A Short Contribution to the Classification of the Group of Units of the Rings $(NCR)_{Z_{pq}}$, $(NCR)_{Z_{2^n}}$ and $(NCR)_{Z_{p^2}}$

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Abstract

In this paper, we study the group of units problem of three different non-commutative logical extensions rings, where we classify the group of units of the rings $(NCR)_{Z_{pq}}$, $(NCR)_{Z_{2^n}}$ and $(NCR)_{Z_{p^2}}$ as semi direct products of well-known abelian groups as the following:

$U(N \subset R)_{Z_{pq}} \cong (Z_{p-1} \times Z_{q-1}) \propto [(Z_p \times Z_q) \propto (Z_{p-1} \times Z_{q-1})$,

$U(NCR)_{Z_{2^n}} \cong (Z_2 \times Z_{2^{n-2}}) \propto (Z_{2^n} \propto (Z_2 \times Z_{2^{n-2}}))$,

$U(N \subset R)_{Z_{p^2}} \cong Z_{p^2-p} \propto (Z_{p^2} \propto Z_{p^2-p})$.

Keywords: Non-commutative logical extension; Group of units; Semi-direct product; Abelian subgroup

1. Introduction

Extending rings to generalized rings is an interesting research direction, where it can produce some generalized structures with more general properties. Inserting logical elements to a classical algebraic ring can generate some special rings. For example the n-cyclic refined rings [3,5] and their group of units classification [4]. The concept of non-commutative logical extension of a ring was suggested in [2] as follows: If R is a ring, then the non-commutative logical extension of R is defined with:

$$(N \subset R)_R = \{x + yN_1 + +zN_2 \ ; x, y, z \in Z_{pq}, {N_1}^2 = N_1N_2 = N_1, {N_2}^2 = N_2N_1 = N_2\}.$$

An interesting open research question was asked in [2]:

How can we classify the group of units of a non-commutative logical extension $(N \subset R)_R$. This question was answered partially in [1] by classifying the group of units of two different non-commutative logical extensions. This has motivated us to study the group of units problem of three different non-commutative logical extensions rings, where we classify the group of units of the rings $(NCR)_{Z_{pq}}$, $(NCR)_{Z_{2^n}}$ and $(NCR)_{Z_{p^2}}$ as semi direct products of well-known abelian groups as the following:

$U(N \subset R)_{Z_{pq}} \cong (Z_{p-1} \times Z_{q-1}) \propto [(Z_p \times Z_q) \propto (Z_{p-1} \times Z_{q-1})$,

$U(NCR)_{Z_{2^n}} \cong (Z_2 \times Z_{2^{n-2}}) \propto (Z_{2^n} \propto (Z_2 \times Z_{2^{n-2}}))$,

$U(N \subset R)_{Z_{p^2}} \cong Z_{p^2-p} \propto (Z_{p^2} \propto Z_{p^2-p})$.

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2. Main Discussion

Definition:

The non- commutative logical extension of Z_{pq} is:

$(N \subset R)_{Z_{pq}} = \{x + yN_1 + +zN_2 \ ; x, y, z \in Z_{pq}, {N_1}^2 = N_1N_2 = N_1, {N_2}^2 = N_2N_1 = N_2\}$.

We denote to its group of units by $G = U(N \subset R)_{Z_{pq}}$.

Remark:

An element $t = t_0 + t_1N_1 + t_2N_2 \in G$, if and only if:

$$\begin{cases} t_0 \in U(Z_{pq}) \\ t_0 + t_1 + t_2 \in U(Z_{pq}) \end{cases}$$

Also, $t^{-1} = {t_0}^{-1} - t_1{t_0}^{-1}(t_0 + t_1 + t_2)^{-1}N_1 + t_2{t_0}^{-1}(t_0 + t_1 + t_2)^{-1}N_2$.

Theorem:

Let $G = U(N \subset R)_{Z_{pq}}$, then:

1] $|G| = pq(p-1)^2(q-1)^2$

2] $Z(G) = U(Z_{pq})$

3] $K = \{1 + x(N_1 - N_2) \ ; x \in Z_{pq}\}$ is a normal subgroup of G.

4] $S = \{1 + xN_1 \ ; 1 + x \in U(Z_{pq})\}$ is a subgroup of G.

5] $T = \{1 + xN_1 + yN_2 \ ; 1 + x + y \in U(Z_{pq})\}$ is a subgroup of G.

