Crossing Cubic Structures Applied to Hoop Algebras

Anas Al-Masarwah[1,*], Fawziah Alharthi[2], Noor Bani Abd Al-Rahman[1]
[1]Department of Mathematics, Faculty of Science, Ajloun National University, P.O. Box 43, Ajloun 26810, Jordan
[2]Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia
Emails: almasarwah85@gmail.com; f.alharthi@qu.edu.sa; noorbaniabdalrahman@gmail.com

Abstract

Recent years have witnessed remarkable developments in fuzzy logic, with interval-valued fuzziness and negative structures emerging as powerful tools for modeling inaccurate phenomena. The crossing cubic structures ($\mathcal{CC}s$), as a generalization of the bipolar fuzziness structures, represent a comprehensive mathematical framework capable of dealing with a wide range of fuzziness and contradictory data, thus expanding research prospects in this area. This paper has made a new contribution to some algebraic structures by investigating the concept of $\mathcal{CC}s$ on algebraic substructures in a hoop algebra. The concepts of crossing cubic sub-hoops ($\mathcal{CC}-\mathcal{SH}s$) and crossing cubic filters ($\mathcal{CCF}s$) are introduced, and a deeper understanding is sought to analyze their characteristics. The effect on the relationship between $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$ is revealed, and the characterizations of $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$ are analyzed.

Keywords: Hoop algebras; Sub-Hoops; Filters; Crossing cubic structures; Crossing cubic sub-hoops; Crossing cubic filters

1 introduction

Bosbach[1,2] proposed a particular class of algebraic structures under the name of complementary semigroups. This structure was aimed to expand the study and analysis of multi-value network-based logical systems using a binary process that achieves interpretability and provides powerful algebraic tools to explore their characteristics. Based on Bosbach's work, in the mid-1970s, Buchi and Owens introduced the term "Hoops" to refer to the commutative complement monoids in their unpublished manuscript entitled "Complemented Monoids and Hoops"[3] . In light of the significant ideas of their manuscript, Blok and Pigozzi[4] provided a study of hoops with dual normal operators analogy with normal Boolean algebras with operators. The first systematic study of hoops has been presented in Ferreirim's thesis[5] . In the last few years, researchers have further enriched the study of hoops in deep-structure theorems (see[6–9]).

The mathematician Zadeh introduced fuzziness structures in the context of his research into systems theory in 1965, to deal with vague and inaccurate information[10] . Based on fuzzy sets, Ronsefeld[11] utilized the idea of fuzzy sets to the theory of groups and constructed the notion of fuzziness subgroups of a group. Since then, numerous papers[12,13] concerning different fuzzy algebraic structures have appeared in the literature and implemented in other algebraic structures such as semigroups, lattices, ideals, rings, vector spaces and modules. The results of hoop studies have a valuable influence on fuzzy logic and represent a pivotal role in modeling non-classical logical systems and deepening understanding of multi-value logical systems. Over time, the concept of these structures has witnessed significant developments at the hands of different researchers, as it has been scaled up and new, more complex species have emerged, such as intuitionistic fuzziness structures[14] , falling shadow theory[15] and interval-valued fuzziness structures[16] . By combining interval-valued fuzzy sets

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and N-structures, Jun et al.[17] introduced a new structure, so called $\mathcal{CC}s$, and they applied it to BCK/BCI-algebras. Additionally, Ozturk et al.[18] developed the concept of crossing cubic on semigroup structures, and commutative ideal in BCK-algebras. In the context of Lie algebras, Al-Masarwah et al.[19] established the idea of crossing cubic Lie subalgebra. They studied the notions of Lie ideals, homomorphisms and certain fundamental results of Lie algebras. In the context of BE-algebras, Al-Masarwah et al.[20] presented the notions of crossing cubic subalgebras and filters, and probed several characteristics of these notions.

In the fuzzification of the hoop algebras, the concept of fuzzy (uncertainty) sets was connected with filter hoop algebras by Borzooei and Aaly Kologani[21] . They studied and discussed the notions of fuzzy filter of hoops and the relation among them. Also, they studied certain characterizations of fuzzy filters and discussed a congruence relation on hoops by a fuzzy filter. Then, filter theory and its fuzzy concepts have been investigated, including studies of numerous features in hoop algebras (see[22]). Merging the ideas of intuitionistic uncertainty sets and hoop algebras, intuitionistic fuzzy filters were implemented by Aaly Kologani et al.[23] . In the context of falling shadow ideas, Borzooei et al.[24] presented the falling shadow sub-hoops and filters in hoops. They discussed the relationship between falling uncertainty sub-hoops and falling uncertainty filters. Based on Lukasiewicz uncertainty structures, Takallo et al.[25] applied the Lukasiewicz uncertainty structures to hoop algebras and investigated some related properties. By expanding the study of[21] and inspired by earlier studies, the notions of $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$ in hoop algebras are displayed by combining the concepts of fuzzy hoop algebras and negative uncertainty model of hoop algebras, and characterizations of sub-hoops and filters according to the features of $\mathcal{CC}s$ are provided in the current study.

