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# Rethinking Strategic Perception: Foundations and Advancements in HyperGame Theory and SuperHyperGame Theory

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## ABSTRACT

Mathematical structures can generally be extended into Hyperstructures and SuperHyperstructures by leveraging powerset and $n$-th iterated powerset constructions (cf. [1, 2, 3]). These frameworks are particularly effective for representing hierarchical systems across various conceptual domains. Game Theory is a mathematical discipline for analyzing strategic interactions among rational agents with conflicting or cooperative objectives and finite choices [4, 5, 6]. HyperGame Theory extends this by modeling situations in which players possess misperceptions or differing beliefs about the game being played [7]. These ideas can be further generalized into hierarchical and dynamic HyperGames [8]. This paper explores the mathematical properties and illustrative examples of both HyperGame Theory and SuperHyperGame Theory. We hope that this investigation contributes to future developments in the theory and application of game-theoretic frameworks.

**Keywords:** Game Theory ▪ HyperGame Theory ▪ SuperHyperGame Theory ▪ Hyperstructure ▪ Superhyperstructure

## 1. PRELIMINARIES AND DEFINITIONS

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. All concepts considered in this paper are assumed to be finite.

### 1.1 Classical Structure, Hyperstructure, and $n$-Superhyperstructure

A *Classical Structure* represents a general mathematical concept, while a *Hyperstructure* can be defined using the power set, and an *$n$-Superhyperstructure* can be defined using the $n$-th powerset [9]. Intuitively, the n-th powerset is obtained by repeatedly applying the powerset operation. Related concepts to the SuperHyperStructure include the superhypergraph [10, 11, 12] and related SuperHyper set structures [9]. Relevant definitions and simple examples are provided below.

**Definition 1.1** (Base Set)**.** A *base set* $S$ is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathscr{P}(S)$ or $\mathscr{P}_n(S)$ originate from the elements of $S$.

**Definition 1.2** (Powerset)**.** [11] The *powerset* of a set $S$, denoted $\mathscr{P}(S)$, is the collection of all possible subsets of $S$, including both the empty set and $S$ itself. Formally, it is expressed as:

$$\mathscr{P}(S) = \{A \mid A \subseteq S\}.$$

**Definition 1.3** ($n$-th Powerset)**.** (cf.[9, 11, 12])

The $n$-th powerset of a set $H$, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the $n$-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of $H$ with the empty set removed.

**Definition 1.4** (Classical Structure)**.** (cf.[9]) A *Classical Structure* is a mathematical framework defined on a non-empty set $H$, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and $H^m$ denotes the $m$-fold Cartesian product of $H$. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

**Definition 1.5** (Hyperoperation)**.** (cf.[13, 14]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set $S$, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \to \mathscr{P}(S),$$

where $\mathscr{P}(S)$ is the powerset of $S$.

**Definition 1.6** (Hyperstructure)**.** (cf.[9, 11]) A *Hyperstructure* extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathscr{H} = (\mathscr{P}(S), \circ),$$

where $S$ is the base set, $\mathscr{P}(S)$ is the powerset of $S$, and $\circ$ is an operation defined on subsets of $\mathscr{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

**Definition 1.7** (SuperHyperOperations)**.** (cf.[9]) Let $H$ be a non-empty set, and let $\mathscr{P}(H)$ denote the powerset of $H$. The $n$-th powerset $\mathscr{P}^n(H)$ is defined recursively as follows:

$$\mathscr{P}^0(H) = H, \quad \mathscr{P}^{k+1}(H) = \mathscr{P}(\mathscr{P}^k(H)), \quad \text{for } k \geq 0.$$

A *SuperHyperOperation* of order $(m,n)$ is an $m$-ary operation:

$$\circ^{(m,n)} : H^m \to \mathscr{P}_*^n(H),$$

where $\mathscr{P}_*^n(H)$ represents the $n$-th powerset of $H$, either excluding or including the empty set, depending on the type of operation:

- If the codomain is
$$\mathscr{P}_*^n(H)$$
excluding the empty set, it is called a *classical-type* $(m,n)$*-SuperHyperOperation*.
- If the codomain is
$$\mathscr{P}^n(H)$$
including the empty set, it is called a *Neutrosophic* $(m,n)$*-SuperHyperOperation*.

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of $n$-th powersets.