6] $K \cong Z_{pq}$, $S \cong Z_{p-1} \times Z_{q-1}$

7] $T \cong (Z_p \times Z_q) \propto (Z_{p-1} \times Z_{q-1})$

8] $G \cong (Z_{p-1} \times Z_{q-1}) \propto [(Z_p \times Z_q) \propto (Z_{p-1} \times Z_{q-1})]$

Proof:

$\boxed{3} - \boxed{4} - \boxed{5}$ can be proved easily by a similar discussion of [1].

1] $x = x_0 + x_1N_1 + x_2N_2 \in G$ if and only if:

$$\begin{cases} x_0 \in U(Z_{pq}) \\ x_0 + x_1 + x_2 \in U(Z_{pq}) \end{cases}$$

For every possible value of $x_0 = t_0, x_0 + x_1 + x_2 = t_1$, then $x_1 + x_2 = t_1 - t_0$, and we get pq different values of x_1.

And for every value of x_1, we have okne corresponding value of x_2.

This implies that:

$|G| = (p-1)(q-1) \times (p-1)(q-1) \times pq = pq(p-1)^2(q-1)^2$.

2] An element $x = x_0 + x_1N_1 + x_2N_2 \in Z(G)$ if and only if for every $y = y_0 + y_1N_1 + y_2N_2 \in G$, we have $xy = yx$.

This implies that:

$x_1y_2 = x_2y_1$. For $y_2 = 0, y_1 = 1 - y_0$, we have:

$x_2(1 - y_0) = 0 \Longrightarrow pq | x_2(1 - y_0)$

$pq \nmid x_2, pq \nmid 1 - y_0$ that is because $x_2, 1 - y_0 < pq$.

We can assume that $p | 1 - y_0 \Longrightarrow y_0 \equiv 1 (mod\ p)$ for all $y_0 \in U(Z_{pq})$ or $q | 1 - y_0 \Longrightarrow y_0 \equiv 1 (mod\ q)$ for all $y_0 \in U(Z_{pq})$.

We know that $y_0 = pq - 1 \in U(Z_{pq})$, that is because $\gcd(y_0, pq) = 1$, and $y_0 \not\equiv 1 (mod\ p)$, $y_0 \not\equiv 1 (mod\ pq)$, thus $pq | x_2$ and $x_2 = 0$.

Now, for $y_1 = 0, y_2 = 1 - y_0$, we have:

$x_1(1 - y_0) = 0 \Longrightarrow pq | x_1(1 - y_0)$.

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If $pq|x_1$, then $x_1 = 0$.

If $pq \nmid x_1$then $pq \nmid 1 - y_0$ that is because $|1 - y_0| < pq$.

We have two possible cases:

$p|1 - y_0$or $q|1 - y_0 \Longrightarrow y_0 \equiv 1(mod\ p)\ or\ y_0 \equiv 1(mod\ q)$ for all $y_0 \in U(Z_{pq})$.

For $y_0 = pq - 1 \in U(Z_{pq})$ and $y_0 \not\equiv 1(mod\ p)$, $y_0 \not\equiv 1(mod\ q)$, thus $pq|x_1$ and $x_1 = 0$.

Hence $x = x_0 \in U(Z_{pq})$ and $Z(G) = U(Z_{pq}) \cong Z_{p-1} \times Z_{q-1}$.

6] It can be proved by a similar discussion of [ ].

7] It is clear that $S \cap K = \{1\}$ and $K \lhd T, S \leq T$with $|T| = |S.K| = |S|.|K| \Longrightarrow T = K \propto S \cong Z_{pq} \propto U(Z_{pq}) \cong \left(Z_p \times Z_q\right) \propto \left(Z_{p-1} \times Z_{q-1}\right)$.

8] It is clear that $Z(G) \cap T = \{1\}$, and $|G| = |Z(G).T| = |Z(G)|.|T| \Longrightarrow G = Z(G).T \Longrightarrow G = Z(G) \propto T \cong \left(Z_{p-1} \times Z_{q-1}\right) \propto \left[\left(Z_p \times Z_q\right) \propto \left(Z_{p-1} \times Z_{q-1}\right)\right]$.

Definition:

The non-commutative logical extension of Z_{2^n} is:

$(NCR)_{Z_{2^n}} = \left\{x + yN_1 + +zN_2\ ; x, y, z \in Z_{2^n}, {N_1}^2 = N_1N_2 = N_1, {N_2}^2 = N_2N_1 = N_2\right\}$.

We denote by $G = U(NCR)_{Z_{2^n}}$.

Remark:

1] $K = \{1 + x(N_1 - N_2)\ ; x \in Z_{2^n}\}$ is a normal subgroup of G.

2] $S = \{1 + xN_1\ ; 1 + x \in U(Z_{2^n})\}$ is a subgroup of G.

