The Characterization of 4-Cyclic Refined Vector Spaces

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Abstract

This paper is dedicated to study 4-cyclic refined vector spaces, where we classify these spaces by using semi-module isomorphisms as direct product of classical complex vector spaces. In addition, we study the inner products defined over these structures and we present sufficient conditions for 4-cyclic refined orthogonality.

Keywords: 4-cyclic refined space; 4-cyclic refined inner product; Orthogonal basis; 4-cyclic refined matrix

1. Introduction

The concept of n-cyclic refined structures has been defined in [12] as a new generalization of classical algebraic structures. These structures have been studied widely, where we can find many applications and theoretical studies in ring theory, group of unit's problem, and Diophantine equations [1-8]. In addition, many authors studied n-cyclic refined structures such as spaces and groups [13-19]. In [20-21], we find many interesting results about 3-cyclic refined matrices and 3-cyclic refined spaces, with 4-cyclic refined matrices.

This has motivated us to study 4-cyclic refined vector spaces. We classify these spaces by using semi-module isomorphisms as the direct product of classical complex vector spaces. We also study the inner products defined over these structures and present sufficient conditions for 4-cyclic refined orthogonality. We refer that all results proved in this paper, can be generalized into 5-cyclic inner spaces and other n-cyclic spaces, and this will be a good research material for future studies.

2. Main Discussion

Definition:

Let V be a vector space over the real field ℝ, the corresponding 4-cyclic refined vector space is defined as follows:

$V_4(I) = \{V_0 + \sum_{i=1}^{4} V_i I_i \ \ ; V_i \in V\}$.

Theorem:

$(V_4(I), +, \cdot)$ is a module over $R_4(I)$(not a space) with:

$\begin{cases} (+): V_4(I) \times V_4(I) \to V_4(I) \\ (\cdot): R_4(I) \times V_4(I) \to V_4(I) \end{cases}$ such that:

$$\left(V_0 + \sum_{i=1}^{4} V_i I_i\right) + \left(w_0 + \sum_{i=1}^{4} w_i I_i\right) = (V_0 + w_0) + \sum_{i=1}^{4} (V_i + w_i) I_i ,$$

$(m_0 + \sum_{i=1}^{4} m_i I_i) \cdot (V_0 + \sum_{i=1}^{4} V_i I_i) = m_0 V_0 + (\sum_{i+j \equiv 1 (mod\ 4)} m_i V_j) I_1 + (\sum_{i+j \equiv 2 (mod\ 4)} m_i V_j) I_2 + \cdots + (\sum_{i+j \equiv 4 (mod\ 4)} m_i V_j) I_4$.

Theorem: [17]

The mapping $g: R_4(I) \to R \times R \times R \times C$ such that:

$g: (x_0 + \sum_{i=1}^{4} x_i I_i) = (x_0, \sum_{i=0}^{4} x_i , x_0 + x_2 + x_4 - x_1 - x_3, x_0 + x_4 - x_2 + i(x_1 - x_3))$ is a ring isomorphism.

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Theorem:

Let $V_4(I)$ be a 4-cyclic refined vector space over $R_4(I)$, then there exists a semi-module isomorphism.

$h: V_4(I) \to V(\mathbb{R}) \times V(\mathbb{R}) \times V(\mathbb{R}) \times V(\mathbb{C})$.

Proof:

Define $h: V_4(I) \to V \times V \times V \times V(\mathbb{C})$ such that:

$h(V_0 + \sum_{i=1}^{4} V_i I_i) = (V_0, \sum_{i=0}^{4} V_i, V_0 + V_2 + V_4 - V_1 - V_3, V_0 + V_4 - V_2 + i(V_1 - V_3))$.

If $V_0 + \sum_{i=1}^{4} V_i I_i = w_0 + \sum_{i=1}^{4} w_i I_i$, then $V_i = w_i$; $0 \leq i \leq 4$.