**Example 1.8** (SuperHyperOperation for Project Phase Grouping)**.** A Project Phase is a distinct stage in a project lifecycle, with specific goals, tasks, deliverables, and timeframes for structured progress (cf.[15]). In practical project management, it is often useful to organize tasks into discrete phases. Consider a small project consisting of three tasks:

$$\begin{aligned} H = \big\{ & T_1 : \text{Requirement Analysis}, \\ & T_2 : \text{Design}, \\ & T_3 : \text{Implementation} \big\}. \end{aligned}$$

We define a $(3,2)$-SuperHyperOperation

$$\begin{aligned} &\circ^{(3,2)} : H^3 \longrightarrow \mathscr{P}_*^2(H), \\ &\quad \mathscr{P}_*^2(H) = \mathscr{P}\big(\mathscr{P}(H)\big) \setminus \{\emptyset\}, \end{aligned}$$

so that

$$\begin{aligned} &\circ^{(3,2)}(T_1, T_2, T_3) \\ &\quad = \big\{ X \subseteq \mathscr{P}(H) \mid X \neq \emptyset, \\ &\quad\quad \textstyle\bigcup X = H,\ X \text{ partitions } H \big\}. \end{aligned}$$

Concretely, $\circ^{(3,2)}(T_1, T_2, T_3)$ comprises exactly the five possible ways to group the three tasks into phases:

$$\begin{aligned} \circ^{(3,2)}(T_1, T_2, T_3) = \{ & \{\{T_1\}, \{T_2\}, \{T_3\}\}, \\ & \{\{T_1, T_2\}, \{T_3\}\}, \\ & \{\{T_1, T_3\}, \{T_2\}\}, \\ & \{\{T_2, T_3\}, \{T_1\}\}, \\ & \{\{T_1, T_2, T_3\}\} \}. \end{aligned}$$

- $\{\{T_1\}, \{T_2\}, \{T_3\}\}$: Three consecutive phases, each dedicated to one task.
- $\{\{T_1, T_2\}, \{T_3\}\}$: Two phases; first phase bundles Analysis and Design, second phase is Implementation.
- $\{\{T_1, T_3\}, \{T_2\}\}$: First phase combines Analysis and Implementation (e.g., rapid prototyping), second phase is Design review.
- $\{\{T_2, T_3\}, \{T_1\}\}$: First phase covers Design and Implementation (e.g., iterative coding), second phase finalizes Analysis.
- $\{\{T_1, T_2, T_3\}\}$: A single monolithic phase handling all tasks together.

This example shows how the $(3,2)$-SuperHyperOperation systematically enumerates every valid "phase grouping" of the task set $H$, illustrating the power of SuperHyperOperations to model multi-level organizational structures.

**Definition 1.9** ($n$-Superhyperstructure)**.** (cf.[9, 11]) An *$n$-Superhyperstructure* further generalizes a Hyperstructure by incorporating the $n$-th powerset of a base set. It is formally described as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where $S$ is the base set, $\mathcal{P}_n(S)$ is the $n$-th powerset of $S$, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

**Example 1.10** (Organizational Hierarchy as an $n$-Superhyperstructure)**.** Organizational Hierarchy is a structured system defining roles, authority, and decision-making levels within an organization for efficient coordination (cf.[16, 17]). Let $S$ be the set of all employees in a company, for instance

$$S = \{\text{Alice, Bob, Carol}\}.$$

We form iterated powersets:

$$\mathcal{P}^0(S) = S, \quad \mathcal{P}^1(S) = \mathcal{P}(S) \quad \big(\text{all possible } \textit{teams}\big),$$

$$\mathcal{P}^2(S) = \mathcal{P}\big(\mathcal{P}^1(S)\big) \quad \big(\text{all possible } \textit{departments}\big),$$

and in general

$$\mathcal{P}^n(S) = \mathcal{P}\big(\mathcal{P}^{n-1}(S)\big).$$

This is the set of all possible "$n$-th level" groupings. Define the Superhyperstructure of order $n$ by

$$\mathcal{SH}_n = \big(\mathcal{P}^n(S), \cup\big),$$

where the operation $\cup$ merges two $n$-level groupings into a larger one.

