Computational Approaches for Nonlinear Fractional Differential Problems Utilizing Chebyshev Polynomial Approximations Space with Neutrosophic Applications

Abdulsalam Al-Dulaimi[1,*], Amirah Azmi[1], Yaseen S. R.[2]
[1]School of Mathematical Sciences, University Sains Malaysia, 11800 Penang, Malaysia
[2]Tikrit university, College of Education for Pure Science, Tikrit, Iraq
Emails: salamdulaimi@student.usm.my; amirahazmi@usm.my; samer2017@tu.edu.iq

Abstract

Applying Chebyshev polynomial approximate results, this paper applies the idea of neutrophilic logic to the approach to partially differential equations (FPDEs). Three elements make up the Neutrosophic technique: Indeterminacy (I), Falsehood (F), and Truth (T). These three elements are appropriate for issues where precise values or distinct limits are lacking since they are utilized to represent ambiguity, vagueness, and imperfect truth in mathematical models. We improve the depiction of real-world occurrences that could contain unclear or ambiguous information by adding these values to the coefficients of FPDEs. In domains like material science, mechanical engineering, and biological phenomena, where uncertainty is inevitable, the use of neutrophilic logic enables a more thorough and precise approximation of approaches to complicated fractional differential equations. The findings show that when working with systems that have unknown characteristics, the Neutrosophic technique increases the accuracy and dependability of computations.

Keywords: Chebyshev Polynomial; Caputo derivatives; Neutrosophic applications; Fractional differential issues

1. Introduction

Fractional differential equations used in many branches of sciences, Mathematics, physics, chemistry and engineering. Debnath and Bhatta [1-2], gave the idea of fractional derivatives and fractional integrals with their basic properties. Several methods including the Laplace transform are discussed in introducing the Riemann-Liouville fractional integral. They used the fractional derivative to solve the celebrated integral equation. Viscoelasticity and other related phenomena, have great importance in the study of mechanical properties of material especially, biological materials. Certain materials show some complex effects in mechanical tests, which cannot be described by standard linear equation (SLE) mostly owing to shape memory effect during deformation. Recently, researchers have been applying fractional calculus in order to probing viscoelasticity of such materials with a high precision. Fractional calculus is a powerful tool for modeling complex phenomenon.

Other applications in the field of signal processing are discussed in the publication [3]. Several newer results can be found e.g., in [4-5]. Due to difficulty for us to solve most of linear and nonlinear fractional partial differential equations exactly, thus the approximation solutions have been used, such as the Adomian decomposition method (ADM) [6], the homotopy perturbation method (HPM) [7], the homotopy analysis method (HAM) [8], the variational iteration method (VIM) [9], a new iterative method (NIM) [10] and the differential quadrature method [11]. Some authors go to complain the linear transformation and an approximate method to solve a wide class of nonlinear fractional partial differential equations, such as in [12], the authors have applied the new Sumudu transform iterative method to get the approximate solution of the time- fractional Cauchy reaction-diffusion equation. Also, in [13], it has been used the iterative method to solve linear and nonlinear initial values problems fractional transport and diffusion- wave equations. The combination of the Elzaki transform and homotopy perturbation method [14] has been used to solve linear or nonlinear system partial differential equations, The Adomian de- composition coupled with the Sumudu transformation [15], were used to get the approximate solutions for more types of nonlinear time-fractional partial differential equations.

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Recently, many authors have shown that a new technique based on the invariant subspace provides an effective rule to find the exact solution of wide class of (FDEs), the advanced of this method is separation the variable of the differential equation. This method was initially proposed by [16-17]. Later, [18] were developed the invariant subspace method.

In (2016) [19], showed how the invariant subspace method could be extended to time fractional partial differential equations (FPDEs) and could construct their exact solutions.

$$D_t^{\alpha} u = F[u].$$

Where $F[.]$ is a nonlinear differential operator, $D_t^{\alpha}$is a fractional time derivative in the Caputo sense. In (2016) [20] developed the invariant subspace method for deriving exact solutions of partial differential equations with fractional space and time derivatives.

$$\sum_{j=0}^{n} \lambda_j D_t^{\alpha+j} u(x,t) = N\left(x, u, D_x^{\beta} u, D_x^{\beta+1} u, \cdots, D_x^{\beta+m} u\right)$$

All fractional partial derivatives are in Caputo sense, and $N[u]$ is a linear - nonlinear operator and $\alpha, \beta \in (0,1], m, n \in N$.

In 2017 [21], used the inverse differential operational method to obtain solutions for differential equations with mixed derivatives of physical problems. The Sumudu transform which can be used to solve the ordinary and partial differential equations with ordinary and fractional order. Sumudu transform has some advantages over the Laplace transform, such as [22]: the function and the Sumudu transform have the same physical units to measure, the constant function and its Sumudu transform are the same. The similarity between the domain of the function and that for its Sumudu transform. Elzaki, T. and Elzaki, S. [23-25] introduced a motivation of the Elzaki transform and used it to solve ordinary and partial differential equations as well for equations which cannot be solved by Sumudu transform.

Neutrosophic logic addresses the truth, indeterminacy, with untruth of information and is a development of fuzzy and classical reasoning. Neutrosophic logic, which was first presented by Smarandache in 1995, is intended to represent knowledge that is ambiguous, imprecise, and uncertain. By combining three elements—truth, indeterminacy, and falsehood—neutrosophic logic permits a more nuanced representation of details than classical logic, which only considers truth values (true or false). These elements are crucial for solving real-world issues when information is prone to volatility or assurance is frequently absent. Neutrosophic logic has been used in a number of domains, such as engineering, artificial intelligence, and decision-making, particularly when the system parameters are erratic or unclear [26-28] also the concept of intuitionistic topological spaces, topological graph space, neutrosophic topological groups, and types of weakly neutrosophic crisp open mappings [29-30].

The main goal of this study is to propose a new method by extending this method or modifying it or ingot with other methods to find the approximate solutions of fractional partial differential equations using concept of Neutrosophic logic. Based on our knowledge, this approach has not been presented before, and that due to complexity of the formula of these equations.

2. Basic Concepts

Definition 2.1 [5]: The Gamma function is a generalization of the factorial function $n!$, i. e.,

$$\Gamma(n) = (n-1)! \quad \text{for } n \in \mathcal{N} \tag{1}$$

For complex arguments with positive real part, it is defined as:

$$\Gamma(\mathrm{z}) = \int_0^{\infty} x^{\mathrm{z}-1} e^{-x} dx, \ \mathrm{Re}\, \mathrm{z} > 0. \tag{2}$$

We listed here some of its properties:

1. $\Gamma(\mathrm{z}+1) = \mathrm{z}\Gamma(\mathrm{z})$
2. $\Gamma(1/2) = \sqrt{\pi}$
3. $\Gamma(n+1/2) = \frac{(2n)!\sqrt{\pi}}{4^n n!} n \in N.$

Definition 2.2 [5]: The Beta function is defined by the definite integral:

$$B(\mathrm{z}, w) = \int_0^1 x^{\mathrm{z}-1}(1-x)^{w-1} dx, \ \ \mathrm{Re}\, \mathrm{z} > 0, \mathrm{Re}\, w > 0. \tag{3}$$

The Beta function can also be defined in terms of Gamma function as:

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$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \quad (4)$$

With the Beta function one can obtain two useful results for the Gamma function

1. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$
2. $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}2^{2z-1}\Gamma(2z)$

Definition 2.3 (Mittag-Leffler) [5]: This function it is a generalization of the exponential function, which is defined as one parameter by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha > 0, z \in \mathbb{C}. \quad (5)$$

Then, the two parameters generalization has been introduced by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha, \beta > 0, z \in \mathbb{C}. \quad (6)$$

Some special cases of the Mittag-Leffler function can be derive as [5]

- $E_0(x) = 1,$
- $E_1(x) = e^x, \ (1-x),$
- $E_2(-x^2) = \cos x,$
- $E_{1,0}(x) = xe^x,$
- $xE_{1,2}(x) = e^x - 1,$
- $xE_{2,2}(x^2) = \sinh x,$
- $E_2(x^2) = \cosh x,$
- $xE_{2,2}(-x^2) = \sin x.$

Using Mittag-Leffler properties, we can derive the following generalized fractional trigonometric functions

$$\begin{aligned} x^\beta E_{2\beta,\beta+1}(-x^{2\beta}) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\beta}}{\Gamma((2k+1)\beta+1)} = \sin_\beta(x), \\ E_{2\beta}(-x^{2\beta}) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\beta}}{\Gamma(2k\beta+1)} = \cos_\beta(x), \\ x^\beta E_{2\beta,\beta+1}(x^{2\beta}) &= \sum_{k=0}^{\infty} \frac{x^{(2k+1)\beta}}{\Gamma((2k+1)\beta+1)} = \sinh_\beta(x), \\ E_{2\beta}(x^{2\beta}) &= \sum_{k=0}^{\infty} \frac{x^{2k\beta}}{\Gamma(2k\beta+1)} = \cosh_\beta(x), \end{aligned} \quad (7)$$

note that with $\beta = 1$, then we have the standard case.

Since the series of the Mittag-Leffler function is uniformly convergent on all compact subset of $\mathbb{C}$ we can differentiate it term by term to get

$$E_{\alpha,\beta}^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!x^k}{k!\Gamma(\alpha k+\alpha n+\beta)}, \ \alpha, \beta > 0, x \in \mathbb{C}, n \in \mathbb{N} \quad (8)$$

Lemma 2.4 [2]: If $\alpha, \beta \in \mathbb{C}\mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, and $n \in \aleph$, then

$$x^n E_{\alpha,\beta+n\alpha}(x) = E_{\alpha,\beta}(x) - \sum_{k=0}^{n-1} \frac{x^k}{\Gamma(\beta+\alpha k)} \quad (9)$$

In special case $n = 1$,

$$E_{\alpha,\beta}(x) = xE_{\alpha,\beta+\alpha}(x) + \frac{1}{\Gamma(\beta)}. \quad (10)$$

Definition 2.5 [5]: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function f is defined as

$$J^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt & \alpha > 0, \ x > 0, \ \alpha, x \in R \\ f(x) & \alpha = 0 \end{cases} \quad (11)$$

Where $\Gamma(.)$ is the well-known Gamma function.