Figure 1 describes the connection between hoop algebras and certain extensions of fuzzy sets, such as intuitionistic fuzzy, falling shadow theory and Lukasiewicz fuzzy sets.

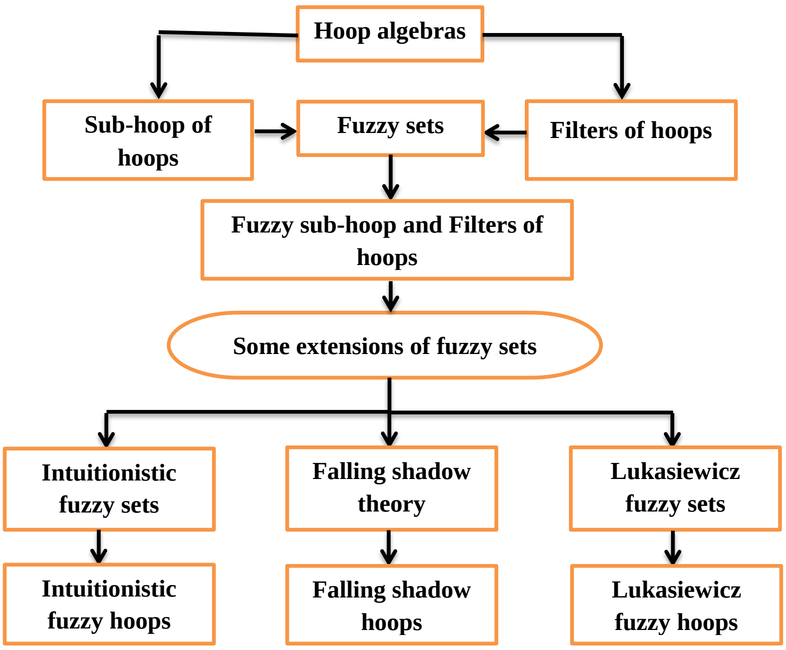

Figure 1: Contributions toward $\mathcal{CC}$ hoop algebras.

This study seeks to assess the performance of $\mathcal{CC}s$ on sub-hoops and filters in hoops. We present the concepts of $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$, and analyze of their properties. We reveal the effect of the relationship between $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$. Also, we probe many theories on $\mathcal{CCF}s$ and prove them. To make it easier to deal with these concepts, we analyze certain characterizations of $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$, and the aim of our precise description of our new concept is to paint a clear mental image of it and to define its limits and areas of application. Table 1 contains certain acronyms and their associated meanings used throughout this present study.

Table 1: List of acronyms.

Acronyms	Representation
$\mathcal{CC}(s)$	Crossing cubic structure(s)
$\mathcal{CC}-\mathcal{SH}s$	Crossing cubic sub-hoop(s)
$\mathcal{CCF}(s)$	Crossing cubic filter(s)

2 Preliminaries

This section covers some fundamental definitions and results of hoops, sub-hoops, filters, and some uncertainty concepts that will be utilized throughout the study.

2.1 Basic Definitions of Hoop Algebras

This subsection includes certain definitions and results related to hoops, such as sub-hoops and filters of a hoop.

Definition 2.1 ([6]). A hoop $(V, \diamond, \hookrightarrow, 1)$ is an algebra where $(V, \diamond, 1)$ forms a commutative monoid and meets the following identities: $\forall \mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V$,

(V_1) $\mathfrak{r} \hookrightarrow \mathfrak{r} = 1$;

(V_2) $\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s}) = \mathfrak{s} \diamond (\mathfrak{s} \hookrightarrow \mathfrak{r})$;

(V_3) $\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow \mathfrak{u}) = (\mathfrak{r} \diamond \mathfrak{s}) \hookrightarrow \mathfrak{u}$.