**Concrete instance for $n = 2$:**

$$\mathcal{P}^1(S) = \{\emptyset, \{\text{Alice}\}, \{\text{Bob}\}, \{\text{Carol}\}, \{\text{Alice, Bob}\}, \dots\},$$

each element represents a team. For example,

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Bob, Carol}\}.$$

Then

$$\mathcal{P}^2(S) = \mathcal{P}\big(\mathcal{P}^1(S)\big) \quad (\text{departments}).$$

Consider two departments:

$$D_1 = \{T_1, \{\text{Carol}\}\}, \quad D_2 = \{T_2, \{\text{Alice}\}\}.$$

Their union under the SuperhyperOperation is

$$D_1 \cup D_2 = \{T_1, \{\text{Carol}\}, T_2, \{\text{Alice}\}\},$$

which is again an element of $\mathcal{P}^2(S)$.

**General interpretation:** Each level $k$ grouping ($\mathcal{P}^k(S)$) corresponds to aggregations such as teams ($k = 1$), departments ($k = 2$), divisions ($k = 3$), and so on. The union operation $\cup$ naturally merges any two such groupings at the $n$-th level, yielding a valid new grouping in $\mathcal{P}^n(S)$. Thus $\mathcal{SH}_n = (\mathcal{P}^n(S), \cup)$ models the hierarchical, multi-level structure of an organization.

## 2. HYPERGAME

HyperGame is a collection of each player's subjective strategic games, capturing asymmetric beliefs about others' strategies and payoffs under adversarial uncertainty contexts [7, 18, 19, 20].

**Definition 2.1** (HyperGame)**.** A *HyperGame* is a collection

$$H = \{G_1, G_2, \dots, G_n\},$$

where each player's perceived game is

$$G_i = (N, S_i, U_i),$$

with

- $N = \{1, 2, \dots, n\}$ the set of players,
- $S_i = \{S_{i1}, S_{i2}, \dots, S_{in}\}$ the strategy spaces as perceived by player $i$,
- $U_i = \{U_{i1}, U_{i2}, \dots, U_{in}\}$ the utility functions as perceived by player $i$.

If for all $i \in N$ one has $S_i = S$ and $U_i = U$, then $H = \{G\}$ reduces to the classical game $G = (N, S, U)$.

**Example 2.2** (Battle of the Generals as a HyperGame)**.** Consider two commanders, $A$ and $B$, who must choose independently whether to attack from the North or the South. However, $A$ mistakenly believes that $B$ has only the Northern option, while $B$ knows that both options are available to both of them.

**Players:** $N = \{A, B\}$.

**Commander $A$'s perceived game $G_A = (N, S_A, U_A)$:**

$$S_A = \{\text{North, South}\}, \quad S_B = \{\text{North}\},$$

and payoff matrix $U_A$ (entries $(u_A(A : x, B : y))$ with $x \in S_A$, $y \in S_B$):

| | $B$ : North |
|---|---|
| $A$ : North | (3) |
| $A$ : South | (0) |

Here $A$ receives payoff 3 if both attack North (successful coordination) and 0 otherwise.

**Commander $B$'s perceived game $G_B = (N, S'_A, S'_B, U_B)$:**

$$S'_A = \{\text{North, South}\}, \quad S'_B = \{\text{North, South}\},$$

and payoff matrix $U_B$ (entries $(u_B(A : x, B : y))$):

| | $B$ : North | $B$ : South |
|---|---|---|
| $A$ : North | 3, 3 | 0, 0 |
| $A$ : South | 0, 0 | 2, 2 |

Here both commanders get their highest payoff (3) by coordinating on North, and a smaller positive payoff (2) if they both coordinate on South; all miscoordination outcomes yield 0.

**Resulting HyperGame:**

$$H = \{G_A, G_B\},$$

which captures the asymmetry of beliefs: $A$ plays as if $B$ can only attack North, while $B$ correctly anticipates both sides' choices.

**Example 2.3** (Rock–Paper–Scissors as a HyperGame)**.** (cf.[21, 22, 23]) Two players, $A$ and $B$, play Rock–Paper–Scissors, but $A$ mistakenly believes that $B$ cannot choose Scissors, while $B$ correctly perceives all three options.

**Players:** $N = \{A, B\}$.

**Player $A$'s perceived game $G_A = (N, S_A, U_A)$:**

$$S_A = \{\mathrm{R}, \mathrm{P}, \mathrm{S}\}, \quad S_B = \{\mathrm{R}, \mathrm{P}\},$$

$$U_A : \begin{array}{c|cc} & B:\mathrm{R} & B:\mathrm{P} \\ \hline A:\mathrm{R} & 0 & -1 \\ A:\mathrm{P} & 1 & 0 \\ A:\mathrm{S} & -1 & 1 \end{array}$$

Here $A$ assigns payoff $+1$ for a win, $-1$ for a loss, and 0 for a tie, but only against $B$'s perceived moves $\{\mathrm{R}, \mathrm{P}\}$.