Properties:

1. J^α is a linear operator i.e $J^\alpha(af(x) + g(x)) = aJ^\alpha f(x) + J^\alpha g(x)$
2. $\lim_{\alpha\to 0} J^\alpha f(x) = f(x)$ for $f(x)$ is a continuous function for $x \geq 0$

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3. $J^{\alpha}J^{\beta}f(x) = J^{\alpha}J^{\beta}f(x) = J^{\beta+\alpha}f(x)$

4. $J^{\alpha}x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}x^{\alpha+\beta}$

Definition 2.6 [5]: The Riemann-Liouville fractional differential operator of order $\alpha > 0$ of a function f is defined as:

$$\mathcal{D}^{\alpha}f(x) = D^{n}J^{n-\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt, & n-1<\alpha<n \in \mathcal{N} \\ f^{(n)}(x), & \alpha = n \in \mathcal{N} \end{cases} \tag{12}$$

for $x > a$, and $\alpha, a, x \in R$.

Equation (12) has the following important properties:

(i) $\mathcal{D}^{\alpha}(af(x)+bg(x)) = a\mathcal{D}^{\alpha}f(x) + b\mathcal{D}^{\alpha}g(x)$.

(ii) $\mathcal{D}^{\alpha}x^{r} = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}x^{r-\alpha}$

(iii) $\mathcal{D}^{\alpha}c = \frac{cx^{-\alpha}}{\Gamma(1-\alpha)}$, c is constant

(iv) $\mathcal{D}^{\alpha}J^{\alpha} = \mathcal{D}^{0} = I$. The operator (12) is the left inverse of the operator (11).

More generally $\mathcal{D}^{\alpha}J^{\beta}f(x) = \mathcal{D}^{\alpha-\beta}f(x)$ for all $\beta > 0$ and if $\alpha < \beta$, then $\mathcal{D}^{\alpha-\beta}f(x) = J^{\beta-\alpha}f(x)$

(v) $J^{\beta}\mathcal{D}^{\alpha}f(x) = \mathcal{D}^{\alpha-\beta}f(x) - \sum_{k=1}^{n}\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)}\mathcal{D}^{\alpha-k}f(0)$

(vi) $J^{n}\mathcal{D}^{n}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{x^k}{k!}f^{(k)}(0)$

(vii) For $n-1<\alpha<n, m-1<\beta<m, n, m, r \in \mathcal{N}$

(a) $\mathcal{D}^{\alpha}\mathcal{D}^{\beta}f(x) = \mathcal{D}^{\alpha+\beta}f(x) - \sum_{k=1}^{m}\frac{x^{-\alpha-k}}{\Gamma(1-\alpha-k)}\mathcal{D}^{\beta-k}f(0)$,

(b) $\mathcal{D}^{r}\mathcal{D}^{\alpha}f(x) = \mathcal{D}^{r+\alpha}f(x) \neq \mathcal{D}^{\alpha}\mathcal{D}^{r}f(x)$

This equality holds when $f^{(i)}(0) = 0,\ i = 0,1,2,\cdots,r$.

Lemma 2.7 [2]: Let $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with positive real parts, then

$$\mathcal{D}^{\gamma}\left[(x-a)^{\beta-1}E_{\alpha,\beta}(\lambda(x-a)^{\alpha})\right] = (x-a)^{\beta-1-\gamma}E_{\alpha,\beta-\gamma}(\lambda(x-a)^{\alpha})$$

In particular, $\gamma = \alpha, \beta = 1$, we have

$$\mathcal{D}^{\alpha}E_{\alpha}(\lambda(x-a)^{\alpha}) = (x-a)^{-\alpha}E_{\alpha,1-\alpha}(\lambda(x-a)^{\alpha}) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} + \lambda E_{\alpha}(\lambda(x-a)^{\alpha})$$

Definition 2.8 [5]: The fractional derivative of $f(x)$ in the Caputo derivative is defined as:

$$D^{\alpha}f(x) = J^{n-\alpha}D^{n}f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt, & n-1<\alpha<n \in \mathcal{N} \\ f^{(n)}(x), & \alpha = n. \end{cases} \tag{13}$$

for $x > a$, and $\alpha, a, x \in R$.

Definition 2.9 [5]: The transform

$$g(\alpha) = \int_a^b f(x)K(\alpha,x)dx \tag{14}$$

is called the integral transform for the function $f(x)$ and $K(\alpha,x)$ is called the kernel of transform.

Many types of transforms we will have by changing the kernel of transform, such as:

- if $K(\alpha,x) = e^{-\alpha x}$ it is known as Laplace transform

$$L(\alpha) = \int_0^{\infty} f(x)e^{-\alpha x}dx$$

$$H(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-\alpha}dx \tag{15}$$

- if $K(\alpha,x) = \frac{1}{\alpha}e^{\frac{-x}{\alpha}}$ it is known as Sumudu transform

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$$S(\alpha) = \frac{1}{\alpha}\int_0^\infty f(x)e^{\frac{-x}{\alpha}}dx \quad (16)$$

- if $K(\alpha, x) = \alpha e^{\frac{-x}{\alpha}}$ it is known as Elzaki transform

$$E(\alpha) = \alpha \int_0^\infty f(x)e^{\frac{-x}{\alpha}}dx \quad (17)$$

Definition 2.10 [24]: For a function $f(x)$ which is of exponential order

$$|f(x)| < \begin{cases} Me^{t/\tau_1} & \text{if} t \le 0 \\ Me^{t/\tau_2} & \text{if} t > 0, \end{cases} \quad (18)$$

the Sumudu transform is defined by

$$S\{f(x)\} = T(u) = \frac{1}{u}\int_0^\infty e^{\frac{-x}{u}}f(x)dx = \int_0^\infty e^{-x}f(ux)dx,\ \tau_1 \le u \le \tau_2 \quad (19)$$

where M is a real finite number and τ_1 and τ_2 can be finite or infinite.

Definition 2.11 [24]: The Sumudu transform of the α-fractional Caputo derivative is defined by

$$S\{D^\alpha f(x)\} = u^{-\alpha}\left[T(u) - \sum_{i=0}^{n-1} u^i f^{(i)}(0)\right],\ \lceil \alpha \rceil = n, n \in N \quad (20)$$

Definition 2.12 [24]: The double Sumudu transform for the function $f(t, x)$ is given by

$$S_2\left\{f(x,t\right\} = T(u,v) = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}f(x,t)dtdx \quad (21)$$

The important properties of double Sumudu transform which are needed:

1. $S_2\{f(ax)g(bt)\} = T(au)G(bv)$, where $G(bv) = S_2 g(bt)$
2. $S_2\{I_x^\alpha f(x,t)\} = u^\alpha T(u,v)$

Theorem 2.13 [24]: If the double Sumudu transform of the function $f(x, t)$ is given by $S_2[f(x,t)] = T(u,v)$, then

$$S_2[x^n f(x,t)] = u^n \sum_{k=0}^{n} a_k^n u^k D_u^k T(u,v),\ n \in N \quad (22)$$

Where $a_0^n = 1, a_n^n = 1, a_1^n = n!\,n, a_{n-1}^n = n^2$, furthermore $a_k^n = a_{k-1}^{n-1} + (n+k)a_k^{n-1}$.

Definition 2.14 [25]: For a function $f(x)$ which is of exponential order

$$|f(x)| < \begin{cases} Me^{t/\tau_1} & if\ t \le 0 \\ Me^{t/\tau_2} & if\ t > 0 \end{cases}$$

the Elzaki transform is defined by

$$E\{f(x)\} = G(u) = u\int_0^\infty e^{\frac{-x}{u}}f(x)dx$$

$$= u^2 \int_0^\infty e^{-x}f(ux)dx,\ u \in (\tau_1, \tau_2), \quad (23)$$

where M is a real finite number and τ_1 and τ_2 can be finite or infinite.

Definition 2.15 [25]: The double Elzaki transform for the function $f(t, x)$ is given by

$$E_2\{f(x,t)\} = G(u,v) = uv\int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}f(x,t)dtdx \quad (24)$$

where $x, t > 0$ and u, v are transform variables for x and t, respectively.

So, by definitions of Sumudu and Elzaki transforms, we can conclude their duality with the Laplace transform $F(p, s)$ of the function $f(x, t)$ as:

$$G(u,v) = uvF\left(\frac{1}{u},\frac{1}{v}\right),\ T(u,v) = \frac{1}{uv}F\left(\frac{1}{u},\frac{1}{v}\right),$$

$$G(u,v) = u^2v^2T(u,v) \quad (25)$$

Double Elzaki transform of fractional Caputo derivatives can be derived as follows:

$$E_2\{D_x^\alpha f(x,t)\} = u^{-\alpha}\left[G(u,v) - \sum_{i=0}^{n-1} u^{i+2}G_i(0,v)\right]$$

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$$E_2\left\{D_t^{\beta} f(x,t)\right\} = v^{-\beta}\left[T(u,v) - \sum_{j=0}^{m-1} v^{j+2} G_j(u,p)\right]$$

$$E_2\left\{D_t^{\beta} D_x^{\alpha} f(x,t)\right\} = u^{-\alpha} v^{-\beta}\left[G(u,v) - \sum_{i=0}^{n-1} u^{i+2} G_i(0,v) - \sum_{j=0}^{m-1} v^{j+2} G_j(u,0) + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} v^{j+2} u^{i+2} \frac{\partial^{i+j}}{\partial t^j \partial x^i} f(0,0)\right] \quad (26)$$

Where $G_i(0,v) = E_2\left\{\frac{\partial^i}{\partial x^i} f(0,t)\right\}$ and $G_j(u,0) = E_2\left\{\frac{\partial^j}{\partial t^j} f(x,0)\right\}$.

3. Analytical solutions on models

Definition 3.1: Let $(\mathcal{N}, \|.\|)$ be a normed space and $\mathcal{S}$ a closed subset of $\mathcal{N}$, we can define the following set $\mathcal{P}(\mathcal{S},n) = inf\{m \in \mathcal{S}; \|m - n\|\}$. Then $\mathcal{S}$ is called proximinal set in $\mathcal{N}$ if $\mathcal{P}(n,\mathcal{S})$ is non-empty and its Chebyshev if $\mathcal{P}(n,\mathcal{S})$ includes just one point. Many research has been complete determine the proximality of closed subset of metric space. Its doesn't our interest to search on this problematic. The characterization of the norm function in conditionally of proximality and Chebyshev of some subspace in $\mathcal{N}$ have been considered.

In this part, some necessary mathematical conditions for some classes of multi fractional differential equations have been presented and studied in details.