So that: $\begin{cases} V_0 = w_0, \sum_{i=0}^{4} V_i = \sum_{i=0}^{4} w_i \\ V_0 + V_2 + V_4 - V_1 - V_3 = w_0 + w_2 + w_4 - w_1 - w_3 \\ V_0 + V_4 - V_2 + i(V_1 - V_3) = w_0 + w_4 - w_2 + i(w_1 - w_3) \end{cases}$

Hence , $h(V_0 + \sum_{i=1}^{4} V_i I_i) = h(w_0 + \sum_{i=1}^{4} w_i I_i)$.

For $X = (V_0 + \sum_{i=1}^{4} V_i I_i), Y = (w_0 + \sum_{i=1}^{4} w_i I_i)$, then:

$h(X + Y) = h[(V_0 + w_0) + \sum_{i=1}^{4}(V_i + w_i) I_i] = (V_0 + w_0, \sum_{i=0}^{4}(V_i + w_i), V_0 + V_2 + V_4 + w_0 + w_2 + w_4 - V_1 - V_3 - w_1 - w_3, V_0 + V_4 - V_2 + w_0 + w_4 - w_2 + i(V_1 - V_3 + w_1 - w_3) = h(X) + h(Y)$.

For $m = m_0 + \sum_{i=1}^{4} m_i I_i \in R_4(I)$, we have:

$h(m \cdot X) = h(a + bI_1 + cI_2 + dI_3 + kI_4)$; where:

$a = m_0 x_0,$

$b = m_0 V_1 + m_1 V_0 + m_1 V_4 + m_4 V_1 + m_2 V_3 + m_3 V_2,$

$c = m_0 V_2 + m_2 V_0 + m_1 V_1 + m_2 V_4 + m_4 V_2 + m_3 V_3,$

$d = m_0 V_3 + m_3 V_0 + m_1 V_2 + m_2 V_1 + m_3 V_4 + m_4 V_3,$

$k = m_0 V_4 + m_4 V_0 + m_1 V_3 + m_3 V_1 + m_2 V_2 + m_4 V_4,$

Thus $h(m \cdot X) = (a, a + b + c + d + k, a + c + k - b - d, a - c + k + i(b - d))$,

We have:

$a + b + c + d + k = (\sum_{i=0}^{4} m_i)(\sum_{i=0}^{4} V_i),$

$a + c + k - b - d = (m_0 + m_2 + m_4 - m_1 - m_3)(V_0 + V_2 + V_4 - V_1 - V_3),$

$a - c + k + i(b - d) = [m_0 - m_2 + m_4 + i(m_1 - m_3)][V_0 - V_2 + V_4 + i(V_1 - V_3)],$

So that $h(m \cdot X) = g(m) \cdot h(X)$.

Assume that $h(X) = 0$, then:

$V_0 = 0, V_1 + V_2 + V_3 + V_4 = 0,$

$V_2 + V_4 - V_1 - V_3 = 0,$

$-V_2 + V_4 = V_1 - V_3 = 0,$

Thus: $V_1 = V_2 = V_3 = V_4$, and $X = 0$.

for $(V_0, V_1, V_2, V_3 + V_4 i) \in V \times V \times V \times V(\mathbb{C})$ there exists:

$X = V_0 + I_1\left[\frac{1}{4}(V_1 - V_2) + \frac{1}{2}V_4\right] + I_2\left[\frac{1}{4}(V_1 + V_2) - \frac{1}{2}V_3\right] + I_3\left[\frac{1}{4}(V_1 - V_2) - \frac{1}{2}V_4\right] + I_4\left[\frac{1}{4}(V_1 + V_2) + \frac{1}{2}V_3 - V_0\right] \in V_4(I)$.

Such that $h(X) = (V_0, V_1, V_2, V_3 + V_4 i)$,

hence (h) is a bijection and then it is a semi-module isomorphism.

Remark:

If $\dim(V) = n$ with $E = \{e_1, \ldots, e_n\}$ as a basis of V over $\mathbb{R}$, then:

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1] $\dim V_4(I) = n^4$.

2] $K = \{h^{-1}(e_i, e_j, e_k, e_m + e_m i)\ ; 1 \le i, j, k, m \le n\}$ is the basis of $V_4(I)$ over $R_4(I)$.

It can be proved by a similar discussion of the 3-cyclic refined case.