**Player $B$'s perceived game $G_B = (N, S'_A, S'_B, U_B)$:**

$$S'_A = \{\mathrm{R}, \mathrm{P}, \mathrm{S}\}, \quad S'_B = \{\mathrm{R}, \mathrm{P}, \mathrm{S}\},$$

$$U_B : \begin{array}{c|ccc} & B:\mathrm{R} & B:\mathrm{P} & B:\mathrm{S} \\ \hline A:\mathrm{R} & 0 & -1 & 1 \\ A:\mathrm{P} & 1 & 0 & -1 \\ A:\mathrm{S} & -1 & 1 & 0 \end{array}$$

Here $B$ correctly assigns standard Rock–Paper–Scissors payoffs against all three of $A$'s possible moves.

**Resulting HyperGame:**

$$H = \{G_A, G_B\},$$

which formalizes the asymmetry in beliefs: $A$ plays as if $B$ cannot choose Scissors, while $B$ knows the full strategy set.

## 3. SUPERHYPERGAME

SuperHyperGame is a recursive, iterative hierarchy of games about games, modeling multiple levels of players' beliefs regarding others' perceived strategies and payoffs [8].

**Definition 3.1** ($n$-SuperHyperGame)**.** [8] Let $N = \{1, 2, \ldots, n\}$ be a finite set of players, and let

$$\Omega(N) = \Big\{G = (N, S, U) \mid S = \{S_1, \ldots, S_n\}, \\ U = \{U_1, \ldots, U_n\}\Big\}$$

denote the collection of all classical strategic games on $N$. Define the iterated powersets of $\Omega(N)$ by

$$P^0\big(\Omega(N)\big) = \Omega(N),$$
$$P^k\big(\Omega(N)\big) = \mathscr{P}\big(P^{k-1}(\Omega(N))\big), \quad k \geq 1.$$

Then an *n-SuperHyperGame* is any element

$$H \in P^n\big(\Omega(N)\big),$$

that is, a set of $(n-1)$-SuperHyperGames. Equivalently, one may write

$$H = \{H^1, H^2, \ldots, H^n\}, \quad H^k \subseteq P^{k-1}\big(\Omega(N)\big),$$

where each $H^k$ is the $k$-th level HyperGame and its members are perceived games

$$G_i^k = \big(N,\, S_i^k,\, U_i^k\big), \quad i \in N,$$

with $S_i^k$ the strategy set and $U_i^k$ the utility function at level $k$ as perceived by player $i$.

In particular:

$$P^1\big(\Omega(N)\big) = \{\text{HyperGames on } N\},$$

and

$$P^2\big(\Omega(N)\big) = \{\text{sets of HyperGames}\},$$

and so on, up to the $n$-th level.

**Example 3.2** (Pricing Competition as a 2-SuperHyperGame)**.** Pricing Competition is a strategic interaction where firms set prices to attract customers, maximize profits, and outperform market rivals (cf.[24, 25, 26]). Consider two firms, $A$ and $B$, competing on price in a market. Each firm chooses either *High* or *Low* price, but they have asymmetric beliefs about each other at two recursive levels.

**Players:** $N = \{A, B\}$.

**Level-1 HyperGame $H^1 = \{G_A^1, G_B^1\}$:**

$$G_A^1 = \big(N,\, S_A^1,\, U_A^1\big), \quad S_A^1 = \{\text{High}, \text{Low}\},\ S_B^1 = \{\text{High}\},$$

$$U_A^1 : S_A^1 \times S_B^1 \to \mathbb{R},$$
$$U_A^1(\text{High}, \text{High}) = 3,$$
$$U_A^1(\text{Low}, \text{High}) = 1.$$

Here $A$ (mistakenly) believes $B$ can only choose *High* and that coordinating on *High* yields payoff 3, while choosing *Low* against *High* yields 1.

$$G_B^1 = \big(N,\, S_A^1,\, S_B^1,\, U_B^1\big), \quad S_A^1 = \{\text{Low}\},\ S_B^1 = \{\text{High}, \text{Low}\},$$

$$U_B^1 : S_A^1 \times S_B^1 \to \mathbb{R},$$
$$U_B^1(\text{Low}, \text{High}) = 1,$$
$$U_B^1(\text{Low}, \text{Low}) = 3.$$

Here $B$ (correctly) believes $A$ can only choose *Low* and that coordinating on *Low* yields payoff 3, while choosing *High* against *Low* yields 1.

