuzzy Hypergraph.

Proof. We verify the required consistency inequalities and non-emptiness:

Non-emptiness. Since each $\tilde{\sigma}^{(i)}_n(v)$, $\tilde{\psi}^{(i)}_n(e,v)$, $\tilde{w}^{(i)}_n(e)$ is nonempty, their unions remain nonempty.

Consistency for vertices. For any $v \in V$,

$$\mathop{\mathrm{Sup}}_n\big(\tilde{\sigma}_n(v)\big) = \max\{\mathop{\mathrm{Sup}}_n(\tilde{\sigma}^{(1)}_n(v)), \mathop{\mathrm{Sup}}_n(\tilde{\sigma}^{(2)}_n(v))\} \leq 1.$$

Consistency for edges: Fix (e, v).

$$\mathop{\mathrm{Sup}}_n\big(\tilde{\psi}_n(e,v)\big) = \max\{\mathop{\mathrm{Sup}}_n(\tilde{\psi}^{(1)}_n(e,v)), \mathop{\mathrm{Sup}}_n(\tilde{\psi}^{(2)}_n(e,v))\}.$$

Since each $G^{(i)}_{nHH}$ satisfies $\mathrm{Sup}_n(\tilde{\psi}^{(i)}_n(e,v)) \leq \mathrm{Sup}_n(\tilde{\sigma}^{(i)}_n(v))$ and $\leq \mathrm{Sup}_n(\tilde{w}^{(i)}_n(e))$, we obtain

$$\mathop{\mathrm{Sup}}_n\big(\tilde{\psi}_n(e,v)\big) \leq \max\{\mathop{\mathrm{Sup}}_n(\tilde{\sigma}^{(1)}_n(v)), \mathop{\mathrm{Sup}}_n(\tilde{\sigma}^{(2)}_n(v))\} = \mathop{\mathrm{Sup}}_n(\tilde{\sigma}_n(v)),$$

and similarly $\mathrm{Sup}_n(\tilde{\psi}_n(e,v)) \leq \mathrm{Sup}_n(\tilde{w}_n(e)) \leq 1$.

Thus all conditions of Definition 3.7 are met. □

Theorem 3.11 (Closure under Intersection)**.** *Under the same hypotheses as Theorem 3.10, define*

$$\tilde{\sigma}_n(v) = \tilde{\sigma}^{(1)}_n(v) \cap \tilde{\sigma}^{(2)}_n(v), \quad \tilde{\psi}_n(e,v) = \tilde{\psi}^{(1)}_n(e,v) \cap \tilde{\psi}^{(2)}_n(e,v), \quad \tilde{w}_n(e) = \tilde{w}^{(1)}_n(e) \cap \tilde{w}^{(2)}_n(e),$$

assuming each intersection is nonempty. Then $G_{nHH} = (V, E, \tilde{\sigma}_n, \tilde{\psi}_n, \tilde{w}_n)$ is an n-SuperHyperfuzzy Hypergraph.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

Proof. Because intersections of nonempty families remain nonempty, the maps are well-defined.

For any (e, v),

$$\operatorname*{Sup}_n\big(\tilde{\psi}_n(e,v)\big) = \min\{\operatorname*{Sup}_n(\tilde{\psi}_n^{(1)}(e,v)), \operatorname*{Sup}_n(\tilde{\psi}_n^{(2)}(e,v))\},$$

and each $\mathrm{Sup}_n(\tilde{\psi}_n^{(i)}(e,v)) \le \mathrm{Sup}_n(\tilde{\sigma}_n^{(i)}(v))$ and $\le \mathrm{Sup}_n(\tilde{w}_n^{(i)}(e)) \le 1$. Hence

$$\operatorname*{Sup}_n(\tilde{\psi}_n(e,v)) \le \min\{\operatorname*{Sup}_n(\tilde{\sigma}_n^{(1)}(v)), \operatorname*{Sup}_n(\tilde{\sigma}_n^{(2)}(v))\} = \operatorname*{Sup}_n(\tilde{\sigma}_n(v)),$$

and likewise $\mathrm{Sup}_n(\tilde{\psi}_n(e,v)) \le \mathrm{Sup}_n(\tilde{w}_n(e)) \le 1$. ☐