Example:

For $V = \mathbb{R}^2$, then:

$$V_4(I) = \left\{(x_0, y_0) + \sum_{i=1}^{4}(x_i, y_i)\, I_i \ \ ; x_i, y_i \in \mathbb{R}\right\},$$

$$\dim(V_4(I)) = 2^4 = 16\,, E = \{e_1 = (1,0), e_2 = (0,1)\}$$

is a basis of V over $\mathbb{R}$.

$h^{-1}(e_1, e_1, e_1, e_1 + e_1 i) = e_1 + I_1\left[\frac{1}{4}(e_1 - e_1) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_1 + e_1) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_1 - e_1) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_1 + e_1) + \frac{1}{2}e_1 - e_1\right] = (1,0) + \left(\frac{1}{2}, 0\right) I_1 + \left(\frac{-1}{2}, 0\right) I_3.$

$$h^{-1}(e_1, e_1, e_1, e_2 + e_2 i) = e_1 + I_1\left[\frac{1}{4}(e_1 - e_1) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_1 + e_1) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_1 - e_1) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_1 + e_1) + \frac{1}{2}e_2 - e_1\right]$$

$= (1,0) + \left(0, \frac{1}{2}\right) I_1 + \left(\frac{1}{2}, \frac{-1}{2}\right) I_2 + \left(0, \frac{-1}{2}\right) I_3 + \left(\frac{-1}{2}, \frac{1}{2}\right) I_4.$

$$h^{-1}(e_1, e_1, e_2, e_1 + e_1 i) = e_1 + I_1\left[\frac{1}{4}(e_1 - e_2) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_1 + e_2) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_1 - e_2) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_1 + e_2) + \frac{1}{2}e_1 - e_1\right]$$

$= (1,0) + \left(\frac{3}{4}, \frac{-1}{4}\right) I_1 + \left(\frac{-1}{4}, \frac{1}{4}\right) I_2 + \left(\frac{-1}{4}, \frac{-1}{4}\right) I_3 + \left(\frac{-1}{4}, \frac{1}{4}\right) I_4.$

$$h^{-1}(e_1, e_1, e_2, e_2 + e_2 i) = e_1 + I_1\left[\frac{1}{4}(e_1 - e_2) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_1 + e_2) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_1 - e_2) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_1 + e_2) + \frac{1}{2}e_2 - e_1\right]$$

$= (1,0) + \left(\frac{1}{4}, \frac{1}{4}\right) I_1 + \left(\frac{1}{4}, -\frac{1}{4}\right) I_2 + \left(\frac{1}{4}, \frac{-3}{4}\right) I_3 + \left(\frac{-3}{4}, \frac{3}{4}\right) I_4.$

$$h^{-1}(e_1, e_2, e_1, e_1 + e_1 i) = e_1 + I_1\left[\frac{1}{4}(e_2 - e_1) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_2 + e_1) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_2 - e_1) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_2 + e_1) + \frac{1}{2}e_1 - e_1\right]$$

$= (1,0) + \left(\frac{1}{4}, \frac{1}{4}\right) I_1 + \left(\frac{-1}{4}, \frac{1}{4}\right) I_2 + \left(\frac{-3}{4}, \frac{1}{4}\right) I_3 + \left(\frac{-1}{4}, \frac{1}{4}\right) I_4.$

$$h^{-1}(e_1, e_2, e_2, e_1 + e_1 i) = e_1 + I_1\left[\frac{1}{4}(e_2 - e_2) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_2 + e_2) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_2 - e_2) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_2 + e_2) + \frac{1}{2}e_1 - e_1\right]$$

$= (1,0) + I_1\left(\frac{1}{2}, 0\right) + I_2\left(\frac{-1}{2}, \frac{1}{2}\right) + I_3\left(\frac{-1}{2}, 0\right) + I_4\left(\frac{-1}{2}, \frac{1}{2}\right).$

$$h^{-1}(e_1, e_2, e_2, e_2 + e_2 i) = e_1 + I_1\left[\frac{1}{4}(e_2 - e_2) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_2 + e_2) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_2 - e_2) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_2 + e_2) + \frac{1}{2}e_2 - e_1\right]$$