Problem Formulation 3.1. Suppose the following system

$$\sum_{j=0}^{k}\left(\mu_j(t) D_t^{\alpha+j} - f_j(x) D_x^{2(j+\beta)}\right)\left(\hat{\mu}_j(t) D_t^{\alpha+j+1} - g_j(t) D_x^{2(j+\beta)}\right)\mu = H(\mu) \quad (27)$$

$$D_i^m \mu(x,0) = W_m(x), m = 0,1,2,\dots,[2(K+\beta)] \quad (28)$$

i. $\mu_j(t), \hat{\mu}_j(t), g_j(t)$ variable functions of t as a coefficients
ii. $f_j(x)$ variable function of x as a coefficients
iii. Fractional partial Caputo derivative $D_t^{\infty+j}, D_t^{\infty+j+1}$
iv. $H(\mu) = H\left(x, \mu, D_t^{\alpha+j}, D_t^{\alpha+j+1}, D_x^{2(j+\beta)}\right)$ is nonlinear function of mixed typed partial derivatives.

The following nonlinear fractional order partial is equivalent to (27), (28) super formulation.

$$\sum_{j=0}^{k} \mu_j(t) D_t^{\alpha+j}\left(\hat{\mu}_j(t) D_t^{\alpha+j+1}\mu\right)$$

$$= \sum_{j=0}^{k}\left[\left(\mu_j(t) g_j(t) D_t^{\alpha+j} + \hat{\mu}_j(t) f_j(x) D_t^{\alpha+j+1}\right) D_x^{2(j+\beta)}\mu\right]$$

$$= \sum_{j=0}^{k}\left[f_j(x) g_j(t) D_x^{2(j+\beta)} D_x^{2(j+\beta)}\mu\right] + H(\mu) \quad (29)$$

and in the operator form $\sum_{j=0}^{k} \mu_j(t) D_t^{\alpha+j}\left(\hat{\mu}_j(t) D_t^{\alpha+j+1}\mu\right)$

$$= \sum_{j=0}^{k}\left[\left(\mu_j(t) g_j(t) D_t^{\alpha+j} + \hat{\mu}_j(t) f_j(x) D_t^{\alpha+j+1}\right) F_j[\mu]\right] - N[\mu] + H(\mu)$$

Where $N[\mu] = \sum_{j=0}^{k}\left[f_j(x) g_j(t) D_x^{2(j+\beta)} D_x^{2(j+\beta)}\mu\right]$ and $F_j[\mu] = D_x^{2(j+B)}\mu$.

Theorem 3.2. Let $I_{n+1} = L\{\Phi_0(x), \Phi_1(x), \dots, \Phi_n(x)\}$ be a finite dimensional linear space which is invariant under the operators $N[\mu]$, $F_j[\mu]$ and $H[\mu]$, then the nonlinear fractional order super PDE with variable functions coefficients has the approximation solution in the form

$$\mu(x,t) = \sum_{i=0}^{k} k_i(t)\Phi_i(t) \approx \sum_{i=0}^{n}\sum_{w=0}^{p} a_{iw} T_w(t)\Phi_i(x)$$

$$= \sum_{i=0}^{n} A_i^T \Phi(t)\Phi_i(x) \quad (30)$$

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Where A_i^T is a $p+1$ vector of constants and $\Phi(t)$ is $p+1$ vector of approximated shifted first kind of Chebyshev polynomials such that $A_i^T\ \Phi(t)$ satisfies the following system of approximated equivalent to nonlinear fractional super PDE (27).

$$\sum_{j=0}^{k}\mu_j(t)\, D_t^{\alpha+j}\big(\widehat{\mu_j}(t)\sum_{i=0}^{n}A_i^T\Delta^{\alpha+j}\Phi(t)\big)=$$

$$\sum_{j=0}^{k}\Bigg[\mu_j(t)g_j(t)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha+j}\Phi(t),A_1^T\Delta^{\alpha+j}\Phi(t),\dots,A_n^T\Delta^{\alpha+j}\Phi(t)\right)+\hat{\mu}_j(t)f_j(x)\sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\left(A_0^T\Delta^{\alpha+j+1}\Phi(t),A_1^T\Delta^{\alpha+j+1}\Phi(t),\dots,A_n^T\Delta^{\alpha+j+1}\Phi(t)\right)\Bigg]$$

$$-\sum_{i=0}^{n}\Psi_i(A_0^T\Phi(t),A_1^T\Phi(t),\dots,A_n^T\Phi(t))+\sum_{i=0}^{n}\Psi_{(j+2)n+1+i}(A_0^T\Phi(t),A_1^T\Phi(t),\dots,A_n^T\Phi(t))\ ,\ n=\ 1,2,\dots,j$$

where

$\{\Psi_0,\Psi_1,\dots,\Psi_n\};\{\Psi_{jn+1},\Psi_{jn+2},\dots,\Psi_{jn+1+n}\},\{\Psi_{(j+1)(n+1),}\Psi_{(j+1)(n+1)+1},\dots,\Psi_{(j+1)(n+1)+n}\}$

are the expansion coefficient of the operator $N[\mu]$, $F_j[\mu]$ and $H[\mu]$ respectively to the base $\{\Phi_0(x),\Phi_1(x),\dots,\Phi_n(x)\}$

Substitute (30) in the left side of (27) and using linearity of Caputo fractional derivatives, we have

$$\sum_{j=0}^{k}\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)D_t^{\alpha j}\mu\right)=\sum_{j=0}^{k}\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)D_t^{\alpha j}\sum_{i=0}^{n}A_i^T\Phi(t)\Phi_i(x)\right)$$

$$=\sum_{j=0}^{k}\left[\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)\sum_{i=0}^{n}A_i^T\Delta^{\alpha j}\Phi(t)\Phi_i(x)\right)\right] \tag{31}$$

As I_{n+1} be invariant subspace with respect to the operator $N[\mu]$, $F_j[\mu]$ and $H[\mu]$ the exist $(n+1)(j+2)$ functions $\{\Psi_0,\Psi_1,\dots,\Psi_n\};\{\Psi_{jn+1},\Psi_{jn+2},\dots,\Psi_{jn+n}\};\{\Psi_{(j+1)n+1,}\Psi_{(j+1)n+2},\dots,\Psi_{(j+1)n+1+n}\}$,$j=\ 1,\dots,k$ such that:

$$N[\mu]=N\big(\sum_{i=0}^{k}k_i(t)\Phi_i(x)\big)=\sum_{i=0}^{n}\Psi_i\big(k_0(t),k_i(t),\dots,k_n(t)\big)\,\Phi_i(x) \quad (4.6)$$

$$F_j[\mu]\ =\ F_j\left(\sum_{i=0}^{n}k_i(t)\Phi_i(x)\right)=\sum_{i=0}^{n}\Psi_{jn+1+i}\big(k_0(t),k_1(t),\dots,k_n(t)\big)\,\Phi_i(x)$$

$$=\sum_{i=0}^{n}\Psi_{jn+1+i}\left(\sum_{w=0}^{p}a_{0w}\,T_w(t),\sum_{w=0}^{p}a_{1w}\,T_w(t),\dots,\sum_{w=0}^{p}a_{nw}\,T_w(t)\right)\Phi_i(x)$$

$$=\sum_{i=0}^{n}\Psi_{jn+1+i}(A_0^T\Phi(t),A_1^T\Phi(t),\dots,A_n^T\Phi(t))\Phi_i(x) \tag{32}$$

$$H[\mu]=H\left(\sum_{i=0}^{n}k_i(t)\Phi_i(x)\right)=\sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\big(k_0(t),k_1(t),\dots,k_n(t)\big)\,\Phi_i(x)$$

$$=\sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\left(\sum_{w=0}^{p}a_{0w}\,T_w(t),\sum_{w=0}^{p}a_{1w}\,T_w(t),\dots,\sum_{w=0}^{p}a_{nw}\,T_w(t)\right)\Phi_i(x)$$

$$=\sum_{i=0}^{n}\Psi_{(j+1)+n+1+i}(A_0^T\Phi(t),A_1^T\Phi(t),\dots,A_n^T\Phi(t))\Phi_i(x) \tag{33}$$

Substitute (30), (31), (32), (33) in (27) we get:

$$\sum_{k=0}^{n}\mu_j(t)\,D_t^{\alpha+j}\big(\widehat{\mu_j}(t)\sum_{i=0}^{n}A_i^T\Delta^{\alpha+j}\Phi(t)\big)\Phi_i(x)=$$
$$\sum_{j=0}^{k}\Big[\mu_j(t)g_j(t)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha+j}\Phi(t),A_1^T\Delta^{\alpha+j}\Phi(t),\dots,A_n^T\Delta^{\alpha+j}\Phi(t)\right)\Phi_i(x)+$$
$$\widehat{\mu_j}(t)f_j(x)\sum_{i=0}^{n}\Psi_{j+n+1+i}\left(A_0^T\Delta^{\alpha+j+1}\Phi(t),A_1^T\Delta^{\alpha+j+1}\Phi(t),\dots,A_n^T\Delta^{\alpha+j+1}\Phi(t)\right)\Big]\Phi_i(x)$$

$$-\ \sum_{i=0}^{n}\Psi_i\big(A_0^T\Phi(t),A_1^T\Phi(t),\dots,A_n^T\Phi(t)\big)\,\Phi_i(x)+$$
$$-\ \sum_{i=0}^{n}\Psi_{(j+1)n+1+i}(A_0^T\Phi(t),A_1^T\Phi(t),\dots,A_n^T\Phi(t))\Phi_i(x) \tag{34}$$

Thus, since, $\Phi_i(x)$, $i=\ 0,1,\dots,n$ are linearly independent, then, we have the following variable coefficients fractional ordinary system.

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$$\sum \mu_j(t) D_t^{\alpha+j}\left(\widehat{\mu_j}(t)\sum_{i=0}^{n} A_i^T \Delta^{\alpha+j}\Phi(t)\right)$$

$$= \sum_{j=0}^{k}\Bigg[\mu_j(t) g_j(t)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha+j}\Phi(t), A_1^T\Delta^{\alpha+j}\Phi(t),\dots,A_n^T\Delta^{\alpha+j}\Phi(t)\right)$$

$$+\widehat{\mu_j}(t) f_j(x)\sum_{i=0}^{n}\Psi_{j+n+1+i}\left(A_0^T\Delta^{\alpha+j+1}\Phi(t), A_1^T\Delta^{\alpha+j+1}\Phi(t),\dots,A_n^T\Delta^{\alpha+j+1}\Phi(t)\right)\Bigg]$$

$$-\textstyle\sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\left(A_0^T\Phi(t), A_1^T\Phi(t),\dots,A_n^T\Phi(t)\right) \quad (35)$$

Also, the initial conditional (26) becomes

$$D_i^m\mu(x,0) = \textstyle\sum_{i=0}^{n} k_i^{(m)}(0)\Phi_i(x) = \sum_{i=0}^{n} A_i^T\Phi^m(0)\,\Phi_i(x) = f_m(x)$$

$$m = 0,1,\dots,[2(\alpha+j)-1] \quad (36)$$

Problem Formulation 3.3. Consider the following nonlinear fractional order super partial differential equation.

$$\sum_{j=0}^{k}\left(\mu_j(t) D_t^{\alpha j} - f_j(x) D_x^{2(j\beta)}\right)\left(\widehat{\mu_j}(t) D_t^{\alpha j} - g_j(t) D_x^{2(j\beta)}\right)\mu = H(\mu) \quad (37)$$

$$D_t^m(x,0) = W_m(x), m = 0,1,2,\dots,[\alpha k] \quad (38)$$

Where

(i) $\mu_j(t)$, $\widehat{\mu_j}(t)$, $g_j(t)$ variable function of t as a coefficient.

