HyperRough Cubic Set and SuperhyperRough Cubic Set

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Abstract

Rough sets provide a mathematical framework for approximating subsets using lower and upper bounds determined by equivalence relations, effectively modeling uncertainty in classification and data analysis. These foundational concepts have been further extended to structures such as Hyperrough Sets and Superhyperrough Sets. In this paper, we introduce the definitions of Hyperrough Cubic Sets and Superhyperrough Cubic Sets, and explore their fundamental properties. We hope that these developments will promote further research into applications such as decision-making based on Rough Set Theory and its extensions.

Keywords: Rough set; Hyperrough Set; Rough Cubic Set; SuperHyperRough Set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this study, all sets are assumed to be finite.

1.1 Rough Set, HyperRough Set, and Superhyperrough Set

A rough set approximates a subset using lower and upper bounds determined by equivalence classes, thereby capturing both certainty and uncertainty in membership.[1,3,8,11,13] The following definitions formalize these concepts.

Definition 1.1 (Rough Set Approximation)**.** [12] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

1. *Lower Approximation*:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

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2. *Upper Approximation*:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying
$$\underline{U} \subseteq U \subseteq \overline{U}.$$

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below.[5,6,8,9]

Definition 1.2 (HyperRough Set)**.** Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product
$$J = J_1 \times J_2 \times \cdots \times J_n.$$
Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as
$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 1.3 (HyperRough Set)**.** Let
$$X = \{x_1, x_2, x_3, x_4\}$$
be a finite universe and define an equivalence relation $R \subseteq X \times X$ by
$$[x_1]_R = \{x_1, x_2\} \quad \text{and} \quad [x_3]_R = \{x_3, x_4\}.$$
Let T_1 and T_2 be two attributes with domains
$$J_1 = \{\text{red}, \text{blue}\} \quad \text{and} \quad J_2 = \{\text{circle}, \text{square}\},$$
respectively. Then the Cartesian product is
$$J = J_1 \times J_2 = \{(\text{red}, \text{circle}), (\text{red}, \text{square}), (\text{blue}, \text{circle}), (\text{blue}, \text{square})\}.$$
Define a mapping
$$F : J \to \mathcal{P}(X)$$
by setting:
$$\begin{aligned} F(\text{red}, \text{circle}) &= \{x_1, x_2\}, \\ F(\text{red}, \text{square}) &= \{x_1\}, \\ F(\text{blue}, \text{circle}) &= \{x_3, x_4\}, \\ F(\text{blue}, \text{square}) &= \varnothing. \end{aligned}$$

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For each $a \in J$, the rough approximations of $F(a)$ are given by

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \varnothing\}.$$

For instance, for $a = (\text{red}, \text{circle})$ we have $F(\text{red}, \text{circle}) = \{x_1, x_2\}$. Since for each $x \in \{x_1, x_2\}$ the equivalence class $[x]_R$ (which equals $\{x_1, x_2\}$) is entirely contained in $F(\text{red}, \text{circle})$, it follows that

$$\underline{F(\text{red}, \text{circle})} = \overline{F(\text{red}, \text{circle})} = \{x_1, x_2\}.$$

Thus, (F, J) constitutes a HyperRough Set over X.

Definition 1.4 (n-SuperHyperRough Set)**.** Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X. An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.
- For each $A \in J$, the lower and upper approximations are defined as

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$,

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 1.5 (n-SuperHyperRough Set)**.** Let

$$X = \{x_1, x_2, x_3\}$$

and consider the equivalence relation $R \subseteq X \times X$ defined by

$$[x_1]_R = \{x_1, x_2\} \quad \text{and} \quad [x_3]_R = \{x_3\}.$$

Let T_1 be an attribute with domain

$$J_1 = \{\text{red}, \text{blue}\}.$$

Then the power set of J_1 is

$$\mathcal{P}(J_1) = \{\varnothing, \{\text{red}\}, \{\text{blue}\}, \{\text{red}, \text{blue}\}\}.$$

Define a mapping

$$F : \mathcal{P}(J_1) \to \mathcal{P}(X)$$

by:

$$\begin{aligned} F(\varnothing) &= \varnothing, \\ F(\{\text{red}\}) &= \{x_1\}, \\ F(\{\text{blue}\}) &= \{x_2, x_3\}, \\ F(\{\text{red}, \text{blue}\}) &= X. \end{aligned}$$

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For each $A \in \mathcal{P}(J_1)$, the rough approximations are given by

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \varnothing\}.$$

Consider $A = \{\text{red}\}$ so that $F(\{\text{red}\}) = \{x_1\}$. Since $[x_1]_R = \{x_1, x_2\}$ is not a subset of $\{x_1\}$, we have

$$\underline{F(\{\text{red}\})} = \varnothing.$$

However, as $[x_1]_R \cap \{x_1\} \neq \varnothing$ and $[x_2]_R = \{x_1, x_2\}$ also intersects $\{x_1\}$, it follows that

$$\overline{F(\{\text{red}\})} = \{x_1, x_2\}.$$

Thus, the pair $(F, \mathcal{P}(J_1))$ defines an n-SuperHyperRough Set (with $n = 1$) over X.