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$= (1,0) + \left(0,\frac{1}{2}\right) I_1 + \left(0,\frac{-1}{2}\right) I_3 + I_4(-1,1).$

$$h^{-1}(e_2, e_1, e_1, e_1 + e_1 i) = e_2 + I_1\left[\frac{1}{4}(e_1 - e_1) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_1 + e_1) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_1 - e_1) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_1 + e_1) + \frac{1}{2}e_1 - e_2\right]$$

$= (0,1) + \left(\frac{1}{2},0\right) I_1 + \left(0,\frac{-1}{2}\right) I_3 + I_4(1,-1).$

$$h^{-1}(e_2, e_2, e_1, e_1 + e_1 i) = e_2 + I_1\left[\frac{1}{4}(e_2 - e_1) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_2 + e_1) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_2 - e_1) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_2 + e_1) + \frac{1}{2}e_1 - e_2\right]$$

$= (0,1) + \left(\frac{1}{4},\frac{1}{4}\right) I_1 + \left(\frac{-1}{4},\frac{1}{4}\right) I_2 + I_3\left(\frac{-3}{4},\frac{1}{4}\right) + I_4\left(\frac{3}{4},\frac{-3}{4}\right).$

$$h^{-1}(e_2, e_2, e_2, e_1 + e_1 i) = e_2 + I_1\left[\frac{1}{4}(e_2 - e_2) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_2 + e_2) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_2 - e_2) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_2 + e_2) + \frac{1}{2}e_1 - e_2\right]$$

$= (0,1) + \left(\frac{1}{2},0\right) I_1 + \left(\frac{-1}{2},\frac{1}{2}\right) I_2 + \left(\frac{-1}{2},0\right) I_3 + \left(\frac{1}{2},\frac{-1}{2}\right) I_4.$

$$h^{-1}(e_2, e_2, e_2, e_2 + e_2 i) = e_2 + I_1\left[\frac{1}{4}(e_2 - e_2) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_2 + e_2) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_2 - e_2) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_2 + e_2) + \frac{1}{2}e_2 - e_2\right]$$

$= (0,1) + \left(0,\frac{1}{2}\right) I_1 + \left(0,\frac{-1}{2}\right) I_3.$

$$h^{-1}(e_1, e_2, e_1, e_2 + e_2 i) = e_1 + I_1\left[\frac{1}{4}(e_2 - e_1) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_2 + e_1) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_2 - e_1) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_2 + e_1) + \frac{1}{2}e_2 - e_1\right]$$

$= (1,0) + \left(\frac{-1}{4},\frac{3}{4}\right) I_1 + \left(\frac{1}{4},\frac{-1}{4}\right) I_2 + \left(\frac{-1}{4},\frac{-1}{4}\right) I_3 + \left(\frac{-3}{4},\frac{3}{4}\right) I_4.$

$$h^{-1}(e_2, e_2, e_1, e_2 + e_2 i) = e_2 + I_1\left[\frac{1}{4}(e_2 - e_1) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_2 + e_1) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_2 - e_1) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_2 + e_1) + \frac{1}{2}e_2 - e_2\right]$$

$= (0,1) + \left(\frac{-1}{4},\frac{3}{4}\right) I_1 + \left(\frac{+1}{4},\frac{-1}{4}\right) I_2 + \left(\frac{-1}{4},\frac{-1}{4}\right) I_3 + \left(\frac{1}{4},\frac{-1}{4}\right) I_4.$

$$h^{-1}(e_2, e_1, e_2, e_1 + e_1 i) = e_2 + I_1\left[\frac{1}{4}(e_1 - e_2) + \frac{1}{2}e_1\right] + I_2\left[\frac{1}{4}(e_1 + e_2) - \frac{1}{2}e_1\right] + I_3\left[\frac{1}{4}(e_1 - e_2) - \frac{1}{2}e_1\right] + I_4\left[\frac{1}{4}(e_1 + e_2) + \frac{1}{2}e_1 - e_2\right]$$