(ii) $H(\mu) = H\left(x,\mu, D_t^{\alpha j}\mu, D_x^{2(j\beta)}\mu, D_t^{2(j\beta)}\right)$ is nonlinear function of mixed type of partial derivative.

The following nonlinear fractional order super partial is equivalent to (37) and (38)

$$\sum_{j=0}^{k}\mu_j(t)\, D_t^{\alpha j}\left(\widehat{\mu_j}(t) D_t^{\alpha j}\mu\right) = \sum_{j=0}^{k}\left[\mu_j(t) g_j(t) D_t^{\alpha j} + \left(\widehat{\mu_j}(t) f_j(t) D_t^{\alpha j}\right) D_x^{2j\beta}\mu\right]$$

$$-\textstyle\sum_{j=0}\left(f_j(x) g_j(t) D_x^{2j\beta} D_x^{2j\beta}\right) + H(\mu) \quad (39)$$

and in the operator form

$$\textstyle\sum_{j=0}^{k}\mu_j(t)\, D_t^{\alpha j}\left(\widehat{\mu_j}(t) D_t^{\alpha j}\mu\right) = \sum_{j=0}^{k}\left[\mu_j(t) g_j(t) D_t^{\alpha j} + \mu_j(t) f_j(x) D_t^{\alpha j}\right]$$

$$F_j[\mu] - N[\mu] + H[\mu]$$

Where $N[\mu] = \sum_{j=0}^{k}\left[f_j(x) g_j(t) D_x^{2j\beta} D_x^{2j\beta}\mu\right]$ and $F_j[\mu] = D_x^{2j\beta}\mu$.

Theorem 3.4. Let $I_{n+1} = L\{\Phi_0(x),\Phi_1(x),\dots,\Phi_n(x)\}$ be a finite dimensional linear space which is invariant under the operators $N[\mu]$, $F_j[\mu]$ and $H[\mu]$, then the nonlinear fractional order super PDE with variable functions coefficients (37) has the approximation solution in the form

$$\mu(x,t) = \textstyle\sum_{i=0}^{k} k_i(t)\Phi_i(t) = \sum_{i=0}^{n}\sum_{w=0}^{p} a_{iw}\, T_w(t)\Phi_i(x) = \sum_{i=0}^{n} A_i^T\Phi(t)\Phi_i(x) \quad (40)$$

Where A_i^T is a $p+1$ vector of constants and $\Phi(t)$ is $p+1$ vector of approximated shifted first kind of Chebyshev polynomials such that $A_i^T\ \Phi(t)$ satisfies the following system of approximated equivalent to nonlinear fractional super PDE (37).

$$\sum_{j=0}^{k}\mu_j(t)\, D_t^{\alpha j}\left(\widehat{\mu_j}(t)\sum_{i=0}^{n} A_i^T\Delta^{\alpha j}\Phi(t)\right)$$

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$$= \sum_{j=0}^{k}\left[\mu_j(t)g_j(t)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha j}\Phi(t), A_1^T\Delta^{\alpha j}\Phi(t), \dots, A_n^T\Delta^{\alpha j}\Phi(t)\right) + \hat{\mu}_j(t)f_j(x)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha j}\Phi(t), A_1^T\Delta^{\alpha j}\Phi(t), \dots, A_n^T\Delta^{\alpha j}\Phi(t)\right)\right]$$

$$-\sum_{i=0}^{n}\Psi_i(A_0^T\Phi(t), A_1^T\Phi(t), \dots, A_n^T\Phi(t)) + \sum_{i=0}^{n}\Psi_{(j+1)n+1+i}(A_0^T\Phi(t), A_1^T\Phi(t), \dots, A_n^T\Phi(t))$$

$n = 1,2,\dots,j$ where $\{\Psi_0, \Psi_1, \dots, \Psi_n\}; \{\Psi_{jn+1}, \Psi_{jn+2}, \dots, \Psi_{(j+1)n+1+n)}\}$

Are expansion coefficients of the operator $N[\mu]$, $F_j[\mu]$ and $H[\mu]$ respectively to the base $\{\Phi_0(x), \Phi_1(x), \dots, \Phi_n(x)\}$.

Proof: Substitute (38) in the left side of (37) and using linearity of Caputo fractional derivatives, we have

$$\sum_{j=0}^{k}\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)D_t^{\alpha j}u\right) = \sum_{j=0}^{k}\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)D_t^{\alpha j}\sum_{i=0}^{n}A_i^T\Phi(t)\Phi_i(x)\right)$$

$$= \sum_{j=0}^{k}\left[\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)\sum_{i=0}^{n}A_i^T\Delta^{\alpha j}\Phi(t)\Phi_i(x)\right)\right] \quad (41)$$

As I_{n+1} be invariant subspace with respect to the operators $N[\mu]$, $F_j[\mu]$ and $H[\mu]$ the exist $(j+1)n+1$ functions $\{\Psi_0, \Psi_1, \dots, \Psi_n\}; \{\Psi_{jn+1}, \Psi_{jn+1+1}, \dots, \Psi_{jn+1+n}\}; \{\Psi_{(j+1)n+1,}\Psi_{(j+1)n+1+1}, \dots, \Psi_{(j+1)n+1+n}\}$, $j = 1, \dots, k$ such that

$$N[u] = N(\textstyle\sum_{i=0}^{n}k_i(t)\Phi_i(x)) = \sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\left(k_0(t), k_i(t), \dots, k_n(t)\right)\Phi_i(x) \quad (42)$$

$$F_j[u] = F_j(\textstyle\sum_{i=0}^{n}k_i(t)\Phi_i(x)) = \sum_{i=0}^{n}\Psi_{jn+1+i}\left(k_0(t), k_i(t), \dots, k_n(t)\right)\Phi_i(x) = \sum_{i=0}^{n}\Psi_{jn+1+i}\left(\sum_{w=0}^{p}a_{0w}T_w(t), \sum_{w=0}^{p}a_{1w}T_w(t), \dots, \sum_{w=0}^{p}a_{nw}T_w(t)\right)\Phi_i(x)$$

$$= \sum_{i=0}^{n}\Psi_{jn+1+i}(A_0^T\Phi(t), A_1^T\Phi(t), \dots, A_n^T\Phi(t))\Phi_i(x) \quad (43)$$

$$H[\mu] = H\left(\sum_{i=0}^{n}k_i(t)\Phi_i(x)\right) = \sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\left(k_0(t), k_1(t), \dots, k_n(t)\right)\Phi_i(x)$$

$$= \sum_{i=0}^{n}\Psi_{(j+1)n+1+i}\left(\sum_{w=0}^{p}a_{0w}T_w(t), \sum_{w=0}^{p}a_{1w}T_w(t), \dots, \sum_{w=0}^{p}a_{nw}T_w(t)\right)\Phi_i(x)$$

$$= \sum_{i=0}^{n}\Psi_{(j+1)n+1+i}(A_0^T\Phi(t), A_1^T\Phi(t), \dots, A_n^T\Phi(t))\Phi_i(x) \quad (44)$$

Thus, since, $\Phi_i(x)$, $i = 0, 1, \dots, n$ are linearly independent, then, we have the following variable coefficients fractional ordinary system.

$$\sum_{j=0}^{k}\mu_j(t)D_t^{\alpha j}\left(\hat{\mu}_j(t)\sum_{i=0}^{n}A_i^T\Delta^{\alpha+j}\Phi(t)\right)$$

$$= \sum_{j=0}^{k}\left[\mu_j(t)g_j(t)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha j}\Phi(t), A_1^T\Delta^{\alpha j}\Phi(t), \dots, A_n^T\Delta^{\alpha j}\Phi(t)\right) + \hat{\mu}_j(t)f_j(x)\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Delta^{\alpha j}\Phi(t), A_1^T\Delta^{\alpha j}\Phi(t), \dots, A_n^T\Delta^{\alpha j}\Phi(t)\right)\right]$$

$$-\sum_{i=0}^{n}\Psi_{jn+1+i}\left(A_0^T\Phi(t), A_1^T\Phi(t), \dots, A_n^T\Phi(t)\right) \quad (45)$$

Also, the initial conditional (38) becomes

$$D_i^m u(x,0) = \sum_{i=0}^{n}k_i^{(m)}(0)\Phi_i(x) = \sum_{i=0}^{n}A_i^T\Phi^m(0)\Phi_i(x) = f_{mi}(x), m = 0, 1, \dots, [2j\alpha - 1] \quad (46)$$

Problem Formulation 3.5. Consider the following nonlinear fractional order super partial differential equation:

$$\sum_{j=0}^{k}\left(\mu_j(t)D_j^{\alpha_1+j}D_j^{\alpha_2+j} - f_j(t)D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\right)\left(\hat{\mu}_j(t)D_t^{\alpha_1+1+j}D_t^{\alpha_2+j+1} - g_j(t)D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\right)\mu$$

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$$= H(\mu) \quad (47)$$

and

$$D_t^m \mu(x,0) = W_m(x), m = 0,1,2,\dots,[\alpha_2 + k]$$
$$D_t^m D_t^{\alpha_2+j+i} \mu(x,0) = R_r(x), r = 0,1,2,\dots,[\alpha_1 + k] \quad (48)$$

i. $\mu_j(t), f_j(t)\, \hat{\mu}_j(t), g_j(t)$ variable functions of x as a coefficient, $j = 1,2,\dots,k$

ii. Caputo derivative $D_t^{\alpha_1+j+1}, D_t^{\alpha_2+j+1}, D_t^{\infty_1+j}, D_t^{\alpha_2+j}, D_x^{2(j+\beta_1)}, D_x^{2(j+\beta_2)}$

iii. $H(\mu) = H\left(x, \mu, D_t^{\alpha_1+j}\mu, D_t^{\alpha_2+j}\mu, D_t^{\alpha_1+j+1}, D_t^{\alpha_2+j+1}\mu, D_x^{2(j+\beta_1)}, D_x^{2(j+\beta_2)}\mu\right)$