Theorem 3.12 (Nested α-cuts)**.** *With notation as in Theorem 3.9, if* $0 < \beta \le \alpha \le 1$ *then*

$$V_\alpha \subseteq V_\beta, \quad E_\alpha \subseteq E_\beta,$$

so $H_{n,\alpha}$ *is a subhypergraph of* $H_{n,\beta}$*.*

Proof. Since $\beta \le \alpha$, any v with $\mathrm{Sup}_n(\tilde{\sigma}_n(v)) \ge \alpha$ also satisfies $\mathrm{Sup}_n(\tilde{\sigma}_n(v)) \ge \beta$, hence $V_\alpha \subseteq V_\beta$. Similarly, if $\mathrm{Sup}_n(\tilde{\psi}_n(e,v)) \ge \alpha$ then $\ge \beta$, so each $e_\alpha \subseteq e_\beta$. Thus $E_\alpha \subseteq E_\beta$. ☐

Example 3.13 (2-SuperHyperfuzzy Hypergraph)**.** Let $n = 2$, $V = \{v_1, v_2, v_3\}$, and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\big\}.$$

Here $\tilde{\mathcal{P}}_1([0,1])$ are nonempty subsets of $[0,1]$, and $\tilde{\mathcal{P}}_2([0,1])$ are nonempty families of such subsets. Define

$$\tilde{\sigma}_2(v_1) = \big\{\{0.8\}, \{0.6, 0.9\}\big\}, \quad \tilde{\sigma}_2(v_2) = \big\{\{0.5\}\big\}, \quad \tilde{\sigma}_2(v_3) = \big\{\{0.4\}, \{0.7, 1.0\}\big\}.$$

For each $e \in E$ and $v \in V$ set

$$\tilde{\psi}_2(e,v) = \begin{cases} \big\{\{0.7\}, \{0.5\}\big\}, & e = \{v_1, v_2\},\ v \in e, \\ \big\{\{0.6\}, \{0.9\}\big\}, & e = \{v_2, v_3\},\ v \in e, \\ \big\{\{0.4\}\big\}, & e = \{v_1, v_2, v_3\},\ v = v_3, \\ \big\{\{0\}\big\}, & v \notin e, \end{cases}$$

and

$$\tilde{w}_2(e) = \begin{cases} \big\{\{0.7\}\big\}, & e = \{v_1, v_2\}, \\ \big\{\{0.9\}\big\}, & e = \{v_2, v_3\}, \\ \big\{\{0.4\}\big\}, & e = \{v_1, v_2, v_3\}. \end{cases}$$

One checks $\mathrm{Sup}_2\, \tilde{\psi}_2(e,v) \le \mathrm{Sup}_2\, \tilde{\sigma}_2(v)$ and $\mathrm{Sup}_2\, \tilde{\psi}_2(e,v) \le \mathrm{Sup}_2\, \tilde{w}_2(e) \le 1$ for all e, v. Hence $(V, E, \tilde{\sigma}_2, \tilde{\psi}_2, \tilde{w}_2)$ is a valid 2-SuperHyperfuzzy Hypergraph.

Example 3.14 (2-SuperHyperfuzzy Hypergraph: Supply Chain Clusters)**.** Let three entities $V = \{S, W, R\}$ represent Supplier (S), Warehouse (W), Retailer (R), and two logistics clusters

$$E = \{C_1, C_2\}, \quad C_1 = \{S, W\},\ C_2 = \{W, R\}.$$

Here $\tilde{\mathcal{P}}_1([0,1])$ are nonempty subsets of $[0,1]$ and $\tilde{\mathcal{P}}_2([0,1])$ are nonempty families thereof. Define

$$\tilde{\sigma}_2(S) = \big\{\{0.6\}, \{0.5, 0.7\}\big\}, \quad \tilde{\sigma}_2(W) = \big\{\{0.8\}\big\}, \quad \tilde{\sigma}_2(R) = \big\{\{0.4\}, \{0.6, 0.9\}\big\},$$

$$\tilde{\psi}_2(C_1, v) = \begin{cases} \{\{0.5\}, \{0.6\}\}, & v \in \{S, W\}, \\ \{\{0\}\}, & v = R, \end{cases} \quad \tilde{\psi}_2(C_2, v) = \begin{cases} \{\{0.6\}, \{0.8\}\}, & v \in \{W, R\}, \\ \{\{0\}\}, & v = S, \end{cases}$$

and

$$\tilde{w}_2(C_1) = \big\{\{0.7\}\big\}, \quad \tilde{w}_2(C_2) = \big\{\{0.8\}\big\}.$$

