Linear-Branch-Decomposition of Digraph

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Abstract

The study of graph width parameters is a well-established field within graph theory. Recently, numerous researchers have been actively extending undirected width parameters to directed graphs, resulting in a wide range of studies on directed width parameters. In this paper, we introduce a new concept called Directed Linear-Branch-Width, which extends the (Undirected) Linear-Branch-Width to digraphs. We also investigate its relationship and hierarchy with Directed Path-width, Directed Cut-width, and Directed Neighbourhood-width.

Keywords: Directed Tree-width; Directed Branch-width; Directed Graph; Branch-width; Linear-branch-width

1 Introduction

1.1 Width Parameters

Graph theory is the study of mathematical structures used to model pairwise relations between objects. Graphs have a wide range of applications, including computer science, biology, social networks, and transportation systems.

To better understand the structure of graphs, researchers have developed various *graph parameters*. One of the most fundamental families of such parameters is the class of *graph width parameters*, which aim to quantify the structural complexity of graphs. These parameters are not only mathematically significant but also play a vital role in analyzing the computational tractability of many graph problems.

Prominent examples include tree-width,[22] path-width,[21] rank-width,[20] linear-width,[7,8] and branch-width.[22] The study of these measures lies at the heart of algorithmic graph theory and continues to influence both theoretical developments and practical applications.

1.2 Directed Width Parameter

Recently, numerous researchers have been actively extending undirected width parameters to directed graphs, resulting in a wide range of studies on directed width parameters.[1,16]

For example, concepts such as Directed Tree-width,[16] D-width,[23] Kelly-width,[15] Kelly-path-width,[19] Directed Path-width,[12] Directed Branch-width,[6] Directed Modular-width,[25] DAG-width,[5] DAG-Pathwidth,[18] Directed

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Clique-width,[9] Directed NLC-width,[13] and Directed Rank-width[17] have been proposed, with extensive research conducted on their mathematical properties, algorithms, and applications.

Furthermore, linear versions of these digraph width parameters are also well-known.[10,11] These restrictions to underlying path-structures are often beneficial in proving results for the general parameters. Moreover, these linear parameters offer valuable structural insights, particularly in the study of special graph classes. As a result, research on linear width parameters is also thriving in the study of both undirected and directed graph width parameters.[2,14]

In the course of this investigation, it was discovered that, while linear concepts are well-defined for undirected graphs, the linear concept of Directed Branch-width has not yet been explicitly defined.

1.3 Hierarkey of Graph Parameters

One active area of research in graph theory involves determining the upper and lower bounds of various graph parameters. Often referred to as studying the hierarchy of these parameters, this research seeks to understand the relationships between different parameters and leverage this understanding to advance graph algorithms and theoretical studies.[11,24]

The study of graph width parameters in directed graphs has become particularly active in recent years, and increasing attention is being focused on their hierarchy. However, I believe that the full scope of these parameters has yet to be fully understood.

1.4 Our Contribution

As noted above, the study of graph width parameters is a central theme in graph theory. However, the investigation of *directed* width parameters remains relatively underexplored, and there are several such parameters that have yet to be formally defined within a unified framework.

In this paper, we introduce a novel concept called *Directed Linear-Branch-Width*, which extends the classical notion of (undirected) linear-branch-width to the setting of directed graphs. Furthermore, we explore the mathematical relationships and hierarchy between this new parameter and existing measures such as Directed Path-Width, Directed Cut-Width, and Directed Neighbourhood-Width.

2 Definitions

This section presents the definitions used throughout this paper. Unless stated otherwise, all graphs considered in this work are assumed to be *simple*, *directed*, and *finite*.

2.1 Basic Notation Used in This Paper

We provide below the basic notation and conventions that will be followed in the rest of this paper.

Definition 2.1 (Directed Graph)**.** A *directed graph* (or *digraph*) is an ordered pair

$$D = (V(D), E(D)),$$

where $V(D)$ is the finite set of vertices and $E(D) \subseteq V(D) \times V(D)$ is the set of directed edges (arcs). For brevity we often write $D = (V, E)$.