3] $T = \{1 + xN_1 + yN_2\ ; 1 + x + y \in U(Z_{2^n})\}$ is a subgroup of G.

4] $K \cong Z_{2^n}, S \cong U(Z_{2^n}) \cong Z_2 \times Z_{2^{n-2}}$

5] $\cap S = \{1\}, K \lhd T, S \leq T$

6] $|T| = 2^n.2^{n-1} = 2^{2n-1} = |K.S| = |K|.|S| \Longrightarrow T = K.S \cong Z_{2^n} \times (Z_2 \times Z_{2^{n-2}})$.

Result:

$H = U(Z_{2^n}) \leq Z(G) \Longrightarrow H \lhd G$, and $|H.T| = 2^{n-1}.2^n.2^{n-1} = 2^{3n-1} = |G|$, hence $G = H.T \cong (Z_2 \times Z_{2^{n-2}}) \propto \left(Z_{2^n} \propto (Z_2 \times Z_{2^{n-2}})\right)$.

Definition:

The non-commutative logical extension of the ring Z_{p^2} with (p) as a prime is defined as follows:

$(N \subset R)_{Z_{p^2}} = \left\{a + bN_1 + cN_2\ ; a, b, c \in Z_{p^2}, {N_1}^2 = N_1N_2 = N_1, {N_2}^2 = N_2N_1 = N_2\right\}$

We denote to the group of units $U(N \subset R)_{Z_{p^2}}$by G.

Remark: [2]

$x = x_0 + x_1N_1 + x_2N_2 \in G$, if and only if:

$$\begin{cases} x_0 \in U\left(Z_{p^2}\right) \\ x_0 + x_1 + x_2 \in U\left(Z_{p^2}\right) \end{cases}$$

$x^{-1} = {x_0}^{-1} - x_1{x_0}^{-1}(x_0 + x_1 + x_2)^{-1}N_1 + x_2{x_0}^{-1}(x_0 + x_1 + x_2)^{-1}N_2$.

Result: [2]

$U\left(Z_{p^2}\right)$ is non abelian group in general.

Theorem:

Let $G = U(N \subset R)_{Z_{p^2}}$, then:

1] $Z(G) = U(Z_{p^2})$

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2] $|G| = p^4(p-1)^2$

Proof:

1] let $x = x_0 + x_1N_1 + x_2N_2 \in Z(G)$, then $\forall\, y = y_0 + y_1N_1 + y_2N_2 \in G$, we have:

$xy = yx \Leftrightarrow x_1y_2 = x_2y_1$.

Take $\begin{cases} y_2 = 0 \\ y_1 = 1 - y_0 \end{cases}$ $then\ x_2y_1 = 0 \Rightarrow x_2(1-y_0) = 0 \Rightarrow p^2|x_2(1-y_0)$

$\Rightarrow \begin{cases} p|x_2 \\ p|1-y_0 \end{cases}$ for all $y_0 \in U(Z_{p^2})$.

Take $\begin{cases} y_1 = 0 \\ y_2 = 1 - y_0 \end{cases}$ $\Rightarrow\ x_1y_2 = 0 \Rightarrow x_1(1-y_0) = 0 \Rightarrow p^2|x_1(1-y_0)$

$\Rightarrow \begin{cases} p|x_1 \\ p|1-y_0 \end{cases}$ for all $y_0 \in U(Z_{p^2})$.

This implies that: $y_0 \equiv 1(mod\ p)$ for all $y_0 \in U(Z_{p^2})$.

We know that $y_0 = p^2 - 1$ is a unit in Z_{p^2}, hence:

$p^2 - 1 \equiv 1(mod\ p) \Rightarrow p^2 \equiv 2(mod\ p)$ which is impossible.

This means that: $p^2|x_1, p^2|x_2 \Rightarrow x_1 = x_2 = 0$.

Thus $x = x_0 \in U(Z_{p^2})$, and $Z(G) = U(Z_{p^2})$.

2] For every $x = x_0 + x_1N_1 + x_2N_2 \in G$, we have:

$\begin{cases} x_0 \in U(Z_{p^2}) \\ x_0 + x_1 + x_2 \in U(Z_{p^2}) \end{cases} \Rightarrow \begin{cases} |\{x_0; x_0 \in U(Z_{p^2})\}| = p(p-1) \\ |\{x_0 + x_1 + x_2 \in U(Z_{p^2})\}| = p(p-1) \end{cases}$

For every possible value: $x_0 = t_0, x_0 + x_1 + x_2 = t_1$.

We have $x_1 + x_2 = t_1 - t_0$(p^2different values of $x_1 + x_2$),

So that:$|G| = p(p-1) \times p(p-1) \times p^2 = p^4(p-1)^2$.