**Level-2 HyperGame $H^2 = \{G_A^2, G_B^2\}$:**

$$G_A^2 = \big(N,\, S_A^2,\, U_A^2\big), \quad S_A^2 = S_B^1,\ S_B^2 = S_A^1, \quad U_A^2 = U_B^1;$$

that is, $A$'s model of $B$'s view is given by swapping the first-level strategy sets and adopting $B$'s first-level payoff.

$$G_B^2 = \big(N,\, S_A^2,\, S_B^2,\, U_B^2\big), \quad U_B^2 = U_A^1;$$

that is, $B$'s model of $A$'s view swaps the first-level strategy sets and adopts $A$'s first-level payoff.

**2-SuperHyperGame:**

$$H = \{H^1, H^2\},$$

capturing each firm's beliefs about the other's perceptions and payoffs at two levels of recursion.

**Example 3.3** (Intersection Crossing as an $n$-SuperHyperGame)**.** Intersection Crossing in game theory models autonomous agents deciding when to proceed or yield at intersections under uncertainty and strategic interaction (cf.[27, 28, 29]). Two autonomous vehicles, $A$ and $B$, approach a single-lane bridge from opposite ends and must decide whether to *Go* or *Wait*. They hold recursive beliefs about each other at $n$ levels.

**Players:** $N = \{A, B\}$.

For each $k = 1, 2, \ldots, n$, define the level-$k$ HyperGame

$$H^k = \{G_A^k, G_B^k\},$$

where:

**Strategy sets:**

$$S_A^k = \begin{cases} \{\text{Go}, \text{Wait}\}, & k \text{ odd}, \\ \{\text{Wait}\}, & k \text{ even}, \end{cases}$$

$$S_B^k = \begin{cases} \{\text{Go}\}, & k \text{ odd}, \\ \{\text{Go}, \text{Wait}\}, & k \text{ even}. \end{cases}$$

**Payoff functions:** Both players' payoffs at level $k$ are given by

$$U_A^k : S_A^k \times S_B^k \to \mathbb{R}, \quad U_B^k : S_A^k \times S_B^k \to \mathbb{R},$$

with the following conventions (undefined combinations are omitted as the corresponding strategy is not in the perceived set):

$$U_A^k(x,y) = \begin{cases} 2, & (x,y) = (\text{Go, Go}), \\ 0, & (x,y) = (\text{Wait, Go}), \\ 1, & (x,y) = (\text{Wait, Wait}), \end{cases}$$

$$U_B^k(x,y) = \begin{cases} 2, & (x,y) = (\text{Go, Wait}), \\ 0, & (x,y) = (\text{Go, Go}), \\ 1, & (x,y) = (\text{Wait, Wait}). \end{cases}$$

**$n$-SuperHyperGame:**

$$H = \{H^1, H^2, \ldots, H^n\},$$

captures each vehicle's beliefs about the other's choices and payoffs at all $n$ recursive levels.

**Example 3.4** (Pricing Competition as a 3-SuperHyperGame)**.** Pricing Competition is a strategic interaction where firms set prices to attract customers, maximize profits, and outperform market rivals (cf.[24, 25, 26]). Consider two competing retailers, $A$ and $B$, who independently set product prices either *High* or *Low*. They hold recursive beliefs about each other's available strategies and payoffs up to three levels.

**Players:** $N = \{A, B\}$.

**Level-1 HyperGame $H^1 = \{G_A^1, G_B^1\}$:**

*Firm A's perceived game*

$$G_A^1 = \big(N,\, S_A^1,\, S_B^1,\, U_A^1\big),$$
$$S_A^1 = \{\text{High}, \text{Low}\}, \quad S_B^1 = \{\text{High}\},$$

$U_A^1$ :

| | $B$ : High |
|---|---|
| $A$ : High | 4 |
| $A$ : Low | 1 |

Here $A$ (incorrectly) believes $B$ can only charge a high price; matching on *High* yields profit 4, undercut (*Low* vs. *High*) yields profit 1.

*Firm B's perceived game*

$$G_B^1 = \big(N,\, S_A'^1,\, S_B'^1,\, U_B^1\big),$$
$$S_A'^1 = \{\text{Low}\}, \quad S_B'^1 = \{\text{High}, \text{Low}\},$$

$U_B^1$ :

| | $B$ : High | $B$ : Low |
|---|---|---|
| $A$ : Low | 3 | 0 |

Here $B$ (incorrectly) believes $A$ can only charge low; matching on *Low* yields profit 3, mismatch yields 0.