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Definition 2.2 (Complement). If X is a subset of $V(D)$ or $E(D)$, then the *complement* is

$$X^c = \begin{cases} V(D) \setminus X, & X \subseteq V(D), \\ E(D) \setminus X, & X \subseteq E(D). \end{cases}$$

Definition 2.3 (Degree). In a digraph $D = (V, E)$, the *in-degree* of $v \in V$ is $\deg^-(v) = |\{(u, v) \in E\}|$ and the *out-degree* is $\deg^+(v) = |\{(v, w) \in E\}|$. The *total degree* is $\deg(v) = \deg^-(v) + \deg^+(v)$.

Definition 2.4 (Directed Tree). A *directed tree* is a connected acyclic digraph. If it has n vertices, then it has exactly $n - 1$ edges.

Definition 2.5 (Leaf and Internal Node). In a directed tree, a *leaf* is a vertex of total degree 1. Any vertex that is not a leaf is called an *internal node*.

Definition 2.6 (Subcubic Directed Tree). A directed tree is *subcubic* if every vertex has total degree at most 3, i.e. $\deg(v) \leq 3$ for all $v \in V$.

Definition 2.7 (Directed Path). A *directed path* in $D = (V, E)$ is a sequence of distinct vertices $(v_0, v_1, \ldots, v_k)$ such that $(v_i, v_{i+1}) \in E$ for $0 \leq i < k$.

Definition 2.8 (Linear Ordering of Vertices). A *linear ordering* of the vertices of $D = (V, E)$ is a bijection $\pi : V \to \{1, 2, \ldots, |V|\}$ such that whenever $(u, v) \in E$, we have $\pi(u) < \pi(v)$.

For further background on these and related concepts, see.[3,28]

2.2 Directed Width parameter

In this subsection, we define directed Width parameters such as directed path-width, directed cut-width, and directed neighbourhood-width.

Definition 2.9. [4] A *directed path-decomposition* of a digraph $G = (V, E)$ is a sequence of vertex subsets $(X_1, X_2, \ldots, X_r)$, called bags, that satisfy:

1. $X_1 \cup \cdots \cup X_r = V$.
2. For each edge $(u, v) \in E$, there exists a pair of indices $i \leq j$ such that $u \in X_i$ and $v \in X_j$.
3. If $u \in X_i$ and $u \in X_j$ for some $u \in V$ and $i \leq j$, then $u \in X_\ell$ for all ℓ with $i \leq \ell \leq j$.

The *width* of a directed path-decomposition is $\max_{1 \leq i \leq r} |X_i| - 1$. The *directed path-width* d-pw(G) is the minimum width over all possible directed path-decompositions of G.

Example 2.10 (Directed Path-Decomposition of the Diamond Digraph). Let $D = (V, E)$ be the digraph with

$$V = \{1, 2, 3, 4\}, \qquad E = \{(1, 2), (1, 3), (2, 4), (3, 4)\},$$

often called the *diamond* or *bi-fan* digraph.

We exhibit a directed path-decomposition of D of width 2:

$$X_1 = \{1\}, \quad X_2 = \{1, 2, 3\}, \quad X_3 = \{2, 3, 4\}.$$

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Verification of the decomposition axioms (cf.[4]):

1. $\bigcup_{i=1}^{3} X_i = \{1\} \cup \{1,2,3\} \cup \{2,3,4\} = V$.
2. For each directed edge $(u,v) \in E$:
   - $(1,2)$: $1 \in X_1$, $2 \in X_2$, and $1 \leq 2$.
   - $(1,3)$: $1 \in X_1$, $3 \in X_2$, and $1 \leq 2$.
   - $(2,4)$: $2 \in X_2$, $4 \in X_3$, and $2 \leq 3$.
   - $(3,4)$: $3 \in X_2$, $4 \in X_3$, and $2 \leq 3$.
3. For each vertex $v \in V$, the set $\{\, i : v \in X_i\}$ is an interval in $\{1,2,3\}$:

$$\begin{aligned} v &= 1 : \{1,2\}, \\ v &= 2 : \{2,3\}, \\ v &= 3 : \{2,3\}, \\ v &= 4 : \{3\}. \end{aligned}$$

   Each is contiguous in the index path 1–2–3.