In the hoop V, we identify a partial order relation "$\leq$" by $\mathfrak{r} \leq \mathfrak{s} \Leftrightarrow \mathfrak{r} \hookrightarrow \mathfrak{s} = 1, \forall \mathfrak{r}, \mathfrak{s} \in V$.

Definition 2.2 ([9]). A non-empty subset $\check{V}$ of a hoop V is called a sub-hoop of V if it is closed under the operations "$\diamond$" and "$\hookrightarrow$".

Note that element 1 is presented in every sub-hoop of a hoop.

Proposition 2.3 ([9]). *Let $(V, \diamond, \hookrightarrow, 1)$ be a hoop. For any $\mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V$, we have the following properties:*

(1) $(V, \leq)$ *is a meet-semilattice with* $\mathfrak{r} \wedge \mathfrak{s} = \mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})$;

(2) $\mathfrak{r} \diamond \mathfrak{s} \leq \mathfrak{u} \Leftrightarrow \mathfrak{r} \leq \mathfrak{s} \hookrightarrow \mathfrak{u}$;

(3) $\mathfrak{r} \diamond \mathfrak{s} \leq \mathfrak{r}, \mathfrak{s}$ *and* $\mathfrak{r}^n \leq \mathfrak{r}, \forall n \in \mathbb{N}$;

(4) $\mathfrak{r} \leq \mathfrak{s} \hookrightarrow \mathfrak{r}$;

(5) $1 \hookrightarrow \mathfrak{r} = \mathfrak{r}$ *and* $\mathfrak{r} \hookrightarrow 1 = 1$;

(6) $\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s}) \leq \mathfrak{s}, \mathfrak{r} \diamond \mathfrak{s} \leq \mathfrak{r} \wedge \mathfrak{s} \leq \mathfrak{r} \hookrightarrow \mathfrak{s}$;

(7) $\mathfrak{r} \hookrightarrow \mathfrak{s} \leq (\mathfrak{s} \hookrightarrow \mathfrak{u}) \hookrightarrow (\mathfrak{r} \hookrightarrow \mathfrak{u})$;

(8) $\mathfrak{r} \leq \mathfrak{s} \Rightarrow \mathfrak{r} \diamond \mathfrak{u} \leq \mathfrak{s} \diamond \mathfrak{u}, \mathfrak{u} \hookrightarrow \mathfrak{r} \leq \mathfrak{u} \hookrightarrow \mathfrak{s}$ *and* $\mathfrak{s} \hookrightarrow \mathfrak{u} \leq \mathfrak{r} \hookrightarrow \mathfrak{u}$;

(9) $\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow \mathfrak{u}) = \mathfrak{s} \hookrightarrow (\mathfrak{r} \hookrightarrow \mathfrak{u})$;

(10) $(\mathfrak{r} \hookrightarrow \mathfrak{s}) \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u}) \leq \mathfrak{r} \hookrightarrow \mathfrak{u}$;

(11) $\mathfrak{r} \leq (\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{s}$.

Definition 2.4 ([9]). A non-empty subset $\hat{V}$ of a hoop V is called a filter of V if it is closed under the operation "$\diamond$" and is upward closed, i.e., $\forall \mathfrak{r}, \mathfrak{s} \in V$,

(1) $\mathfrak{r}, \mathfrak{s} \in \hat{V} \Rightarrow \mathfrak{r} \diamond \mathfrak{s} \in \hat{V}$;

(2) $\mathfrak{r} \in \hat{V}, \mathfrak{r} \leq \mathfrak{s} \Leftrightarrow \mathfrak{s} \in \hat{V}$.

Lemma 2.5 ([9]). *A subset $\hat{V}$ of a hoop V is a filter of V if and only if for any $\mathfrak{r}, \mathfrak{s} \in V$ the following hold:*

(1) $1 \in \hat{V}$;

(2) $\mathfrak{r} \in \hat{V}, \mathfrak{r} \hookrightarrow \mathfrak{s} \in \hat{V} \Rightarrow \mathfrak{s} \in \hat{V}$.

2.2 Basic Definitions of Crossing Cubic Structures

This subsection covers some basic definitions and results of certain uncertainty concepts.

Definition 2.6 ([16]). Let *V* be a nonempty set. An interval valued fuzzy set $\widetilde{\Psi}_\Omega$ on *V* is a function

$$\widetilde{\Psi}_\Omega : V \rightarrow \mathbb{I}_\Psi[0,1],$$

where $\mathbb{I}_\Psi[0,1]$ denotes the set containing every closed subinterval of $[0,1]$, and the members of $\mathbb{I}_\Psi[0,1]$ are called interval numbers. If $\widetilde{\hbar} \in \mathbb{I}_\Psi[0,1]$, $\widetilde{\hbar} = [\hbar^-, \hbar^+]$ with $0 \leq \hbar^- \leq \hbar^+ \leq 1$.