Problem formulation on (3.5) equivalent to $\sum_{j=0}^{k} \mu_j(t)\hat{\mu}_j(t) D_t^{\alpha_1+j} D_t^{\alpha_2+j} D_t^{\alpha_1+1+j} D_t^{\alpha_2+1+j}\mu$

$$= \sum_{j=1}^{k}\left(D_t^{\alpha_1+j}D_t^{\alpha_2+j}\left(f_j(t)D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\mu\right) + D_t^{\alpha_1+j+1}D_t^{\alpha_2+j+1}\left(g_j(t)D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\mu\right)\right) - \sum_{j=1}^{k} f_j(t)D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\left(g_j(t)D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\right)\mu + H(u) \quad (49)$$

Thus,

$$\sum_{j=1}^{k} \mu_j(t)\hat{\mu}_j(t) D_t^{\alpha_1+j} D_t^{\alpha_2+j} D_t^{\alpha_1+1+j} D_t^{\alpha_2+1+j}\mu$$

$$= \sum_{j=1}^{k}\left(D_t^{\alpha_1+j}D_t^{\alpha_2+j} f_j(t)F_j(\mu) + D_t^{\alpha_1+j+1}D_t^{\alpha_2+j+1} g_j(t)F_j(u)\right) - \sum_{j=1}^{k} f_j(t)g_j(t)N\,[\mu] + H(u)$$

Where

$F_j(u) = D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}\mu$, $N[u] = D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}D_x^{2(j+\beta_1)}D_x^{2(j+\beta_2)}$.

Theorem 3.6. Let $I_{n+1} = L\{\Phi_0(x), \Phi_1(x), \dots, \Phi_n(x)\}$ is a finite dimensional linear space which is invariant under the operators $N[u]$, $F_j[u]$ and $H[u]$, then the problem formulation (3.5) has the approximate solution in the form n

$$u(x,t) = \sum_{i=0}^{n} k_i(t)\, \Phi_i(x)$$

Where $k_i(t)$ satisfies the following system of fractional ordinary differential equation

$$\sum_{j=1}^{k}\left(\mu_j(t)\hat{\mu}_j(t)D_t^{\alpha_1+j}D_t^{\alpha_2+j}D_t^{\alpha_1+j+1}D_t^{\alpha_2+j+1}u\right)$$
$$= \sum_{j=1}^{k}\mu_j(t)\hat{\mu}_j(t)D_t^{\alpha_1+j}D_t^{\alpha_2+j}D_t^{\alpha_1+j+1}D_t^{\alpha_2+j+1}\sum_{i=0}^{n} k_i(t)\,\Phi_i(x)$$
$$= \sum_{j=1}^{k}\sum_{i=0}^{n}\mu_j(t)\hat{\mu}_j(t)\left(D_t^{\alpha_1+j}D_t^{\alpha_2+j}D_t^{\alpha_1+j+1}D_t^{\alpha_2+j+1}k_i(t)\right)\Phi_i(x) \quad (50)$$

As I_{n+1} be invariant subspace with respect to the operators $N[u]$, $F_j[u]$ and $H[u]$ the exist $(j+1)n+1+n$ functions.

$\{\Psi_0, \Psi_1, \dots, \Psi_n\}; \{\Psi_{jn+1}, \Psi_{jn+2}, \dots, \Psi_{jn+1+n}\}; \{\Psi_{(j+1)n+1,} \Psi_{(j+1)n+2}, \dots, \Psi_{(j+1)n+1+n}\}$

$j = 1,\dots,k$ such that

$$N[u] = N\left(\sum_{i=0}^{n} k_i(t)\Phi_i(x)\right) = \sum_{i=0}^{n}\Psi_{(j+2)n+i}\left(k_0(t), k_1(t), \dots, k_n(t)\right)\Phi_i(x) \quad (51)$$

$$F_j[u] = F_j\left(\sum_{i=0}^{n} k_i(t)\Phi_i(x)\right) = \sum_{i=0}^{n}\Psi_{jn+1+i}\left(k_0(t), k_1(t), \dots, k_n(t)\right)\Phi_i(x) \quad (52)$$

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$$H[u] = \sum_{i=0}^{n} \Psi_i\big(k_0(t), k_1(t), \dots, k_n(t)\big)\, \Phi_i(x) \tag{53}$$

Substitute (51) – (53) in (49), we get

$$\sum_{j=1}^{k} \sum_{i=0}^{n} \big(\mu_j(t)\hat{\mu}_j(t) D_t^{\alpha_1+j} D_t^{\alpha_2+j} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} k_i(t)\big)\, \Phi_i(x)$$

$$= \sum_{j=1}^{k} \sum_{i=0}^{n} \left(D_t^{\alpha_1+j} D_t^{\alpha_2+j} f_j(t) \Psi_{jn+1+i}\big(k_0(t), k_1(t), \dots, k_n(t)\big)\right) \Phi_i(x) +$$
$$\sum_{j=1}^{k} \sum_{i=0}^{n} \left(D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}\big(k_0(t), k_1(t), \dots, k_n(t)\big)\right) \Phi_i(x)$$
$$- \sum_{j=1}^{k} \sum_{i=0}^{n} f_j(t) g_j(t) \Psi_{(j+1)n+1+i}(k_0, k_1, \dots, k_n) + \sum_{i=0}^{n} \Psi_i(k_0, k_1, \dots, k_n)\, \Phi_i(x)$$

Thus, since, $\Phi_i(x)$, $i = 1, \dots, n$ are linearly independent, then, we have the following variable coefficients super fractional ordinary system.

$$\sum_{j=1}^{k} \sum_{i=0}^{n} \big(\mu_j(t)\hat{\mu}_j(t) D_t^{\alpha_1+j} D_t^{\alpha_2+j} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} k_i(t)\big)$$
$$= \sum_{j=1}^{k} \sum_{i=0}^{n} \left(D_t^{\alpha_1+j} D_t^{\alpha_2+j} f_j(t) \Psi_{jn+1+i}\big(k_0(t), k_1(t), \dots, k_n(t)\big)\right) +$$
$$\sum_{j=1}^{k} \sum_{i=0}^{n} \left(D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}\big(k_0(t), \dots k_n(t)\big)\right)$$
$$- \sum_{j=1}^{k} \sum_{i=0}^{n} f_j(t) g_j(t) \Psi_{(j+1)n+1+i}(k_0(t), \dots, k_n(t)) + \sum_{i=0}^{n} \Psi_i(k_0(t), \dots, k_n(t)) \tag{54}$$

Subject to the initial conditions which can be derived from (48), applying the Elzaki transform of both sides of (54), we get

$$E\big(\sum_{j=1}^{k} \sum_{i=0}^{n} \mu_j(t)\hat{\mu}_j(t)\big(D_t^{\alpha_1+j} D_t^{\alpha_2+j} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} k_i(t)\big)\big)$$
$$= E\big(\sum_{j=1}^{k} \sum_{i=0}^{n} D_t^{\alpha_1+j} D_t^{\alpha_2+j} \Psi_{jn+1+i}\big(k_0(t), k_1(t), \dots, k_n(t)\big)\big)$$
$$+E\left(\sum_{j=1}^{k} \sum_{i=0}^{n} \left(D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}\big(k_0(t), \dots k_n(t)\big)\right)\right)$$
$$+E\left(\sum_{j=1}^{k} \sum_{i=0}^{n} \left(D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}\big(k_0(t), \dots k_n(t)\big)\right)\right)$$
$$-E\big(\sum_{j=1}^{k} \sum_{i=0}^{n} f_j(t) g_j(t) \Psi_{(j+1)n+1+i}(k_0(t), \dots, k_n(t))\big) + E\big(\sum_{i=0}^{n} \Psi_i(k_0(t), \dots, k_n(t))\big)$$

Using Elzaki transformation properties of product super fractional Caputo derivatives to have that

$$G_i(v) = H(v) + \mu(v) E\left[\sum_{i=0}^{n} \mu_j(t)\hat{\mu}_j(t) \left(D_t^{\alpha+j} D_t^{\alpha_2+j} \Psi_{jn+1+i}(k_0, \dots, k_n)\right) + \right.$$
$$\sum_{i=0}^{n} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}(k_0, \dots k_n) + \sum_{i=0}^{n} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}(k_0, \dots k_n) -$$
$$\left. \sum_{i=0}^{n} f_j(t) g_j(t) \Psi_{(j+1)n+1+i}(k_0, \dots, k_n) + \sum_{i=0}^{n} \Psi_i(k_0, \dots, k_n)\right] \tag{55}$$

Where $G_i(v) = E\big(k_i(t)\big), H(v)$ is a function arises from the Elzaki transformation of the initial conditions and $\mu(v)$ is arises from this computation.