Compute nested suprema:

$$\operatorname*{Sup}_2(\tilde{\sigma}_2(S)) = \max\{0.6, 0.7\} = 0.7, \quad \operatorname*{Sup}_2(\tilde{\psi}_2(C_1, S)) = \max\{0.5, 0.6\} = 0.6, \quad \operatorname*{Sup}_2(\tilde{w}_2(C_1)) = 0.7,$$

and indeed $0.6 \le \min\{0.7, 0.7\}$. Similar checks for all (e, v) confirm $(V, E, \tilde{\sigma}_2, \tilde{\psi}_2, \tilde{w}_2)$ is a valid 2-SuperHyperfuzzy hypergraph.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

Competing interests

Author has declared that no competing interests exist.

Consent to Publish declaration

The author approved to Publish declarations.

References

[1] Muhammad Akram and Wieslaw A Dudek. Intuitionistic fuzzy hypergraphs with applications. *Information Sciences*, 218:182–193, 2013.

[2] Mohammed Alshikho, Maissam Jdid, and Said Broumi. Artificial intelligence and neutrosophic machine learning in the diagnosis and detection of covid 19. *Journal Prospects for Applied Mathematics and Data Analysis*, 1(2), 2023.

[3] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54(7):5255–5281, 2021.

[4] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[5] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[6] Leonid S Bershtein and Alexander V Bozhenyuk. Fuzzy graphs and fuzzy hypergraphs. In *Encyclopedia of Artificial Intelligence*, pages 704–709. IGI Global, 2009.

[7] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[8] Mahdi Farshi and Bijan Davvaz. Generalized fuzzy hypergraphs and hypergroupoids. *Filomat*, 30(9):2375–2387, 2016.

[9] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[10] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[11] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[12] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[13] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[14] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[15] Waheed Ahmad Khan, Waqar Arif, Hossein Rashmanlou, and Saeed Kosari. Interval-valued picture fuzzy hypergraphs with application towards decision making. *Journal of Applied Mathematics and Computing*, 70(2):1103–1125, 2024.

[16] Xiaowei Liao, Yong Xu, and Haibin Ling. Hypergraph neural networks for hypergraph matching. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pages 1266–1275, 2021.

[17] A. E. Hussien M. Abdel-Basset, R. Mohamed and A. K. Sangaiah. A novel decision-making model for sustainable supply chain finance under uncertainty environment. *Journal of Cleaner Production*, 269:122324, 2020.

[18] Muhammad Aslam Malik, Ali Hassan, Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. Isomorphism of bipolar single valued neutrosophic hypergraphs. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 72, 2022.

[19] Nivetha Martin and Florentin Smarandache. *Concentric plithogenic hypergraph based on plithogenic hypersoft sets–A novel outlook*. Infinite Study, 2020.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025

[20] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[21] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[22] Sovan Samanta and Madhumangal Pal. Bipolar fuzzy hypergraphs. *International Journal of Fuzzy Logic Systems*, 2(1):17–28, 2012.

[23] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[24] Muhammad Shoaib, Waqas Mahmood, Weded E Albalawi, and Faria Ahmad Shami. Notion of complex spherical fuzzy graph with application. *Journal of Function Spaces*, 2022.

[25] Prem Kumar Singh et al. Dark data analysis using intuitionistic plithogenic graphs. *International Journal of Neutrosophic Sciences*, 16(2):80–100, 2021.

[26] S Sivasankar and Said Broumi. Balanced neutrosophic graphs. *Neutrosophic Sets and Systems*, 50:309–319, 2022.

[27] Florentin Smarandache. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*. Infinite Study, 2017.

[28] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[29] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

Received November 20, 2024, Revised December 24, 2024 Accepted January 14,2025