For each two interval numbers $\widetilde{\hbar} = [\hbar^-, \hbar^+]$ and $\widetilde{\aleph} = [\aleph^-, \aleph^+]$, we define

(1) $\widetilde{\hbar} \preceq \widetilde{\aleph}$(or $\widetilde{\aleph} \succeq \widetilde{\hbar}) \Leftrightarrow \hbar^- \leq \aleph^-, \hbar^+ \leq \aleph^+$(or $\hbar^- \geq \aleph^-, \hbar^+ \geq \aleph^+$;

(2) $\widetilde{\hbar} = \widetilde{\aleph} \Leftrightarrow \widetilde{\hbar} \preceq \widetilde{\aleph}$ and $\widetilde{\aleph} \succeq \widetilde{\hbar}$;

(3) $\widetilde{min}\{\widetilde{\hbar}, \widetilde{\aleph}\} = [min\{\hbar^-, \aleph^-\}, min\{\hbar^+, \aleph^+\}]$;

(4) $\widetilde{max}\{\widetilde{\hbar}, \widetilde{\aleph}\} = [max\{\hbar^-, \aleph^-\}, max\{\hbar^+, \aleph^+\}]$.

Definition 2.7 ([17]). Let *V* be a nonempty set. A $\mathcal{CC}$ of *V* is an object having the following form:

$$\Omega = \{(\mathfrak{r}, \widetilde{\Psi}_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r})) \mid \mathfrak{r} \in V\},$$

where $\widetilde{\Psi}_\Omega = [\Psi^-_\Omega, \Psi^+_\Omega] : V \rightarrow \mathbb{I}_\Psi[0,1]$ is an interval valued fuzzy set on *V* and $\eth_\Omega : V \rightarrow [-1,0]$ is an *N*-function on *V*.

For the purpose of simplicity, we will employ the symbol $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ for the $\mathcal{CC}$ $\Omega = \{(\mathfrak{r}, \widetilde{\Psi}_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r})) \mid \mathfrak{r} \in V\}$.

Definition 2.8 ([17]). Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ over *V*. Then, $(\widetilde{\omega}, \sigma)$-level of $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is the crisp set in *V* denoted by $\Gamma\Big(\widetilde{\Psi}_\Omega; (\widetilde{\omega}, \sigma)\Big)$ and is defined as

$$\Gamma\Big(\widetilde{\Psi}_\Omega; (\widetilde{\omega}, \sigma)\Big) = \Big\{\mathfrak{r} \in V \mid \widetilde{\Psi}_\Omega(\mathfrak{r}) \succeq \widetilde{\omega}, \eth_\Omega(\mathfrak{r}) \leq \sigma\Big\},$$

where $\widetilde{\omega} \in \mathbb{I}_\Psi[0,1]$ and $\sigma \in [-1,0]$. For $\widetilde{\omega} \in \mathbb{I}_\Psi[0,1]$ and $\sigma \in [-1,0]$. The $\widetilde{\omega}$-level $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and σ-level $\Gamma\Big(\eth_\Omega; \sigma\Big)$ subsets of Ω can be defined as:

$$\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) = \Big\{\mathfrak{r} \in V \mid \widetilde{\Psi}_\Omega(\mathfrak{r}) \succeq \widetilde{\omega}\Big\} \text{ and } \Gamma\Big(\eth_\Omega; \sigma\Big) = \Big\{\mathfrak{r} \in V \mid \eth_\Omega(\mathfrak{r}) \leq \sigma\Big\}.$$

3 Crossing Cubic Sub-Hoops

In this section, we apply the $\mathcal{CC}s$ to sub-hoops of hoop algebras. We originate the notion of a $\mathcal{CC}-\mathcal{SH}$, and discuss its certain characterizations. We refer to $V \neq \emptyset$ as a hoop, that is $(V, \diamond, \hookrightarrow, 1)$ is a hoop, unless otherwise provided in the text.

Definition 3.1. A $\mathcal{CC}$ $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ in V is called a $\mathcal{CC}-\mathcal{SH}$ of V if the following states are verified, for all $\mathfrak{r}, \mathfrak{s} \in V$.