1.2 Rough Cubic Sets

The concept of cubic sets has been extensively studied in various contexts such as fuzzy and neutrosophic frameworks.[4,7,10,14] Similarly, research has also been conducted on the application of cubic sets within rough set theory.[2] We present the formal definition of Rough Cubic Sets below.

Definition 1.6 (Rough Cubic Sets)**.** [2] Let X be a nonempty set and let

$$R = \langle R, r \rangle : X \times X \to ([0,1] \times [0,1])$$

be a *cubic relation* on X. A *cubic set* on X is given by a mapping

$$A = \langle A, \lambda \rangle : X \to ([0,1] \times [0,1]),$$

where for each $x \in X$, $A(x)$ (or equivalently, $\langle A(x), \lambda(x) \rangle$) represents the membership degree of x in an interval-valued and crisp manner.

Using the cubic relation R, we define two *rough operators* on cubic sets as follows. For every cubic set $A = \langle A, \lambda \rangle$ on X, define the *lower rough approximation* $N(A)$ and the *upper rough approximation* $H(A)$ by

$$N(A)(x) = \bigcup_{y \in X} \Big(A(y)\, t\, R^c(y, x) \Big), \quad \forall x \in X,$$

$$H(A)(x) = \bigcap_{y \in X} \Big(A(y)\, u\, R(x, y) \Big), \quad \forall x \in X.$$

Here, t denotes a chosen t-norm (e.g., the minimum operation), u denotes the corresponding t-conorm (e.g., the maximum operation), and R^c represents the complement of the cubic relation R.

If $N(A) = H(A)$ (that is, if the lower and upper approximations coincide), then A is called a *definable cubic set*. Otherwise, A is termed *undefinable*, and the pair $(N(A), H(A))$ is referred to as a *rough cubic set*.

2 Results of This Paper

This section presents the results obtained in this paper.

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2.1 Hyperrough Cubic Sets

We provide the definition of Hyperrough Cubic Sets as follows.

Definition 2.1 (Hyperrough Cubic Set)**.** Let X be a nonempty set and let

$$R = \langle R, r\rangle : X \times X \to \big([0,1] \times [0,1]\big)$$

be a *cubic relation* on X. Denote by $\mathcal{CS}(X)$ the collection of all cubic sets on X (each cubic set A being given by a mapping $A = \langle A, \lambda\rangle : X \to ([0,1] \times [0,1])$). Let J be a nonempty set (interpreted as an index set of attribute value combinations). A *Hyperrough Cubic Set* over X is defined as a mapping

$$F : J \to \mathcal{CS}(X).$$

For each $a \in J$, denote $A_a = F(a)$ and define the lower and upper rough approximations of A_a with respect to the cubic relation R by

$$N(A_a)(x) = \bigcup_{y\in X} \Big(A_a(y)\, t\, R^c(y,x)\Big), \quad \forall x \in X,$$

$$H(A_a)(x) = \bigcap_{y\in X} \Big(A_a(y)\, u\, R(x,y)\Big), \quad \forall x \in X,$$

where t is a chosen t-norm (for instance, the minimum operation), u is the corresponding t-conorm (for instance, the maximum operation), and R^c denotes the complement of the cubic relation R.

The pair $\big(N(A_a), H(A_a)\big)$ constitutes the rough approximation of the cubic set A_a. Thus, the structure (F, J) is called a *Hyperrough Cubic Set* on X.

Example 2.2 (Hyperrough Cubic Set Example)**.** Let $X = \{x_1, x_2, x_3\}$ and suppose a cubic relation $R = \langle R, r\rangle$ on X is given (for simplicity, assume that for each pair (x_i, x_j), $R(x_i, x_j)$ provides an interval and a crisp value). Let $J = \{a\}$ be a singleton index set. Define a cubic set A on X by

$$A(x_1) = \big([0.3, 0.7], 0.5\big), \quad A(x_2) = \big([0.4, 0.8], 0.6\big), \quad A(x_3) = \big([0.2, 0.6], 0.4\big).$$

Then define the hyperrough cubic set $F : J \to \mathcal{CS}(X)$ by setting $F(a) = A$. The rough approximations $N(A)$ and $H(A)$ are computed from A using the operators

$$N(A)(x) = \bigcup_{y\in X} \Big(A(y)\, t\, R^c(y,x)\Big), \quad H(A)(x) = \bigcap_{y\in X} \Big(A(y)\, u\, R(x,y)\Big).$$

Thus, $(F, \{a\})$ constitutes a Hyperrough Cubic Set that, by definition, generalizes the rough cubic set $(N(A), H(A))$ and, if the cubic structure were degenerate, would coincide with a hyperrough set.