Width. The bag sizes are $|X_1| = 1$, $|X_2| = 3$, $|X_3| = 3$, so the width is

$$\max_{i=1,2,3} \bigl(|X_i| - 1\bigr) = \max\{1-1,\, 3-1,\, 3-1\} = 2.$$

Hence $\text{d-pw}(D) \leq 2$. One can show no decomposition of smaller width exists, so in fact $\text{d-pw}(D) = 2$.

Definition 2.11. [11] The directed cut-width of a digraph $G = (V,E)$ is defined as:

$$\text{d-cutw}(G) = \min_{\varphi \in \Phi(G)} \max_{1 \leq i \leq |V|} \left|\{(u,v) \in E \mid u \in L(i,\varphi,G), v \in R(i,\varphi,G)\}\right|,$$

where:

- $\Phi(G)$ is the set of all possible layouts (linear orderings) of the vertices.
- $L(i,\varphi,G)$ is the set of vertices in the left partition (preceding vertices) up to position i in the layout φ.
- $R(i,\varphi,G)$ is the set of vertices in the right partition (succeeding vertices) from position $i+1$ onward.
- The directed cut-width represents the minimum, over all possible vertex layouts, of the maximum number of edges crossing the cut between these two partitions.

Example 2.12 (Directed Cut-Width of the Diamond Digraph)**.** Let $D = (V,E)$ be the digraph with

$$V = \{1,2,3,4\}, \qquad E = \{(1,2),(1,3),(2,4),(3,4)\},$$

known as the *diamond* digraph.

Vertex layout. Consider the vertex ordering

$$\varphi = (1,2,3,4).$$

For $1 \leq i \leq 3$, define

$$L(i) = \{\varphi(1),\ldots,\varphi(i)\}, \quad R(i) = V \setminus L(i).$$

Crossing edges.

$$\begin{aligned} i=1: &\quad L(1) = \{1\}, & R(1) = \{2,3,4\}, &\quad \text{crossing } \{(1,2),(1,3)\}, & \bigl|\cdot\bigr| = 2, \\ i=2: &\quad L(2) = \{1,2\}, & R(2) = \{3,4\}, &\quad \text{crossing } \{(1,3),(2,4)\}, & \bigl|\cdot\bigr| = 2, \\ i=3: &\quad L(3) = \{1,2,3\}, & R(3) = \{4\}, &\quad \text{crossing } \{(2,4),(3,4)\}, & \bigl|\cdot\bigr| = 2. \end{aligned}$$

Since the maximum number of edges crossing any cut is 2, we have

$$\bigl|\{(u,v) \in E \mid u \in L(i),\, v \in R(i)\}\bigr| \leq 2 \quad \text{for all } i,$$

so $\text{d-cutw}(D) \leq 2$.

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Optimality. One checks that no vertex ordering can reduce the maximum below 2. Hence the directed cut-width of the diamond digraph is

$$\text{d-cutw}(D) = 2.$$

Definition 2.13. [11] The directed neighbourhood-width of a digraph $G = (V, E)$ is defined as:

$$\text{d-nw}(G) = \min_{\varphi \in \Phi(G)} \max_{1 \leq i \leq |V|} |N(L(i, \varphi, G), R(i, \varphi, G))|,$$

where:

- $\Phi(G)$ is the set of all possible layouts (linear orderings) of the vertices.
- $L(i, \varphi, G)$ is the set of vertices in the left partition (preceding vertices) up to position i in the layout φ.
- $R(i, \varphi, G)$ is the set of vertices in the right partition (succeeding vertices) from position $i + 1$ onward.
- $N(L(i, \varphi, G), R(i, \varphi, G))$ represents the set of all directed neighbourhoods of the vertices in $L(i, \varphi, G)$ into $R(i, \varphi, G)$, accounting for both out-neighbours and in-neighbours.

Example 2.14 (Directed Neighbourhood-Width of the Diamond Digraph)**.** Let $D = (V, E)$ be the digraph with

$$V = \{1, 2, 3, 4\}, \qquad E = \{(1, 2), (1, 3), (2, 4), (3, 4)\},$$

commonly called the *diamond* digraph.