$= (0,1) + \left(\frac{3}{4},\frac{-1}{4}\right) I_1 + \left(\frac{-1}{4},\frac{1}{4}\right) I_2 + \left(\frac{-1}{4},\frac{-1}{4}\right) I_3 + \left(\frac{3}{4},\frac{-3}{4}\right) I_4.$

$$h^{-1}(e_2, e_1, e_2, e_2 + e_2 i) = e_2 + I_1\left[\frac{1}{4}(e_1 - e_2) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(e_1 + e_2) - \frac{1}{2}e_2\right] + I_3\left[\frac{1}{4}(e_1 + e_2) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(e_1 - e_2) + \frac{1}{2}e_2 - e_2\right]$$

$= (0,1) + \left(\frac{1}{4},\frac{1}{4}\right) I_1 + \left(\frac{1}{4},\frac{-1}{4}\right) I_2 + \left(\frac{1}{4},\frac{-1}{4}\right) I_3 + \left(\frac{1}{4},\frac{-3}{4}\right) I_4.$

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$$h^{-1}(e_2, e_1, e_1, e_2 + e_2 i) = e_2 + I_1\left[\frac{1}{4}(0) + \frac{1}{2}e_2\right] + I_2\left[\frac{1}{4}(2e_1) - \frac{1}{2}e_2\right] +$$
$$I_3\left[\frac{1}{4}(0) - \frac{1}{2}e_2\right] + I_4\left[\frac{1}{4}(2e_1) + \frac{1}{2}e_2 - e_2\right]$$
$= (0,1) + \left(0, \frac{1}{2}\right) I_1 + \left(\frac{1}{2}, -\frac{1}{2}\right) I_2 + \left(0, -\frac{1}{2}\right) I_3 + \left(\frac{1}{2}, -\frac{1}{2}\right) I_4.$

So that, the basis of $V_4(I)$ is:

$$\{\left(1 + \frac{1}{2}I_1 - \frac{1}{2}I_3, 0\right), \left(1 + \frac{1}{2}I_2 - \frac{1}{2}I_4, \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 + \frac{1}{2}I_4\right),$$
$$\left(1 + \frac{3}{4}I_1 - \frac{1}{4}I_2 - \frac{1}{4}I_3 - \frac{1}{4}I_4, -\frac{1}{4}I_1 + \frac{1}{4}I_2 - \frac{1}{4}I_3 + \frac{1}{4}I_4\right),$$
$$\left(1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{1}{2}I_4, \frac{1}{2}I_2 + \frac{1}{2}I_4\right), \left(1 + \frac{1}{4}I_1 + \frac{1}{4}I_2 + \frac{1}{4}I_3 - \frac{3}{4}I_4, \frac{1}{4}I_1 - \frac{1}{4}I_2 - \frac{3}{4}I_3 + \frac{3}{4}I_4\right),$$
$$\left(\frac{1}{2}I_1 + I_4, 1 - \frac{1}{2}I_3 - I_4\right), \left(1 + \frac{1}{4}I_1 - \frac{1}{4}I_2 - \frac{3}{4}I_3 - \frac{1}{4}I_4, \frac{1}{4}I_1 + \frac{1}{4}I_2 + \frac{1}{4}I_3 + \frac{1}{4}I_4\right),$$
$$\left(1 - I_4, \frac{1}{2}I_1 - \frac{1}{2}I_3 + I_4\right), \left(\frac{1}{4}I_1 - \frac{1}{4}I_2 - \frac{3}{4}I_3 + \frac{3}{4}I_4, 1 + \frac{1}{4}I_1 + \frac{1}{4}I_2 + \frac{1}{4}I_3 - \frac{3}{4}I_4\right),$$
$$\left(\frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 + \frac{1}{2}I_4, 1 + \frac{1}{2}I_2 - \frac{1}{2}I_4\right), \left(0, 1 + \frac{1}{2}I_1 - \frac{1}{2}I_3\right),$$
$$\left(1 - \frac{1}{4}I_1 + \frac{1}{4}I_2 - \frac{1}{4}I_3 - \frac{3}{4}I_4, \frac{3}{4}I_1 - \frac{1}{4}I_2 - \frac{1}{4}I_3 + \frac{3}{4}I_4\right), \left(-\frac{1}{4}I_1 + \frac{1}{4}I_2 - \frac{1}{4}I_3 + \frac{1}{4}I_4, 1 + \frac{3}{4}I_1 - \frac{1}{4}I_2 - \frac{1}{4}I_3 - \frac{1}{4}I_4\right),$$
$$\left(\frac{3}{4}I_1 - \frac{1}{4}I_2 - \frac{1}{4}I_3 + \frac{3}{4}I_4, 1 - \frac{1}{4}I_1 + \frac{1}{4}I_2 - \frac{1}{4}I_3 - \frac{3}{4}I_4\right), \left(\frac{1}{4}I_1 + \frac{1}{4}I_2 + \frac{1}{4}I_3 + \frac{1}{4}I_4, 1 + \frac{1}{4}I_1 - \frac{1}{4}I_2 - \frac{1}{4}I_3 - \frac{3}{4}I_4\right),$$
$$\left(\frac{1}{2}I_2 + \frac{1}{2}I_4, 1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{1}{2}I_4\right)\}.$$