**Level-2 HyperGame $H^2 = \{G_A^2, G_B^2\}$:**
Each firm models the other's first-level beliefs by swapping roles:

$$G_A^2 = \big(N,\, S_A^2,\, S_B^2,\, U_A^2\big), \quad S_A^2 = S_B'^1,\, S_B^2 = S_A'^1,\, U_A^2 = U_B^1,$$

$$G_B^2 = \big(N,\, S_A^2,\, S_B^2,\, U_B^2\big), \quad S_A^2 = S_B^1,\, S_B^2 = S_A^1,\, U_B^2 = U_A^1.$$

**Level-3 HyperGame $H^3 = \{G_A^3, G_B^3\}$:**
Each then models the other's second-level view similarly:

$$G_A^3 = \big(N,\, S_A^3,\, S_B^3,\, U_A^3\big), \quad S_A^3 = S_B^2,\, S_B^3 = S_A^2,\, U_A^3 = U_B^2 = U_A^1,$$

$$G_B^3 = \big(N,\, S_A^3,\, S_B^3,\, U_B^3\big), \quad U_B^3 = U_A^2 = U_B^1.$$

**3-SuperHyperGame:**

$$H = \{H^1, H^2, H^3\},$$

which captures each retailer's beliefs and meta-beliefs about strategy availability and profits at three recursive levels.

**Example 3.5** (SuperHyper Cops and Robbers as an $n$-SuperHyperGame)**.** Cops and Robbers is a pursuit–evasion game played on a graph where cops try to capture a moving robber over time (cf.[30, 31]). Consider the classic pursuit–evasion game on a simple graph with two vertices, but with recursive beliefs at $n$ levels.

**Players:** Cop ($C$) and Robber ($R$), so $N = \{C,R\}$.

**Environment:** Each player chooses a vertex in the set $V = \{v_1, v_2\}$.

For each $k = 1,2,\ldots,n$, define the level-$k$ HyperGame

$$H^k = \{G^k_C, G^k_R\}, \quad G^k_i = \bigl(N,\, S^k_C,\, S^k_R,\, U^k_i\bigr),$$

where the perceived strategy sets alternate by level:

$$S^k_C = \begin{cases} \{v_1,v_2\}, & k \text{ odd}, \\ \{v_1\}, & k \text{ even}, \end{cases} \qquad S^k_R = \begin{cases} \{v_1\}, & k \text{ odd}, \\ \{v_1,v_2\}, & k \text{ even}. \end{cases}$$

That is, at odd levels the Cop believes the Robber can only occupy $v_1$, while the Robber believes the Cop has full mobility; at even levels these beliefs are reversed.

**Payoff functions:** For any $c \in S^k_C$, $r \in S^k_R$,

$$U^k_C(c,r) = \begin{cases} 1, & c = r \quad \text{(capture)}, \\ 0, & c \neq r \quad \text{(escape)}, \end{cases}$$

$$U^k_R(c,r) = \begin{cases} 0, & c = r, \\ 1, & c \neq r. \end{cases}$$

Thus the Cop gains 1 if he lands on the same vertex as the Robber, otherwise 0; the Robber's payoffs are the complement.

**$n$-SuperHyperGame:**

$$H = \{H^1, H^2, \ldots, H^n\},$$

which captures each player's $k$-th level beliefs about the opponent's available moves and payoffs.

**Theorem 3.6** (Reduction to HyperGame)**.** *Let $N = \{1,2,\ldots,n\}$. If $n = 1$, then*

$$P^1\bigl(\Omega(N)\bigr) = \mathscr{P}\bigl(\Omega(N)\bigr)$$

*is exactly the set of all HyperGames on $N$. Hence every 1-SuperHyperGame is equivalent to a single HyperGame.*

*Proof.* By Definition 3.1,

$$P^1\bigl(\Omega(N)\bigr) = \mathscr{P}\bigl(\Omega(N)\bigr),$$

and an element of $\mathscr{P}(\Omega(N))$ is by definition a set of classical games on $N$, i.e. a HyperGame. Thus 1-SuperHyperGames coincide with HyperGames. □