Vertex layout. Consider the ordering

$$\varphi = (1, 2, 3, 4).$$

For each $1 \leq i \leq 3$, define

$$L(i) = \{\varphi(1), \ldots, \varphi(i)\}, \quad R(i) = V \setminus L(i).$$

Neighbourhood sets. Recall

$$N\big(L(i), R(i)\big) = \{\, v \in R(i) \mid \exists\, u \in L(i) : (u, v) \in E \text{ or } (v, u) \in E\}.$$

We compute:

$$\begin{aligned}
i = 1 : L(1) = \{1\}, \qquad & R(1) = \{2, 3, 4\}, N(L(1), R(1)) = \{2, 3\}, \quad |N| = 2,\\
i = 2 : L(2) = \{1, 2\}, \qquad & R(2) = \{3, 4\}, N(L(2), R(2)) = \{3, 4\}, \quad |N| = 2,\\
i = 3 : L(3) = \{1, 2, 3\}, \qquad & R(3) = \{4\}, N(L(3), R(3)) = \{4\}, \quad |N| = 1.
\end{aligned}$$

Since the maximum neighbourhood size over all cuts is 2, the directed neighbourhood-width is

$$\text{d-nw}(D) = \min_{\varphi} \max_{i} |N(L(i), R(i))| = 2.$$

2.3 Linear Width of a Directed Graph

We now introduce the *linear width* of a digraph. This parameter is the linear analogue of the directed branch-width defined by Bumpus and Oum[6] and extends the linear width for undirected graphs studied in.[26,27]

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Definition 2.15 (cf.[6])**.** Let $D = (V, E)$ be a digraph. For any edge set $X \subseteq E$ define

$$f_D(X) \;=\; \bigl|\, S_V(X) \cup S_V\bigl(E \setminus X\bigr)\bigr|,$$

where $S_V(X)$ denotes the (directed) vertex-separator induced by the edge set X and $S_V\bigl(E \setminus X\bigr)$ is defined analogously.

A *directed branch-decomposition* of D is a pair $(T, \mathcal{L})$ in which T is a tree whose leaves are in one-to-one correspondence with the edges of D (the bijection is denoted $\mathcal{L}$). Each edge e of T partitions the leaves—and hence the edge set E—into two parts; the *width* of the decomposition is the maximum value of $f_D(X)$ taken over all such partitions.

If the underlying tree T is a path, the decomposition is called a *directed linear-branch decomposition*. Equivalently, a linear-branch decomposition corresponds to a linear ordering

$$(e_1, e_2, \ldots, e_m)$$

of the edges. For every cut between e_i and e_{i+1}, $1 \le i < m$, we evaluate

$$X_i = \{e_1, \ldots, e_i\}, \qquad E \setminus X_i = \{e_{i+1}, \ldots, e_m\}.$$

The *directed linear-branch width* of D, denoted $\mathrm{lbw}(D)$, is the minimum—over all linear orderings of the edges—of the maximum value of $f_D(X_i)$ encountered along the ordering.

Example 2.16 (Directed Linear-Branch Width of a 3-Vertex Directed Path)**.** Let $D = (V, E)$ be the digraph with

$$V = \{1, 2, 3\}, \qquad E = \{e_1, e_2\},$$

where $e_1 = (1, 2)$ and $e_2 = (2, 3)$.

For any $X \subseteq E$, define the *vertex-separator*

$$S_V(X) = \{\, v \in V \mid v \text{ is an endpoint of some edge in } X \,\}.$$

A directed linear-branch decomposition corresponds to the single linear ordering

$$(e_1, e_2)$$

of the edges. We must evaluate

$$f_D(X_i) = \bigl|\, S_V(X_i) \cup S_V(E \setminus X_i)\bigr|$$

for each cut $1 \le i < 2$.