Definition:

Let $X = x_0 + \sum_{i=1}^{4} x_i I_i$, $Y = y_0 + \sum_{i=1}^{4} y_i I_i \in V_4(I)$ with:

$h(X) = (x_0, \sum_{i=0}^{4} x_i, x_0 + x_4 + x_2 - x_1 - x_3, x_0 + x_4 - x_2 + i(x_1 - x_3)),$

$h(Y) = (y_0, \sum_{i=0}^{4} y_i, y_0 + y_4 + y_2 - y_1 - y_3, y_0 + y_4 - y_2 + i(y_1 - y_3)),$

assume that $\langle , \rangle: V \times V \to \mathbb{R}$ is an inner product on V, we define $\varphi: V_4(I) \times V_4(I) \to R_4(I)$ such that:

$\varphi(X, Y) = g^{-1}(\langle h(X) \rangle, \langle h(Y) \rangle)$; where:

$\langle h(X), h(Y) \rangle = (\langle x_0, y_0 \rangle, \langle \sum_{i=0}^{4} x_i, \sum_{i=0}^{4} y_i \rangle, \langle x_0 + x_2 + x_4 - x_1 - x_3, y_0 + y_2 + y_4 - y_1 - y_3 \rangle, \langle x_0 + x_4 - x_2, y_0 + y_4 - y_2 \rangle + \langle x_1 - x_3, y_1 - y_3 \rangle + i(-\langle x_0 + x_4 - x_2, y_1 - y_3 \rangle + \langle x_1 - x_3, y_0 + y_4 - y_2 \rangle)).$

(φ) is called the 4-cyclic refined inner product.

Example:

For $V = \mathbb{R}^2$, and $X = (1,0) + (1,-1)I_1 + (0,1)I_2 + (1,1)I_3 + (0,1)I_4$,

$Y = (0,1) + (2,1)I_1 + (1,1)I_2 + (-1,1)I_3 + (1,-1)I_4$, we have:

$$\begin{cases} x_0 = (1,0),\ y_0 = (0,1), \langle x_0, y_0 \rangle = 0 \\ \sum_{i=0}^{4} x_i = (3,2), \sum_{i=0}^{4} y_i = (3,3), \langle \sum_{i=0}^{4} x_i, \sum_{i=0}^{4} y_i \rangle = 15 \\ x_0 + x_2 + x_4 - x_1 - x_3 = (1,2) - (2,0) = (-1,2) \\ y_0 + y_2 + y_4 - y_1 - y_3 = (2,1) - (1,2) = (1,-1) \end{cases}$$

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And: $\begin{cases} \langle x_0 + x_2 + x_4 - x_1 - x_3, y_0 + y_2 + y_4 - y_1 - y_3 \rangle = -1 - 2 = -3 \\ x_0 - x_2 + x_4 = (1,0) - (0,1) + (0,1) = (1,0) \\ y_0 - y_2 + y_4 = (0,1) + (1,-1) - (1,1) = (0,-1) \\ x_1 - x_3 = (0,-2) \\ y_1 - y_3 = (3,0) \end{cases}$