So, by taking inverse Elzaki transform for both sides of (55), we get

$$k_i(t) = n(t) + E^{-1}\left[\mu(v) E\left[\sum_{i=0}^{n} \mu_j(t)\hat{\mu}_j(t) D_t^{\alpha_1+j} D_t^{\alpha_2+j} \Psi_{jn+1+i}(k_0, \dots, k_n) + \right.\right.$$
$$\sum_{i=0}^{n} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}(k_0, \dots k_n) + \sum_{i=0}^{n} D_t^{\alpha_1+j+1} D_t^{\alpha_2+j+1} g_j(t) \Psi_{jn+1+i}(k_0, \dots k_n) -$$
$$\left.\left. \sum_{i=0}^{n} f_j(t) g_j(t) \Psi_{(j+1)n+1+i}(k_0, \dots, k_n) + \sum_{i=0}^{n} \Psi_i(k_0, \dots, k_n)\right]\right] \tag{56}$$

Where $n(t) = E^{-1}(H(v))$

$$N[\mu] = N\big(\sum_{i=0}^{n} k_i(t)\Phi_i(x)\big) = \sum_{i=0}^{n} \Psi_i\big(k_0(t), k_1(t), \dots, k_n(t)\big)\, \Phi_i(x) \tag{57}$$

$$F_j[\mu] = F_j\big(\sum_{i=0}^{n} k_i(t)\Phi_i(x)\big) =$$
$$\sum_{i=0}^{n} \Psi_{jn+1+i}\big(k_0(t), k_1(t), \dots, k_n(t)\big)\, \Phi_i(x) \tag{58}$$

$$H[\mu] = H\big(\sum_{i=0}^{n} k_i(t)\Phi_i(x)\big) = \sum_{i=0}^{n} \Psi_{(j+1)n+1+i}\big(k_0(t), k_1(t), \dots, k_n(t)\big)\, \Phi_i(x) \tag{59}$$

Substitute (56) – (59) in (49), we get

$$\sum_{j=0}^{k} \sum_{i=0}^{n} \left(D_t^{j+\alpha_1} D_t^{j+\alpha_2} D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} k_i(t)\right) \Phi_i(x)$$

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$= \sum_{j=0}^{k}\big[D_t^{j+\alpha_1} D_t^{j+\alpha_2} \sum_{j=0}^{k}\big[D_t^{j+\alpha_1} D_t^{j+\alpha_2} \sum_{i=0}^{n} \Psi_{jn+1+i}\big(k_0(t),\dots,k_n(t)\big)\Phi_i(x) +$
$D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} \sum_{i=0}^{n} \Psi_{jn+1+i}\big(k_0(t),k_1(t),\dots,k_n(t)\big)\,\Phi_i(x) +$
$\sum_{i=0}^{n} \Psi_i\big(k_0(t),k_1(t),\dots,k_n(t)\big)\Phi_i(x) + \sum_{i=0}^{n} \Psi_{(j+1)n+1+i}\big(k_0(t),\dots,k_n(t)\big)\Phi_i(x)\big]\big]$

Linearity of Caputo derivative lead to

$\sum_{j=0}^{k}\sum_{i=0}^{n}\left(D_t^{j+\alpha_1} D_t^{j+\alpha_2} D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} k_i(t)\right)\Phi_i(x) =$
$\sum_{j=0}^{k}\big[\sum_{i=0}^{n} \Psi_{jn+1+i}\,\big[D_t^{j+\alpha_1} D_t^{j+\alpha_2} k_0(t), D_t^{j+\alpha_1} D_t^{j+\alpha_2} k_1(t),\dots, D_t^{j+\alpha_1} D_t^{j+\alpha_2} k_n(t)\big]\Phi_i(x) +$
$\sum_{i=0}^{n} \Psi_{jn+1+i}\,\big[D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} k_0(t), D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} k_1(t),\dots, D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} k_n(t)\big]\Phi_i(x) +$
$\sum_{i=0}^{n} \Psi_i\,\big(k_0(t),k_1(t),\dots,k_n(t)\big)\Phi_i(x) + \sum_{i=0}^{n} \Psi_{(j+1)+n+1+i}\,\big(k_0(t),\dots,k_n(t)\big)\Phi_i(x)\big]$

Thus, since, $\Phi_i(x)$, $i = 0,1,...,n$ are linearly independent, then, we have the following variable coefficients super fractional ordinary system.

$\sum_{j=0}^{k}\left(D_t^{j+\alpha_1} D_t^{j+\alpha_2} D_t^{1+j+\alpha_1} D_t^{1+j+\alpha_2} k_i(t)\right) =$
$\sum_{j=0}^{n} \Psi_{jn+1+i}\left(D_t^{j+\alpha_1} D_t^{j+\alpha_2} k_0(t), D_t^{j+\alpha_1} D_t^{j+\alpha_2} k_1(t),\dots, D_t^{j+\alpha_1} D_t^{j+\alpha_2} k_n(t)\right).$

4.4. Numerical Results

We provide instances pertaining to the aforementioned subjects in this section. We begin with the example below:
Example 4.1: We adjust the formulation's coefficients to take uncertainty into account with Neutrosophic values. The formula turns into:

$$u(t) + \omega(t)\,u(t) = h(t),$$

where $\omega(t) = T_\omega(t) + I_\omega(t) + F_\omega(t)$, and $h(t) = T_h(t) + I_h(t) + F_h(t)$

The truth portion of $\omega(t)$ is represented by $T_\omega(t)$, the indeterminacy portion of $\omega(t)$ is represented by $I_\omega(t)$.

The untruth portion of $\omega(t)$ is represented by $F_\omega(t)$.The parts of truth, indeterminacy, and falsity of $h(t)$ are represented by $T_h(t), I_h(t)$ and $F_h(t)$, respectively.

Fractional derivatives are involved in the following equation:

$$2e^{-t}D_t^{\alpha}u(t,x) - \cos(x)\,D_t^{\beta}u(t,x) + \sin(t)\,D_t^{\gamma}u(t,x) - t^2 D_t^{\delta}u(t,x) = e^{-2t} - e^{-t} \quad (60)$$

Neutrosophic values turn this into:

$$2e^{-t}D_t^{\alpha}u(t,x) - \cos(x)\,D_t^{\beta}u(t,x) + \sin(t)\,D_t^{\gamma}u(t,x) - t^2 D_t^{\delta}u(t,x) = \left(e^{-2t} + T_h(t) + I_h(t) + F_h(t)\right) - \left(e^{-t} + T_h(t) + I_h(t) + F_h(t)\right) \quad (61)$$

The uncertainty in this case is represented by the neutrophilic components $T_h(t), I_h(t)$ and $F_h(t)$.

Let's assume a simple form for $W_v(x)$ that demonstrates how the initial conditions can be specified:

$$W_v(x) = sin(v\pi x), for\ \ 0 \le x \le 1, and\ \ v = 0,1,2,\dots,[2(K+\beta)]. \quad (62)$$

For $u(t,x)$, the Chebyshev polynomial approximations are still:

$$u(t,x) \approx \sum_{k=0}^{N} \omega_k(t)\,T_k(x)$$

Where $T_k(x) = cos(kcos^{-1}(x))$

When Neutrosophic values are taken into account, the updated integral equation over the domain $[-1,1]$. changes to:

$$\int_{-1}^{1}(2e^{-t}T_n(x)D_t^{\alpha}u(t,x) - cos(x)T_n(x)D_t^{\beta}u(t,x) + sin(t)T_n(x)D_t^{\gamma}u(t,x) - t^2 \quad T_n(x)D_t^{\delta}u(t,x))dx = \int_{-1}^{1}(e^{-2t} - e^{-t} + T_h(t) + I_h(t) + F_h(t))\,T_n(x)dx \quad (63)$$

We approximate $u(t,x)$ by a finite sum of Chebyshev polynomials: $u(t,x) \approx \sum_{k=0}^{N} a_k\,(t)T_k(x)$ where $T_k(x) = cos(k\cos^{-1}(x))$ are the Chebyshev polynomials and $a_n(t)$ are time-dependent coefficients.

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Multiply the given fractional differential equation by $T_n(x)$ and integrate over $[-1,1]$. For each term involving derivatives with respect to t, you'll use the property of orthogonality of Chebyshev polynomials. For each derivative order $\alpha, \beta, \gamma, \delta$, handle the term accordingly: $\int_{-1}^{1} T_n(x) D_t^{\alpha}\, dx\, (\sum_{k=0}^{N} a_k(\mathrm{t})\, T_k(x)) dx$ where D_t^{α} denotes the fractional derivative of order α with respect to t.

The equation involves terms like:

$$2e^{-2t} T_n(x) D_t^{\alpha} u(t,x) - \cos(x)\, T_n(x) D_t^{\beta} u(t,x) + \sin(t)\, T_n(x) D_t^{\gamma} u(t,x) - t^2 T_n(x) D_t^{\delta} u(t,x)$$

Let's dive deeper into why the integrals involving different Chebyshev polynomials simplify due to their orthogonality.

The Chebyshev polynomials $T_n(x)$ satisfy the orthogonality property:

$$\int_{-1}^{1} T_n(x) T_k(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{Bmatrix} 0 & if\ k \neq n \\ \pi\ if\ k = n \neq 0 \\ \frac{\pi}{2} & if\ k = n \end{Bmatrix}$$

To solve the integral, we can use the orthogonality property of Chebyshev polynomials. Let's denote the Chebyshev polynomials of the first kind as $T_n(x)$. Then, the orthogonality property is given by:

Now, let's use this property to solve the given integral:

$$\int_{-1}^{1} 2e^{-2t} T_n(x) D_t^{\alpha} u(t,x) - \cos(x)\, T_n(x) D_t^{\beta} u(t,x) + \sin(t)\, T_n(x) D_t^{\gamma} u(t,x) - t^2 T_n(x) D_t^{\delta} u(t,x)$$

$$= 2e^{-2t} D_t^{\alpha} u(t,x) \int_{-1}^{1} T_n(x) dx - cos(x) D_t^{\beta} u(t,x) \int_{-1}^{1} T_n(x) dx + sin(t) D_t^{\gamma} u(t,x) \int_{-1}^{1} T_n(x) dx$$
$$- t^2 D_t^{\delta} u(t,x) \int_{-1}^{1} T_n(x) dx = 2e^{-2t} \pi \delta_{n0} - cos(x) \frac{\pi}{2} \delta_{n0} + sin(t) \frac{\pi}{2} \delta_{n0} - t^2 \pi \delta_{n0}$$
$$= \pi(2e^{-t} - t^2 + T_h(t) + I_h(t) + F_h(t)) \delta_{n0}$$

Where δ_{n0} is the Kronecker delta, which equals 1 if $n = 0$ and 0 otherwise.

Now, for the right-hand side of the equation:

$$\int_{-1}^{1} (e^{-2t} - e^{-t} + T_h(t) + I_h(t) + F_h(t))\, T_n(x) dx = (2\pi e^{-2t} - \pi e^{-t} + T_h(t) + I_h(t) + F_h(t)) \delta_{n0}$$

So, equating both sides of the equation, we get:

$$\pi(2e^{-t} - t^2 + T_h(t) + I_h(t) + F_h(t)) \delta_{n0} = (2\pi e^{-2t} - \pi e^{-t} + T_h(t) + I_h(t) + F_h(t)) \delta_{n0}$$

This equation holds true if and only if $2e^{-t} - t^2 = e^{-2t} - e^{-t}$, which simplifies to:

$e^{-2t} - 2e^{-t} + 1 = 0$This is a quadratic equation in e−te−t. Solving it, we find:

$e^{-t} = -1,\ e^{-2t} - e^{-t}$Thus, the solution is $e^{-t} = -1,\ e^{-2t} - e^{-t}$, which implies $t = -\frac{1}{3} ln(2)$.

Therefore, taking into account the Neutrosophic uncertainty of the terms, the integral formula is fulfilled for $t = -\frac{1}{3} ln(2)$.