**Theorem 3.7** (Cardinality of $n$-SuperHyperGames)**.** *For each integer $n \geq 1$,*

$$\bigl|P^n(\Omega(N))\bigr| = 2^{\left|P^{n-1}(\Omega(N))\right|}.$$

*In particular, if $\bigl|\Omega(N)\bigr| = M$, then*

$$\bigl|P^n(\Omega(N))\bigr| = 2^{2^{\cdot^{\cdot^{\cdot^{2^M}}}}} \qquad \textit{(tower of height n).}$$

*Proof.* By the definition of the powerset, for any set $X$ we have $\bigl|\mathscr{P}(X)\bigr| = 2^{|X|}$. Since

$$P^n(\Omega(N)) = \mathscr{P}\bigl(P^{n-1}(\Omega(N))\bigr),$$

it follows immediately that

$$\bigl|P^n(\Omega(N))\bigr| = \bigl|\mathscr{P}(P^{n-1}(\Omega(N)))\bigr| = 2^{\left|P^{n-1}(\Omega(N))\right|}.$$

Iterating this formula gives the stated tower of exponentials. □

**Theorem 3.8** (Canonical Embedding)**.** *For each $n \geq 1$, the map*

$$\iota_n : P^{n-1}(\Omega(N)) \longrightarrow P^n(\Omega(N)), \quad \iota_n(G) = \{G\},$$

*is injective. Hence every $(n-1)$-SuperHyperGame embeds as a singleton $n$-SuperHyperGame.*

*Proof.* Let $G, H \in P^{n-1}(\Omega(N))$ and suppose $\iota_n(G) = \iota_n(H)$. Then

$$\{G\} = \{H\}$$

implies $G = H$. Therefore $\iota_n$ is injective, exhibiting $P^{n-1}(\Omega(N))$ as a subset of $P^n(\Omega(N))$. □

**Theorem 3.9** (Powerset Functoriality)**.** *For all nonnegative integers $m,k$,*

$$P^m\bigl(P^k(\Omega(N))\bigr) = P^{m+k}\bigl(\Omega(N)\bigr).$$

*Proof.* We proceed by induction on $m$.

*Base case ($m = 0$):* By definition $P^0(X) = X$ for any set $X$. Hence

$$P^0\bigl(P^k(\Omega(N))\bigr) = P^k(\Omega(N)) = P^{0+k}(\Omega(N)).$$

*Inductive step:* Assume

$$P^m\bigl(P^k(\Omega(N))\bigr) = P^{m+k}(\Omega(N)).$$

Then, using the definition $P^{m+1}(X) = \mathscr{P}\bigl(P^m(X)\bigr)$, we have

$$\begin{aligned} &P^{m+1}\bigl(P^k(\Omega(N))\bigr) \\ &\quad = \mathscr{P}\Bigl(P^m\bigl(P^k(\Omega(N))\bigr)\Bigr) \\ &\quad = \mathscr{P}\bigl(P^{m+k}(\Omega(N))\bigr) \\ &\quad = P^{(m+k)+1}(\Omega(N)) \\ &\quad = P^{(m+1)+k}(\Omega(N)), \end{aligned}$$

completing the induction. □

**Theorem 3.10** (Projection and Retraction)**.** *Define*

$$p_n : P^n(\Omega(N)) \longrightarrow P^{n-1}(\Omega(N)), \qquad p_n(H) = \bigcup_{G \in H} G.$$

*Then:*

1. *$p_n$ is surjective.*
2. *If $\iota_n\colon P^{n-1}(\Omega(N)) \to P^n(\Omega(N))$ is given by $\iota_n(G) = \{G\}$, then*
$$p_n \circ \iota_n = \mathrm{id}_{P^{n-1}(\Omega(N))}.$$

*Proof.* First, for any $H \in P^n(\Omega(N))$, each $G \in H$ lies in $P^{n-1}(\Omega(N))$, so their union $p_n(H)$ is indeed an element of $P^{n-1}(\Omega(N))$.

To see surjectivity, let $G_0 \in P^{n-1}(\Omega(N))$. Then $p_n(\{G_0\}) = \bigcup\{G_0\} = G_0$, so every element of $P^{n-1}(\Omega(N))$ has a preimage under $p_n$.