Also: $\begin{cases} \langle x_0 - x_2 + x_4, y_0 + y_4 - y_2 \rangle = 0 \\ \langle x_1 - x_3, y_1 - y_3 \rangle = 0 \\ \langle x_1 - x_3, y_0 + y_4 - y_2 \rangle - \langle x_0 + x_4 - x_2, y_1 - y_3 \rangle = 2 - 3 = -1 \end{cases}$

Thus $\varphi(X,Y) = g^{-1}(0,15,-3,0-i) = 0 + I_1\left[\frac{1}{4}(18) - \frac{1}{2}\right] + I_2\left[\frac{1}{4}(12) - \frac{1}{2}(0)\right] + I_3\left[\frac{1}{4}(18) + \frac{1}{2}\right] + I_4\left[\frac{1}{4}(12) + \frac{1}{2}(0) - 0\right] = 4I_1 + 3I_2 + 5I_3 + 3I_4$.

Definition:

Let $m = m_0 + \sum_{i=1}^{4} m_i I_i \in R_4(I)$, we define:

$\bar{m} = g^{-1}\left(m_0, \sum_{i=0}^{4} m_i, m_0 + m_2 + m_4 - m_1 - m_3, m_0 + m_4 - m_2 - i(m_1 - m_3)\right) = g^{-1}(\overline{g(m)})$.

Example:

Consider $m = 1 + 2I_1 + I_2 + I_3 + 2I_4$, then:

$g(m) = (1,7,1,2+i)$, $\overline{g(m)} = (1,7,1,2-i)$

$\bar{m} = g^{-1}(1,7,1,2-i) = 1 + I_1\left[\frac{1}{4}(6) - \frac{1}{2}\right] + I_2\left[\frac{1}{4}(8) - \frac{1}{2}(2)\right] + I_3\left[\frac{1}{4}(6) + \frac{1}{2}\right] + I_4\left[\frac{1}{4}(8) + \frac{1}{2}(2) - 1\right] = 1 + I_1 + I_2 + 2I_3 + 2I_4$.

Remark:

For $m = m_0 + \sum_{i=1}^{4} m_i I_i \in R_4(I)$, we have:

$g^{-1}\left(\overline{g(m)}\right) = g^{-1}\left(m_0, \sum_{i=0}^{4} m_i, m_0 + m_2 + m_4 - m_1 - m_3, m_0 - m_2 + m_4 - i(m_1 - m_3)\right) = m_0 + I_1\left[\frac{1}{4}(2m_1 + 2m_3) - \frac{1}{2}(m_1 - m_3)\right] + I_2\left[\frac{1}{4}(2m_0 + 2m_2 + 2m_4) - \frac{1}{2}(m_0 - m_2 + m_4)\right] + I_3\left[\frac{1}{4}(2m_1 + 2m_3) + \frac{1}{2}(m_1 - m_3)\right] + I_4\left[\frac{1}{4}(2m_0 + 2m_2 + 2m_4) + \frac{1}{2}(m_0 - m_2 + m_4) - m_0\right] = m_0 + m_3 I_1 + m_2 I_2 + m_1 I_3 + m_4 I_4$.

Definition:

Let $m = m_0 + \sum_{i=1}^{4} m_i I_i$, $n = n_0 + \sum_{i=1}^{4} n_i I_i \in R_4(I)$, we define:

$m \geq n$ if and only if:

$$\begin{cases} g(m), g(n) \in R \times R \times R \times R \quad i.e.\ \ m_1 = m_3, n_1 = n_3 \\ m_0 \geq n_0\ ,\ \sum_{i=0}^{4} m_i \geq \sum_{i=0}^{4} n_i \\ m_0 + m_2 + m_4 - 2m_1 \geq n_0 + n_2 + n_4 - 2n_1 \\ m_0 - m_2 + m_4 \geq n_0 - n_2 + n_4 \end{cases}$$

It is easy to check that $(\leq)$ is a partial order relation on $C(R_4(I)) = \{m = m_0 + \sum_{i=1}^{4} m_i I_i \in R_4(I)\ ; m = \bar{m}, i.e.\ m_1 = m_3\}$.