The following figure probably shows the initial configuration of the differential equation that requires solving. It might feature a depiction of the mathematical model or the initial boundary conditions, emphasizing the formulation of the problem for later transformation via Chebyshev polynomials. The figure could contain schematic diagrams or initial data plots essential for grasping the beginning of the numerical solution approach.

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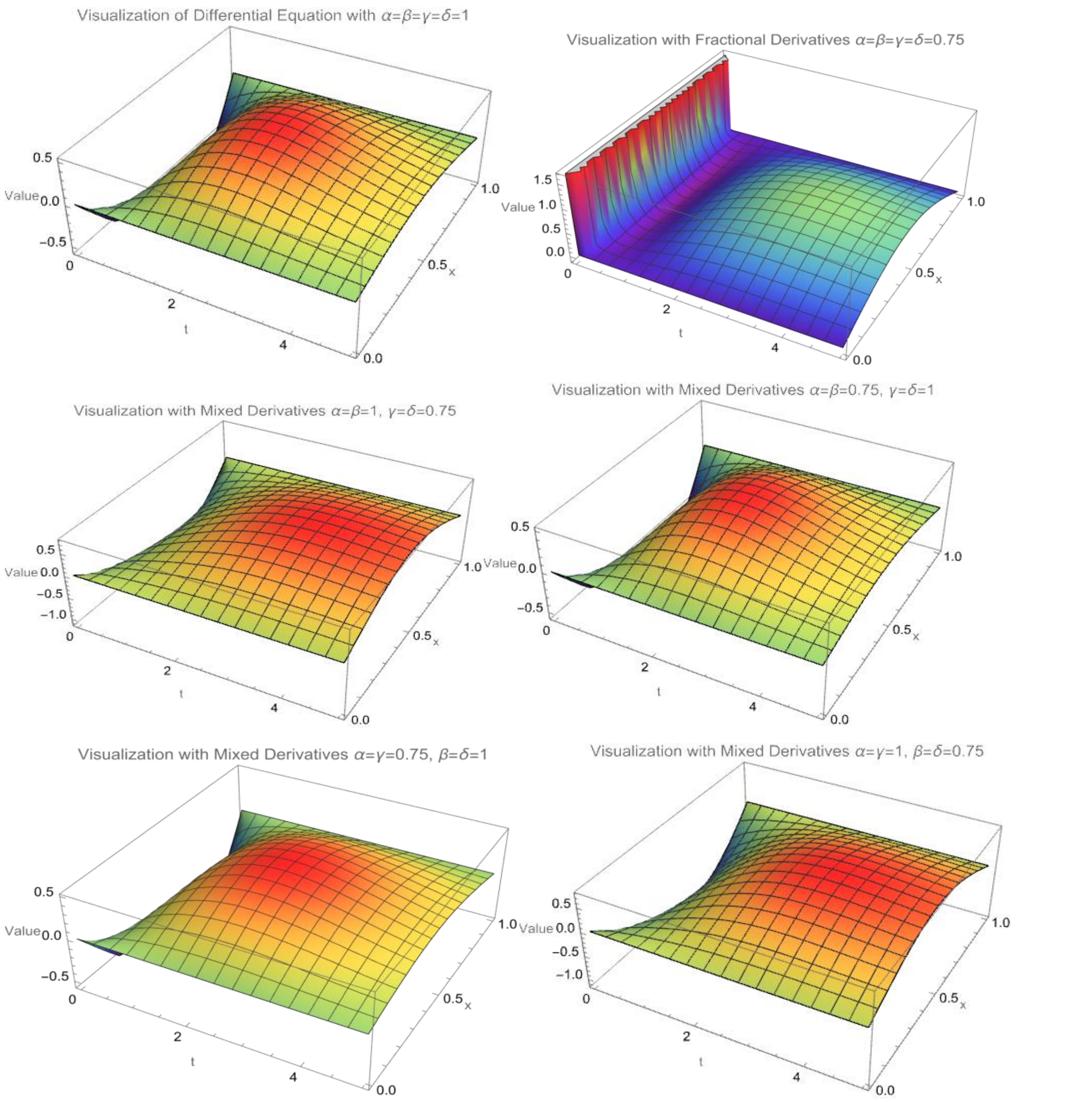

Figure 1. Initial setup and problem formulation for Chebyshev polynomial approximation

Example 4.2. We begin by updating the parameter $g(x,t)$ as follows after adding neutrophilic values to the equation:

$$g(x,t) = T_g(x,t) + I_g(x,t) + F_g(x,t) \tag{64}$$

In this case, the truth, unpredictability, and falsity components of $g(x,t)$ are represented by $T_g(x,t), I_g(x,t)$ and $F_g(x,t)$, respectively. The altered equation consequently changes to:

$$\mu(x,t)D_x^{0.9}u(x,t) - f(x,t)D_t^{0.5}u(x,t) = \mu_6(t)D_x^{0.9}u(x,t) - \left(T_g(x,t) + I_g(x,t) + F_g(x,t)\right)D_t^{0.5}u(x,t) \tag{65}$$

with the initial condition:

$$u(x,0) = w(x) \tag{66}$$

When the function $w(x)$has been established.

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Choose basis functions for the invariant subspace L. Assume: $L = \{\phi_1(x) = x, \phi_2(x) = x^2\}$. This choice is arbitrary and for illustrative purposes; in practice, these should be chosen based on the problem specifics.

According to Theorem 3.4, the solution $u(x,t)$ can be approximated by:

$$u(x,t) = k_1(t)\phi_1(x) + k_2(t)\phi_1(x) = k_1(t)x + k_1(t)x^2 \quad (67)$$

Substitute the approximate solution back into the differential equation. For each term, apply the fractional derivatives to the basic functions:

- $D_x^{0.9}(x)$and $D_x^{0.9}(x^2)$
- $D_t^{0.5}(x)$and $D_x^{0.5}(x^2)$

We now give each term in the formula that calculates these fractional derivatives:

$$D_x^{0.9}(x) = \mathcal{M}_1(x)$$

$$D_x^{0.9}(x^2) = \mathcal{M}_2(x)$$

$$D_x^{0.5}(x) = \mathcal{N}_1(x)$$

$$D_x^{0.5}(x^2) = \mathcal{N}_2(x)$$

The following is the following formula:

$$\mu(t)\mathcal{M}_1(t)k_1(t) + \mu(t)\mathcal{M}_2(t)k_2(t) - f(x)\mathcal{N}_1(t)k_1(t) - (T_g(x,t) + I_g(x,t) + F_g(x,t))B2(t)k_2(t) = 0$$

To create a collection for ODEs involving $k_1(t)$ and $k_2(t)$, we now align the coefficients of similar powers of x on both sides of the calculation:

The parameters of x: $\mathcal{M}_1(t)k_1(t) + \mathcal{N}_1(t)k_2(t) = \mathcal{K}_1(t)$

The parameters of x^2: $\mathcal{M}_2(t)k_1(t) + \mathcal{N}_2(t)k_2(t) = \mathcal{K}_2(t)$

where $\mathcal{M}_i(t), \mathcal{N}_i(t), \mathcal{K}_i(t)$are functions derived from the differential equation, involving $\mu(t), \mu_6(t), f(x), g(t)$, and the computed fractional derivatives.

Solve the system of ODEs for $k_1(t)$ and $k_2(t)$ using suitable numerical methods. This will provide the coefficients for the approximate solution $u(x,t)$.

Last but not least, we use the starting condition $u(x,0) = w(x)$ to find any constants or particular forms needed in the solution. In this way, we may ascertain the values of $k_1(t)$ and $k_2(t)$ at $t = 0$.

With the Neutrosophic values included, the system's result will look like this:

$$u(x,t) = k_1(t)x + k_2(t)x^2$$

where $k_1(t)$ and $k_1(t)$ are the solutions to the system of ODEs, and the coefficients are adjusted to account for the uncertainty and the Neutrosophic components $T_g(x,t)$, $I_g(x,t)$ and $F_g(x,t)$.

Figure 2 likely illustrates the transformation method for fractional differential equations using Chebyshev polynomials. This figure might display a flowchart or sequences of equations that demonstrate how the original differential system is broken down or reformatted for resolution using polynomial approximations. It could also feature graphs or matrices that detail the transformation steps, centering on how the Chebyshev polynomial technique is applied to reformat the problem for numerical solutions.

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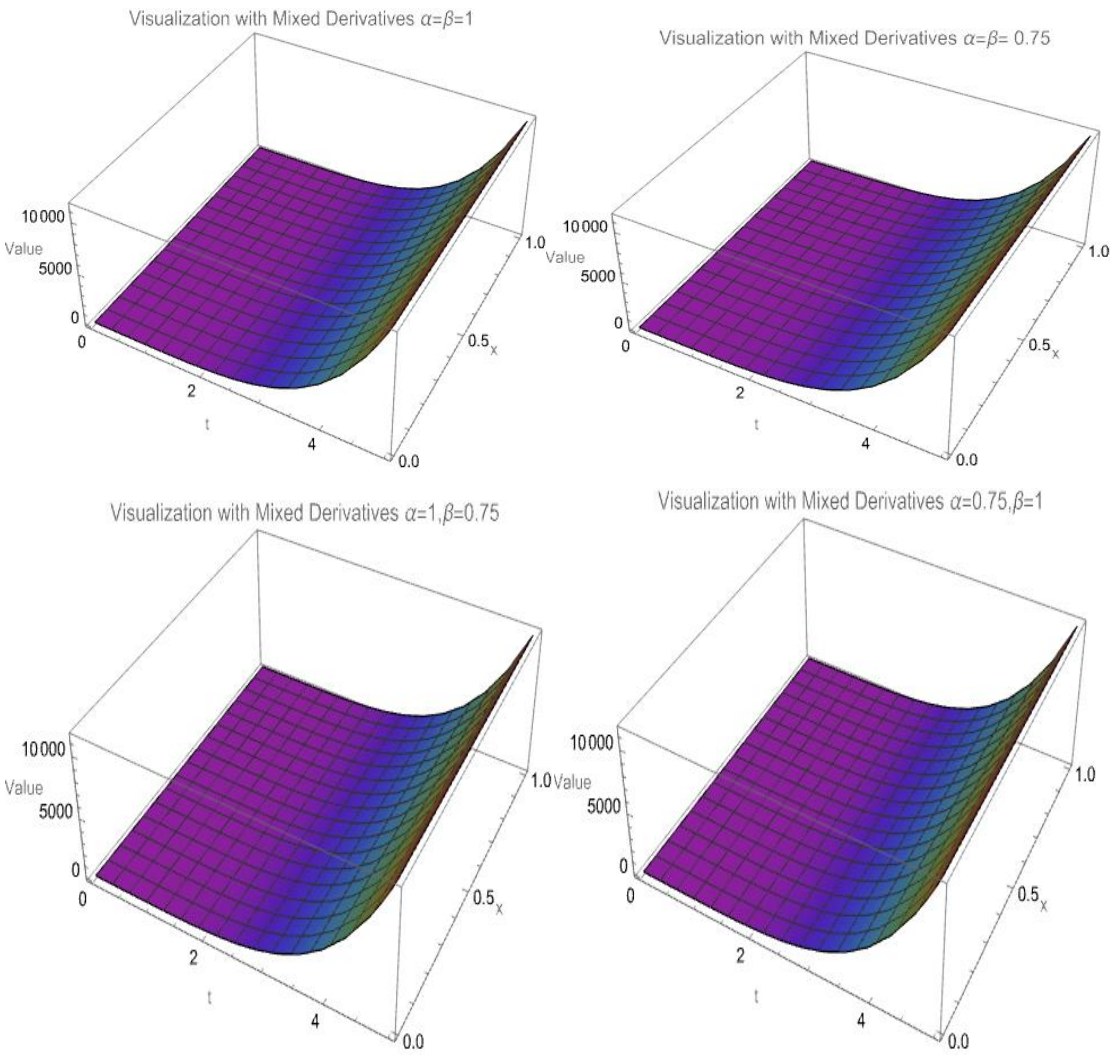