Cut after e_1.

$$X_1 = \{e_1\}, \quad E \setminus X_1 = \{e_2\}.$$

Then

$$S_V(X_1) = \{1, 2\}, \quad S_V(E \setminus X_1) = \{2, 3\},$$

so

$$f_D(X_1) = \bigl|\{1, 2\} \cup \{2, 3\}\bigr| = 3.$$

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Cut after e_2.

$$X_2 = \{e_1, e_2\}, \quad E \setminus X_2 = \emptyset.$$

Then

$$S_V(X_2) = \{1, 2, 3\}, \quad S_V(\emptyset) = \emptyset,$$

so

$$f_D(X_2) = \big|\{1, 2, 3\} \cup \emptyset\big| = 3.$$

Since these are the only cuts, the maximum value of $f_D(X_i)$ is $\max\{3, 3\} = 3$. Therefore, the directed linear-branch width of D is

$$\text{lbw}(D) = 3.$$

Example 2.17 (Directed Linear-Branch Width of a Four-Vertex Directed Star)**.** Let $D = (V, E)$ be the digraph with

$$V = \{r, a, b, c\}, \qquad E = \{e_1, e_2, e_3\},$$

where $e_1 = (r, a),\ e_2 = (r, b),\ e_3 = (r, c)$.

We consider the unique linear ordering

$$(e_1, e_2, e_3)$$

of the edges. For each cut $1 \leq i < 3$ define

$$X_i = \{e_1, \ldots, e_i\}, \quad E \setminus X_i = \{e_{i+1}, \ldots, e_3\},$$

and compute

$$f_D(X_i) = \big|S_V(X_i) \ \cup \ S_V(E \setminus X_i)\big|.$$

Cut after e_1.

$$X_1 = \{e_1\}, \quad E \setminus X_1 = \{e_2, e_3\},$$
$$S_V(X_1) = \{r, a\}, \quad S_V(\{e_2, e_3\}) = \{r, b, c\},$$
$$f_D(X_1) = \big|\{r, a\} \cup \{r, b, c\}\big| = 4.$$

Cut after e_2.

$$X_2 = \{e_1, e_2\}, \quad E \setminus X_2 = \{e_3\},$$
$$S_V(X_2) = \{r, a, b\}, \quad S_V(\{e_3\}) = \{r, c\},$$
$$f_D(X_2) = \big|\{r, a, b\} \cup \{r, c\}\big| = 4.$$

Since these are the only cuts, the maximum of $f_D(X_i)$ is $\max\{4, 4\} = 4$. Therefore the directed linear-branch width of D is

$$\text{lbw}(D) = 4.$$

The next observation follows immediately from the definitions.

Theorem 2.18. *For every digraph D,*

$$\text{lbw}(D) \ \geq \ \text{bw}(D),$$

where $\text{bw}(D)$ is the directed branch-width of D.

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Proof. Recall that a *directed branch-decomposition* of a digraph $D = (V, E)$ is any layout of the edge set on a tree T, whereas a *directed linear-branch decomposition* is the special case in which T is restricted to be a path. Consequently,

$$\{\text{directed linear-branch decompositions of } D\} \subseteq \{\text{directed branch decompositions of } D\}.$$

Let $\mathcal{D}_{\mathrm{lbw}}$ be an optimal directed linear-branch decomposition of D; that is,

$$\mathrm{width}(\mathcal{D}_{\mathrm{lbw}}) = \mathrm{lbw}(D).$$

Because its underlying tree is a path, $\mathcal{D}_{\mathrm{lbw}}$ is also a valid directed branch decomposition. Hence

$$\mathrm{bw}(D) = \min_{\mathcal{D}} \mathrm{width}(\mathcal{D}) \leq \mathrm{width}(\mathcal{D}_{\mathrm{lbw}}) = \mathrm{lbw}(D).$$

In other words, $\mathrm{lbw}(D) \geq \mathrm{bw}(D)$, as claimed. □

3 Result of this paper

The results of this paper are presented below.

3.1 Directed linear-width vs Directed path-width

In this subsection, we will examine the relationship between Directed Linear-Width and Directed Path-Width.

Theorem 3.1. *Let $D = (V, E)$ be a digraph. Denote by $\mathrm{dpw}(D)$ its directed path-width and by $\mathrm{lbw}(D)$ its directed linear-branch width. Let $\Delta^+(D) = \max_{v \in V} \deg^+(v)$ be the maximum out-degree in D. Then*

$$\mathrm{lbw}(D) \leq \mathrm{dpw}(D) + 1, \qquad \mathrm{dpw}(D) \leq \mathrm{lbw}(D) + \Delta^+(D).$$

Proof. We prove each inequality in turn.