Theorem:

Let $G = U(N \subset R)_{Z_{p^2}}$, then:

1] $T = \{1 + a_1N_1 + a_2N_2\,; 1 + a_1 + a_2 \in U(Z_{p^2})\}$ is a subgroup of G.

2] $K = \{1 + a(N_1 - N_2)\,; a \in Z_{p^2}\}$ is a normal subgroup of G.

3] $S = \{1 + aN_1\,; 1 + a \in U(Z_{p^2})\}$ is a subgroup of G.

4] $K \cong Z_{p^2}, S \cong U(Z_{p^2})$

5] $G \cong Z(G) \propto T \cong U(Z_{p^2}) \propto (U(Z_{p^2}) \propto Z_{p^2})$

Proof:

1] By a similar discussion of [1], we get that T is a subgroup of G.

2] It can be proved as [1].

3] It is similar to the proof mentioned in [1].

4] Define $f: K \to Z_{p^2}, g: S \to U(Z_{p^2})$ such that:

$f(1 + a(N_1 - N_2)) = a, g(1 + aN_1) = 1 + a$,

(f) and (g) are well defined group homomorphisms, that is because:

If $\begin{cases} x = 1 + a(N_1 - N_2) = y = 1 + b(N_1 - N_2) \\ x' = 1 + aN_1 = y' = 1 + bN_1 \end{cases}$

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Then: $\begin{cases} a = b \\ 1 + a = 1 + b \end{cases} \Rightarrow \begin{cases} f(x) = f(y) \\ g(x') = g(y') \end{cases} \Rightarrow \{ f, g \text{ are well defined}.$

$\begin{cases} x.y = 1 + (a+b)(N_1 - N_2) \\ x'.y' = 1 + (a + b + ab)N_1 \end{cases} \Rightarrow \begin{cases} f(xy) = a + b = f(x) + f(y) \\ g(x'.y') = 1 + a + b + ab = (1+a)(1+b) = g(x')g(y') \end{cases}$

If $f(x) = 0 \Rightarrow x = \{1\} \Rightarrow k_{er}(f) = \{1\}$.

If $g(x') = 1 \Rightarrow a = 0 \Rightarrow x' = \{1\} \Rightarrow k_{er}(g) = \{1\}$.

For each $a \in Z_{p^2}$, there exists $x = 1 + a(N_1 - N_2) \in K$ such that $f(x) = a$.

For each $a \in U(Z_{p^2})$, there exists $x' = 1 + (a-1)N_1 \in S$ such that $g(x') = a$.

This means that f,g are group isomorphisms, hence:

$(K, x) \cong (Z_{p^2}, +), (S, x) \cong U(Z_{p^2})$.

5] It is clear that $S \cap K = \{1\}$ and $s \leq T, K \lhd T$.

Also, $|T| = \left|\left\{(a_1, a_2) \in (Z_{p^2})^2 ; 1 + a_1 + a_2 \in U(Z_{p^2})\right\}\right| = p^2.p(p-1) = p^3(p-1) = |S|.|K| = |S.K|$,

Hence $T = S \propto K$ and $T \cong U(Z_{p^2}) \propto Z_{p^2}$.

On the other hand, we have:

$Z(G) \cap T = \{1\}, |Z(G).T| = |Z(G)|.|T| = (p(p-1)).p^3(p-1) = p^4(p-1)^2 = |G| \Rightarrow G = Z(G).T$, hence:

$G \cong U(Z_{p^2}) \propto \left(Z_{p^2} \propto U(Z_{p^2})\right)$

Remark:

Since $U(Z_{p^2}) \cong Z_{p^2-p}$, then:

$G \cong Z_{p^2-p} \propto (Z_{p^2} \propto Z_{p^2-p})$.

3. Conclusion

In this paper, we studied the group of units problem of three different non-commutative logical extensions rings, where we classified the group of units of the rings $(NCR)_{Z_{pq}}$, $(NCR)_{Z_{2^n}}$ and $(NCR)_{Z_{p^2}}$ as semi direct products of well-known abelian groups as the following:

$U(N \subset R)_{Z_{pq}} \cong (Z_{p-1} \times Z_{q-1}) \propto [(Z_p \times Z_q) \propto (Z_{p-1} \times Z_{q-1})$,

$U(NCR)_{Z_{2^n}} \cong (Z_2 \times Z_{2^{n-2}}) \propto (Z_{2^n} \propto (Z_2 \times Z_{2^{n-2}}))$,

$U(N \subset R)_{Z_{p^2}} \cong Z_{p^2-p} \propto (Z_{p^2} \propto Z_{p^2-p})$.

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