Theorem:

Let $(V_4(I), \varphi)$ be a 4-cyclic refined real inner product space, then:

1] $\varphi(X,X) \in C(R_4(I))$, and $\varphi(X,X) \geq 0$

2] If $\varphi(X,X) = 0$, then $X = 0$

3] $\varphi(X,Y) = \overline{\varphi(Y,X)}$

4] $\begin{cases} \varphi(X+Y,Z) = \varphi(X,Z) + \varphi(Y,Z) \\ \varphi(X,Y+Z) = \varphi(X,Y) + \varphi(X,Z) \end{cases}$

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5] $\begin{cases} \varphi(m \cdot X, Y) = m\,\varphi(X,Y) \\ \varphi(X, m \cdot Y) = \overline{m}\,\varphi(X,Y) \end{cases}$

Where $X, Y, Z \in V_4(I), m \in R_4(I)$.

Proof:

It is similar to 3-cyclic refined case.

Definition:

Let $X, Y \in V_4(I)$, then:

1] $X \perp Y$ if and only if $\varphi(X,Y) = 0$.

2] $\|X\|^2 = \varphi(X,X)$.

Remark:

From the definition of φ, we get:

$$X \perp Y \text{ if and only if:} \begin{cases} x_0 \perp y_0\,, \sum_{i=0}^{4} x_i \perp \sum_{i=0}^{4} y_i \\ x_0 + x_2 + x_4 - x_1 - x_3 \perp y_0 + y_2 + y_4 - y_1 - y_3 \\ \langle x_0 + x_4 - x_2, y_0 + y_4 - y_2 \rangle + \langle x_1 - x_3, y_1 - y_3 \rangle = 0 \\ \langle x_1 - x_3, y_0 + y_4 - y_2 \rangle = \langle x_0 + x_4 - x_2, y_1 - y_3 \rangle \end{cases}$$

Also, if $X \perp Y$, then: $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$.

Remark:

$$\|X\| = g^{-1}\left(\|x_0\|, \left\|\sum_{i=0}^{4} x_i\right\|, \|x_0 + x_4 + x_2 - x_1 - x_3\|, \sqrt{\|x_0 + x_4 - x_2\|^2 + \|x_1 - x_3\|^2}\right)$$

For example, take:

$X = (1,1) + (1,0)I_1 + (1,0)I_2 + (-1,3)I_3 + (0,1)I_4$, we can see:

$$x_0 = (1,1), \|x_0\| = \sqrt{2}\,, \sum_{i=0}^{4} x_i = (2,5), \left\|\sum_{i=0}^{4} x_i\right\| = \sqrt{29}\,,$$

$x_0 + x_4 + x_2 - x_1 - x_3 = (2,2) - (0,3) = (2,-1), \|x_0 + x_4 + x_2 - x_1 - x_3\| = \sqrt{5}\,,$

$x_0 + x_4 - x_2 = (1,2) - (1,0) = (0,2), \|x_0 + x_4 - x_2\|^2 = 4\,, x_1 - x_3 = (2,-3), \|x_1 - x_3\|^2 = 13\,,$

$$\|X\| = g^{-1}(\sqrt{2}, \sqrt{29}, \sqrt{5}, \sqrt{17}) = \sqrt{2} + I_1\left[\frac{1}{4}(\sqrt{2} - \sqrt{5}) + \frac{\sqrt{17}}{2}\right] + I_2\left[\frac{1}{4}(\sqrt{29} + \sqrt{5}) - \frac{\sqrt{5}}{2}\right] + I_3\left[\left(\frac{\sqrt{29}-\sqrt{5}}{4}\right) - \frac{\sqrt{17}}{2}\right] + I_4\left[\left(\frac{\sqrt{29}+\sqrt{5}}{4}\right) + \frac{\sqrt{5}}{2} - \sqrt{2}\right].$$

3. Conclusion

In this paper, we studied the 4-cyclic refined vector spaces, where we classified these spaces by using semi-module isomorphisms as direct product of classical complex vector spaces. In addition, we studied the inner products defined over these structures and we present sufficient conditions for 4-cyclic refined orthogonality.

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