(i) $\mathrm{lbw}(D) \leq \mathrm{dpw}(D) + 1$. Let $(X_1, X_2, \ldots, X_r)$ be an optimal directed path-decomposition of D of width $\mathrm{dpw}(D) = w$. Thus

$$|X_i| \leq w + 1 \quad \text{for all } 1 \leq i \leq r.$$

We now build a linear ordering of the edges of D whose linear-branch width is at most $w + 1$.

Assign to each edge $e \in E$ the index

$$\ell(e) = \min\{\, i \mid e = (u, v) \text{ with } u, v \in X_i\}.$$

Order the edges so that $\ell(e)$ is nondecreasing. Call this ordering $(e_1, e_2, \ldots, e_m)$.

For each cut between e_i and e_{i+1}, consider the partition

$$X = \{e_1, \ldots, e_i\}, \quad E \setminus X = \{e_{i+1}, \ldots, e_m\}.$$

By construction, every vertex involved in an edge crossing the cut must lie in $X_{\ell(e_i)}$. Hence

$$f_D(X) = \big|S_V(X) \cup S_V(E \setminus X)\big| \leq |X_{\ell(e_i)}| \leq w + 1.$$

Taking the maximum over all cuts shows $\mathrm{lbw}(D) \leq w + 1 = \mathrm{dpw}(D) + 1$.

(ii) $\mathrm{dpw}(D) \leq \mathrm{lbw}(D) + \Delta^+(D)$. Let $(e_1, e_2, \ldots, e_m)$ be an optimal linear ordering achieving $\mathrm{lbw}(D) = k$. Define a sequence of vertex–bags $\big(Y_0, Y_1, \ldots, Y_m\big)$ by

$$Y_0 = \emptyset, \qquad Y_i = \{\, u \mid \exists j \leq i : e_j = (u, v)\,\} \cup \{\, v \mid \exists j < i : e_j = (u, v)\,\}.$$

That is, Y_i contains all vertices incident to the first i edges, plus those vertices that appear as heads of edges among the first $i - 1$.

One checks easily that $\big(Y_i\big)_{i=0}^{m}$ satisfies the directed path-decomposition axioms:

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- Every vertex v appears in exactly those Y_i from its earliest incident edge to its latest, so contiguity holds.
- Every edge $(u, v) = e_i$ satisfies $\{u, v\} \subseteq Y_i$.

Moreover, since each new edge $e_i = (u, v)$ can introduce at most the out-neighbors of u into the bag,

$$|Y_i| \ \leq \ f_D\big(\{e_1, \ldots, e_i\}\big) \ + \ \deg^+(u) \ \leq \ k \ + \ \Delta^+(D).$$

It follows that $\mathrm{dpw}(D) \leq k + \Delta^+(D) = \mathrm{lbw}(D) + \Delta^+(D)$. □

3.2 Directed linear-width vs Directed cut-width

In this subsection, we will examine the relationship between Directed Linear-Width and Directed cut-Width.

Theorem 3.2. *For any digraph D,*

$$\mathrm{lbw}(D) \ \leq \ \text{d-cutw}(D).$$

Proof. Let φ be a vertex layout of $D = (V, E)$ that realises the directed cut-width,

$$\text{d-cutw}(D) = \max_{1 \leq i < |V|} \big|\{(u, v) \in E \mid \varphi(u) \leq i < \varphi(v)\}\big|.$$

Label the vertices so that $\varphi^{-1}(1), \varphi^{-1}(2), \ldots, \varphi^{-1}(n)$ is the order of φ. We now construct a linear-branch decomposition of the edges whose width does not exceed this value.

Edge ordering. List the edges in nondecreasing order of the minimum endpoint index under φ. Concretely, for each edge $e = (u, v)$ set

$$\sigma(e) = \min\{\varphi(u),\, \varphi(v)\},$$

and order $E = \{e_1, \ldots, e_m\}$ so that $\sigma(e_1) \leq \sigma(e_2) \leq \cdots \leq \sigma(e_m)$.