Figure 2. Differential system transformation using Chebyshev polynomials

Example 4.3. The nonlinear a fractional- super partial differential formula is presented to us:

$$\partial^{\alpha} u(x,t) = g(x,t), \tag{68}$$

with the initial condition updated as:

$$u(x,0) = T_u(x,0) + I_u(x,0) + F_u(x,0)$$

where truth, indeterminacy, and falsity components of $u(x,0)$ are represented by $T_u(x,0), I_u(x,0)$ and $F_u(x,0)$, respectively. By adding uncertainty, this addition improves the model's suitability for real-world scenarios with imprecisely known parameters.

The equation is changed to:

$$2\mu(t)D_x^{0.9}u(x,t) - f(x)D_t^{0.5}u(x,t) =$$

$$3\mu_6(t)D_x^{0.9}u(x,t) - g(t)D_t^{0.5}u(x,t) \tag{69}$$

where:

- $\mu(t) = e^{-t}, \mu_6(t) = e^{-2t}$
- $f(x) = sin(x), g(t) = cos(t)$
- $D_x^{0.9}$ and $D_t^{0.5}$ denote fractional derivatives with respect to x and t, respectively.

Choose a set of basis functions for the invariant subspace L. Let's choose: $L = \{\phi_1(x) = x, \phi_2(x) = x^2\}$. Using Theorem 3.5, we represent the solution as a linear combination of the basic functions:

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$u(x,t) = k_1(t)\phi_1(x) + k_2(t)\phi_1(x) = k_1(t)x + k_2(t)x^2$

The Neutrosophic components in this formula for $u(x,t)$ accommodate for the model's uncertainty.

Replace $u(x,t)$) in the Eq. (69):

Substitute $u(x,t)$ into the differential equation and equate terms involving like powers of x:

Calculate $D_x^{0.9}(x)$, $D_x^{0.9}(x^2)$, $D_t^{0.5}(x)$ and $D_t^{0.5}(x^2)$ for each $\phi_i(x)$**.** Since these are fractional derivatives, they might require numerical methods or special functions for evaluation. It may be necessary to evaluate these fractional derivatives using specific functions or numerical techniques. Let's use the following notation to represent the fractional derivatives for the basis functions for simplicity's sake:

- $D_x^{0.9}(x)= \mathcal{M}_1(t)$
- $D_x^{0.9}(x^2)= \mathcal{M}_2(t)$
- $D_t^{0.5}(x)= \mathcal{N}_1(t)$
- $D_t^{0.5}(x^2)= \mathcal{N}_2(t)$

These fractional derivatives are now entered into the equation, and terms involving corresponding powers of x (constant term and x-term independently) are equated.

Regarding the equation:

$$2\mu(t)D_x^{0.9}u(x,t) - f(x)D_t^{0.5}u(x,t) = 3\mu_6(t)D_x^{0.9}u(x,t) - g(t)D_t^{0.5}u(x,t)$$

Following the substitution of the approximate solution, $u(x,t) = k_1(t)x + k_2(t)x^2$,we obtain:

Solve the system of ODEs for $k_1(t)$ and $k_2(t)$.This may require numerical methods for integration, especially due to the complexity introduced by fractional derivatives.

Apply the initial condition $u(x,0) = x^2$ to solve for the constants in $k_1(t)$ and $k_2(t)$: $k_1(0) + k_2(0)x = x^2$**.** Thus, $k_1(0) = 0$ and $k_2(0) = 1$. The approximate answer is obtained by calculating the system for ODEs involving $k_1(t)$ and $k_2(t)$:

$u(x,t) = k_1(t)x + k_2(t)x^2$

The figure below is intended to showcase the outcomes of the numerical methods employed, particularly displaying the computed solutions of the differential equations using adaptive Chebyshev polynomial methods. It might feature graphs or contour maps that detail the solution across the problem's domain, depicting the behavior of the solution under different conditions or parameters. Additionally, the figure could emphasize the efficacy of the adaptive methods in reaching the required accuracy or stability, presenting the final results or the dynamic modifications implemented during the computational process.

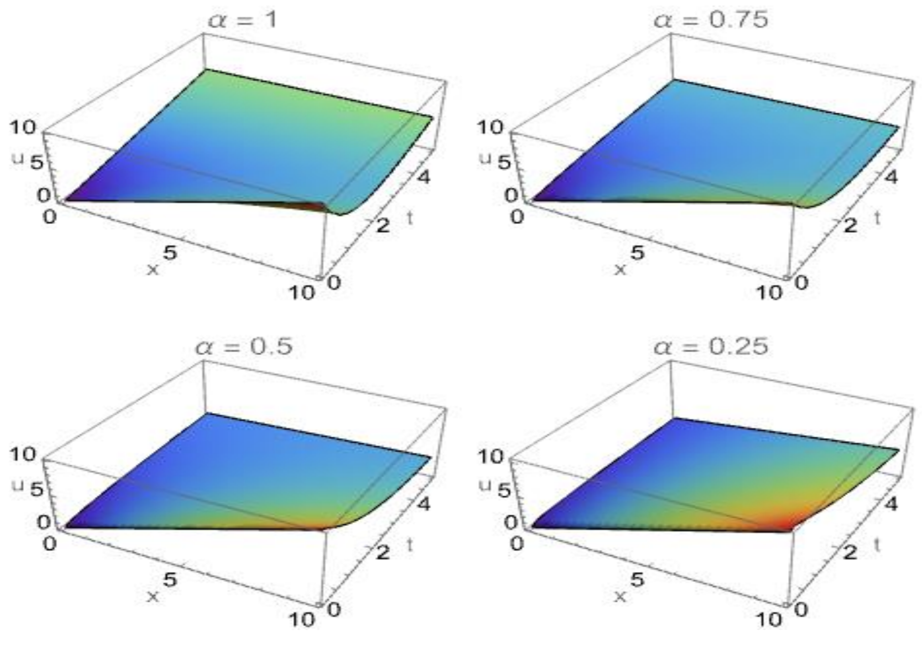

Figure 3. Computed solution profile using adaptive Chebyshev methods

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4. Results and Conclusion

We obtain the following results in our work:

1. Effectiveness of Numerical Analysis Techniques: This study showed how successful it may be to approximate solutions to partial differential equations involving fractions using approximation techniques, especially Chebyshev polynomials. We took into consideration the uncertainty presents in many real-world structures by introducing Neutrosophic logic within the approximation’s procedure. In addition to offering precise answers, this method guarantees that the outcomes take into account both the system's known and unknown components. Neutrosophic values, particularly when the parameters show ambiguity or indeterminacy, help to cut down on calculation time without sacrificing precision and dependability of the findings.

2. Practical Application of Fractional Methods: Complex systems with dynamics that are challenging to represent with traditional differential equations can be studied well by applying fractional differentiation as well as integration, improved by neutrophilic logic. A more sophisticated model that may capture the intrinsic uncertainty in intricate physical systems, such viscoelasticity and the motion of biological materials, is provided by neutrophilic fractional equations. By combining the truth, indeterminacy, with falsity of system characteristics, these models offer more realistic depictions of physical processes.

3. Stability and Accuracy: According to the study, Chebyshev polynomials in conjunction with neutrophilic logic offer a high degree of computational stability as well as accuracy when used to approximate solutions to fractional equations. By tackling the instability frequently seen with traditional approaches, neutrophilic logic improves the solution's resilience. Neutrosophic component integration increases the findings' dependability, especially in scientific and engineering applications when system parameters are erratic or unpredictable. This method greatly improves the accuracy of the model and the stability of the results.

4. Diversity of applications: The study showed that the suggested approaches, which use Neutrosophic logic, have not only efficient but also adaptable. They may be used in a variety of domains, such as biophysical systems, material mechanics, and signal processing, while taking into consideration the unpredictability and variability seen in real-world data. Because neutrophilic fractional equations directly incorporate uncertainty into the solving process, they enable more precise modeling of these systems. Understanding how materials behave and natural phenomena work is enhanced by having the capacity to solve these kinds of equations, which provide insights into complicated systems whose parameters are not always exactly known.

This paper concludes by demonstrating the potent combination of Neutrosophic logic and Chebyshev polynomials to overcome fractional differential equations, especially in systems where uncertainty is a significant factor. The suggested techniques improve computational solutions' correctness and stability by using Neutrosophic values, providing a more accurate representation of intricate physical processes. This method's adaptability makes it possible to use it in a variety of domains, including signal processing, material science, and biophysics, and it significantly improves the representation of systems with unknown characteristics. The findings imply that these approaches provide a viable way to address difficult issues in engineering and scientific study.

Future development: It is evident from the results that integrating Neutrosophic logic with additional approximation and numerical approaches has a great deal of promise to advance methods for solving fractional equations. Creating increasingly sophisticated techniques that can manage a wider variety of fractional equations is part of the future scope, especially in cutting-edge disciplines like finance, biology, and the social sciences. We may solve the ambiguity and complexity associated with models in these domains by using Neutrosophic logic, offering more dependable and durable solutions for practical uses.

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