Theorem 2.3. *A Hyperrough Cubic Set generalizes both Rough Cubic Sets and Hyperrough Sets.*

Proof. If the index set J is a singleton (say, $J = \{a_0\}$), then the mapping F reduces to $F(a_0) = A_{a_0}$, and the Hyperrough Cubic Set (F, J) becomes the rough cubic set $(N(A_{a_0}), H(A_{a_0}))$ induced by the cubic relation R.

Conversely, if the cubic structure degenerates (i.e., if the secondary component of each cubic set is constant so that every cubic set reduces to an interval-valued fuzzy set), then the rough approximations $N(A_a)$ and $H(A_a)$ coincide with those defined for hyperrough sets. Hence, a Hyperrough Cubic Set encompasses both rough cubic sets (when J is trivial) and hyperrough sets (when the cubic structure is degenerate). □

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2.2 n-Superhyperrough Cubic Sets

We provide the definition of n-Superhyperrough Cubic Sets as follows.

Definition 2.4 (n-Superhyperrough Cubic Set)**.** Let X be a nonempty set and let

$$R = \langle R, r \rangle : X \times X \to \big([0,1] \times [0,1]\big)$$

be a cubic relation on X. Suppose that $T_1, T_2, \ldots, T_n$ are n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. Define the attribute domain as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n),$$

where $\mathcal{P}(J_i)$ denotes the power set of J_i. An *n-Superhyperrough Cubic Set* over X is defined as a mapping

$$F : J \to \mathcal{CS}(X),$$

such that for each $A = (A_1, A_2, \ldots, A_n) \in J$, if we denote $A_A = F(A)$ and define its lower and upper rough approximations (with respect to the cubic relation R) by

$$N(A_A)(x) = \bigcup_{y \in X} \Big(A_A(y) \, t \, R^c(y,x)\Big), \quad \forall x \in X,$$

$$H(A_A)(x) = \bigcap_{y \in X} \Big(A_A(y) \, u \, R(x,y)\Big), \quad \forall x \in X,$$

then the pair $\big(N(A_A), H(A_A)\big)$ constitutes the rough approximation of the cubic set A_A. The structure (F, J) is called an *n-Superhyperrough Cubic Set* on X.

Example 2.5 (n-Superhyperrough Cubic Set)**.** Let $X = \{x_1, x_2, x_3, x_4\}$ and let $R = \langle R, r \rangle$ be a given cubic relation on X. Assume there are two attributes T_1 and T_2 with domains $J_1 = \{p, q\}$ and $J_2 = \{r, s\}$, respectively. Then, define the attribute domain as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

For instance, an element of J could be $A = (\{p\}, \{r, s\})$. For each $A \in J$, define a cubic set A_A on X; for example, let

$$A_{(\{p\},\{r,s\})}(x_1) = \big([0.2, 0.5], 0.4\big), \quad A_{(\{p\},\{r,s\})}(x_2) = \big([0.3, 0.6], 0.5\big),$$

and similarly define $A_{(\{p\},\{r,s\})}(x_3)$ and $A_{(\{p\},\{r,s\})}(x_4)$. Then, define the mapping

$$F : J \to \mathcal{CS}(X)$$

by setting $F(A) = A_A$ for each $A \in J$. For every $A \in J$, compute the lower and upper rough approximations $N(A_A)$ and $H(A_A)$ using the cubic relation R. The pair (F, J) is then an n-Superhyperrough Cubic Set on X, which, by our previous theorem, generalizes both the Hyperrough Cubic Set and the n-Superhyperrough Set.

Theorem 2.6. *An n-Superhyperrough Cubic Set generalizes both Hyperrough Cubic Sets and n-Superhyperrough Sets.*

Proof. If we restrict the attribute domain J to a singleton (i.e., when each J_i is a singleton so that J itself is a singleton), then the mapping F reduces to a mapping into $\mathcal{CS}(X)$, and the n-Superhyperrough Cubic Set reduces to a Hyperrough Cubic Set.

Alternatively, if the cubic structure is degenerate (so that every cubic set becomes an interval-valued fuzzy set), then the mapping $F : J \to \mathcal{CS}(X)$ effectively maps into the collection of fuzzy sets, and the rough approximations coincide with those defined for n-Superhyperrough Sets. Thus, an n-Superhyperrough Cubic Set encompasses both Hyperrough Cubic Sets and n-Superhyperrough Sets. □

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Author Contributions

The paper has been solely authored by the corresponding author at this stage.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

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Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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