Width of the resulting decomposition. For each cut between e_i and e_{i+1}, let

$$X_i = \{e_1, \ldots, e_i\}, \qquad E \setminus X_i = \{e_{i+1}, \ldots, e_m\}.$$

By definition of σ, every edge in X_i has at least one endpoint among the first $\sigma(e_i)$ vertices of φ, while every edge in $E \setminus X_i$ has at least one endpoint among the last $n - \sigma(e_i)$ vertices. Hence the only edges that cross the cut in the vertex layout φ are those with one endpoint in $\varphi^{-1}(1), \ldots, \varphi^{-1}(\sigma(e_i))$ and the other in $\varphi^{-1}(\sigma(e_i) + 1), \ldots, \varphi^{-1}(n)$. By construction,

$$f_D(X_i) = \big|S_V(X_i) \cup S_V(E \setminus X_i)\big|$$

counts exactly the vertices incident to these crossing edges. Since φ realises the directed cut-width,

$$\big|\{(u, v) \in E \mid \varphi(u) \leq \sigma(e_i) < \varphi(v)\}\big| \ \leq \ \text{d-cutw}(D),$$

and each such edge contributes at most one new vertex to the separator union, it follows that

$$f_D(X_i) \ \leq \ \text{d-cutw}(D).$$

Taking the maximum over all i shows that the directed linear-branch width, which is the minimum over all edge-orderings of the maximum of $f_D(X_i)$, satisfies

$$\mathrm{lbw}(D) \ \leq \ \text{d-cutw}(D).$$

□

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3.3 Directed linear-width vs Directed neighbourhood-width

In this subsection, we will examine the relationship between Directed Linear-Width and Directed neighbourhood-width.

Theorem 3.3. *For any digraph $D = (V, E)$,*

$$\mathrm{lbw}(D) \ \leq \ \text{d-nw}(D),$$

where $\mathrm{lbw}(D)$ *is the directed linear-branch width and* $\text{d-nw}(D)$ *is the directed neighbourhood-width.*

Proof. Let φ be a vertex ordering of D that realises the directed neighbourhood-width:

$$\text{d-nw}(D) = \max_{1 \leq i < |V|} \left| N\big(L(i), R(i)\big) \right|,$$

where

$$L(i) = \{\varphi^{-1}(1), \ldots, \varphi^{-1}(i)\}, \quad R(i) = V \setminus L(i),$$

and

$$N\big(L(i), R(i)\big) = \{v \in V \mid \exists\, u \in L(i), (u, v) \in E \text{ or } (v, u) \in E\}.$$

Edge Ordering. Define for each edge $e = (u, v) \in E$ the index

$$\sigma(e) = \max\{\varphi(u),\ \varphi(v)\}.$$

Order the edges so that $\sigma(e_1) \leq \sigma(e_2) \leq \cdots \leq \sigma(e_m)$. This gives a directed linear-branch decomposition $(e_1, \ldots, e_m)$.

Width Analysis. For each cut between e_i and e_{i+1}, let

$$X_i = \{e_1, \ldots, e_i\}, \quad E \setminus X_i = \{e_{i+1}, \ldots, e_m\}.$$

By the definition of σ, every edge in X_i has both endpoints in the prefix $\varphi^{-1}(1), \ldots, \varphi^{-1}(k)$ with $k = \sigma(e_i)$, and every edge in $E \setminus X_i$ has at least one endpoint in $\varphi^{-1}(k+1), \ldots, \varphi^{-1}(n)$. It follows that a vertex v lies in the *vertex-separator*

$$S_V(X_i) = \big\{\, v \in V \mid \exists\, e \in X_i,\ e' \in E \setminus X_i : v \in e \cap e' \big\}$$

if and only if v has a neighbour across the cut in the ordering φ, i.e. $v \in N(L(k), R(k))$. Therefore

$$|S_V(X_i)| = \big|N(L(k), R(k))\big| \ \leq \ \text{d-nw}(D).$$

Since this holds for every cut i, the directed linear-branch width $\mathrm{lbw}(D)$, which is the minimum over all linear edge orderings of the maximum $|S_V(X_i)|$, also satisfies $\mathrm{lbw}(D) \leq \text{d-nw}(D)$. □

4 Conclusion and Future Work

In this paper, we examined the relationships and hierarchical ordering of directed path-width, directed cut-width, and directed neighbourhood-width. As future work, we plan to extend additional width and length parameters to the directed setting, including directed Boolean-width, directed tree-length, and directed branch-length.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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