Finally, for any $G \in P^{n-1}(\Omega(N))$,
$$(p_n \circ \iota_n)(G) = p_n(\{G\}) = \bigcup\{G\} = G,$$
establishing that $p_n \circ \iota_n$ is the identity on $P^{n-1}(\Omega(N))$. $\square$

**Theorem 3.11** (Strict Monotonicity)**.** *If $|\Omega(N)| \geq 1$, then for every $k \geq 0$,*
$$P^k(\Omega(N)) \subsetneq P^{k+1}(\Omega(N)).$$

*Proof.* We show two facts: $P^k(\Omega(N)) \subseteq P^{k+1}(\Omega(N))$ and the inclusion is proper.

*(i) Inclusion:* By definition $P^{k+1}(\Omega(N)) = \mathscr{P}(P^k(\Omega(N)))$, so every element of $P^k(\Omega(N))$ is a subset of $P^k(\Omega(N))$ (via the singleton embedding $\iota_{k+1}$), hence lies in its powerset.

*(ii) Properness:* Since $|\Omega(N)| \geq 1$, the set $P^k(\Omega(N))$ is nonempty, and its powerset $\mathscr{P}(P^k(\Omega(N)))$ always contains the empty set, which is not a singleton of an element of $P^k(\Omega(N))$. Concretely,
$$\emptyset \in P^{k+1}(\Omega(N)), \quad \emptyset \notin P^k(\Omega(N)),$$
ensuring $P^k(\Omega(N)) \subsetneq P^{k+1}(\Omega(N))$. $\square$

**Theorem 3.12** (Boolean Algebra Structure)**.** *For each $n \geq 1$, the set*
$$P^n\big(\Omega(N)\big)$$
*equipped with the operations of union ($\cup$), intersection ($\cap$), and complement*
$$H^c = P^{n-1}\big(\Omega(N)\big) \setminus H, \quad H \subseteq P^{n-1}(\Omega(N)),$$
*together with the least element $\varnothing$ and greatest element $P^{n-1}(\Omega(N))$, forms a complete Boolean algebra.*

*Proof.* It is a standard fact that for any set $X$, its powerset $\mathscr{P}(X)$ is a Boolean algebra under $\cup$, $\cap$, and complement relative to $X$. Since
$$P^n\big(\Omega(N)\big) = \mathscr{P}\big(P^{n-1}(\Omega(N))\big),$$
all Boolean-algebra axioms (commutativity, associativity, distributivity, identity, complements, and De Morgan's laws) hold, and arbitrary joins (unions) and meets (intersections) exist, making it complete. $\square$

**Theorem 3.13** (Composition of Projections)**.** *For any integers $0 \leq \ell < m < n$, define the projection*
$$p_{n,m} = p_{m+1} \circ p_{m+2} \circ \cdots \circ p_n : P^n\big(\Omega(N)\big) \longrightarrow P^m\big(\Omega(N)\big).$$
*Then for all $0 \leq \ell < m < n$,*
$$p_{m,\ell} \circ p_{n,m} = p_{n,\ell},$$
*and in particular $p_{n,n-1} = p_n$.*

*Proof.* By definition $p_{k+1} : P^{k+1}(\Omega(N)) \to P^k(\Omega(N))$ is the union map. Composition of these unions is associative, so
$$p_{m,\ell} \circ p_{n,m} = \big(p_{\ell+1} \circ \cdots \circ p_m\big) \circ \big(p_{m+1} \circ \cdots \circ p_n\big) = p_{n,\ell}.$$
The case $\ell = m-1$ recovers $p_{n,n-1} = p_n$. $\square$

**Theorem 3.14** (Galois Connection between Embedding and Projection)**.** *For each $n \geq 1$, the singleton-embedding*
$$\iota_n : P^{n-1}(\Omega(N)) \longrightarrow P^n(\Omega(N)), \quad \iota_n(G) = \{G\},$$
*and the projection*
$$p_n : P^n(\Omega(N)) \longrightarrow P^{n-1}(\Omega(N)), \quad p_n(H) = \bigcup H,$$
*form a Galois connection:*
$$\iota_n(G) \subseteq H \iff G \subseteq p_n(H),$$
*for all $G \in P^{n-1}(\Omega(N))$ and $H \in P^n(\Omega(N))$.*

*Proof.* $(\Rightarrow)$ If $\{G\} = \iota_n(G) \subseteq H$, then $G \in H$, so
$$G \subseteq \bigcup H = p_n(H).$$
$(\Leftarrow)$ Conversely, if $G \subseteq p_n(H)$, then every element of $G$ lies in $\bigcup H$, so $G \in H$ implies $\{G\} \subseteq H$. Thus $\iota_n(G) \subseteq H$. $\square$

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