(i)

$$\left(\mathfrak{r} \diamond \mathfrak{s} \in V \Rightarrow \left(\begin{array}{c} \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond \mathfrak{s}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{r} \diamond \mathfrak{s}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \end{array} \right) \right);$$

(ii)

$$\left(\mathfrak{r} \hookrightarrow \mathfrak{s} \in V \Rightarrow \left(\begin{array}{c} \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \end{array} \right) \right).$$

Example 3.2. Consider a set $V = \{0, 1, \ell, \xi\}$. The following tables display the binary operations "$\diamond$" and "$\hookrightarrow$" on V.

Table 2: Cayley tables for the binary operations "$\diamond$" and "$\hookrightarrow$"

$\diamond$	0	1	ℓ	ξ
0	0	0	0	0
1	0	0	1	1
ℓ	0	1	ℓ	ℓ
ξ	0	1	ℓ	ξ

$\hookrightarrow$	0	1	ℓ	ξ
0	ξ	ξ	ξ	ξ
1	1	ξ	ξ	ξ
ℓ	0	1	ξ	ξ
ξ	0	1	ℓ	ξ

Then, $(V, \diamond, \hookrightarrow, \xi)$ is a hoop. Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V which is demonstrated in the following table.

Table 3: $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is represented tabularly

V	$\widetilde{\Psi}_\Omega(\mathfrak{r})$	$\eth_\Omega(\mathfrak{r})$
0	[0.37, 0.64]	-0.69
1	[0.33, 0.61]	-0.58
ℓ	[0.27, 0.58]	-0.37
ξ	[0.41, 0.74]	-0.77

It is typical to check $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CC}-\mathcal{SH}$ of $(V, \diamond, \hookrightarrow, \xi)$.

Proposition 3.3. *Every $\mathcal{CC}-\mathcal{SH}$ $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ of V satisfies:*

$$(\boldsymbol{A}) \quad (\forall \mathfrak{r} \in V)((1, \mathfrak{r}) \in \mathcal{C}(\Omega)),$$

where $\mathcal{C}(\Omega) = \{(\mathfrak{r}, \mathfrak{s}) | \widetilde{\Psi}_\Omega(\mathfrak{r}) \succeq \widetilde{\Psi}_\Omega(\mathfrak{s}), \eth_\Omega(\mathfrak{r}) \leq \eth_\Omega(\mathfrak{s})\}$.

Proof. This is instantly the result of the amalgamation of (V_1) and Definition 3.1. □

Theorem 3.4. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ over a hoop V. Then, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CC}-\mathcal{SH}$ of V if and only if the nonempty sets $\Gamma\left(\widetilde{\Psi}_\Omega; \widetilde{\omega}\right)$ and $\Gamma\left(\eth_\Omega; \sigma\right)$ are sub-hoops of V for all $\widetilde{\omega} = [\omega^-, \omega^+] \in \mathbb{I}_\Psi[0, 1]$ and $\sigma \in [-1, 0]$.*

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CC} - \mathcal{SH}$ of V and $\widetilde{\omega} = [\omega^-, \omega^+]$ and $\sigma \in [-1, 0]$ be such that $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\Gamma\Big(\eth_\Omega; \sigma\Big)$ are nonempty. Let $\mathfrak{r}, \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$. Then, $\widetilde{\Psi}_\Omega(\mathfrak{r}) \succeq \widetilde{\omega}, \widetilde{\Psi}_\Omega(\mathfrak{s}) \succeq \widetilde{\omega}, \eth_\Omega(\mathfrak{r}) \leq \sigma$ and $\eth_\Omega(\mathfrak{s}) \leq \sigma$. Hence,

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond \mathfrak{s}) &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\} \succeq \widetilde{\omega}, \\ \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\} \succeq \widetilde{\omega}, \end{aligned}$$

that is, $\mathfrak{r} \diamond \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\mathfrak{r} \hookrightarrow \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$. Also, we have

$$\begin{aligned} \eth_\Omega(\mathfrak{r} \diamond \mathfrak{s}) &\leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \leq \sigma, \\ \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) &\leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \leq \sigma, \end{aligned}$$

that is, $\mathfrak{r} \diamond \mathfrak{s} \in \Gamma\Big(\eth_\Omega; \sigma\Big)$ and $\mathfrak{r} \hookrightarrow \mathfrak{s} \in \Gamma\Big(\eth_\Omega; \sigma\Big)$. Therefore, $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\Gamma\Big(\eth_\Omega; \sigma\Big)$ are sub-hoops of V.

Conversely, suppose that $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\Gamma\Big(\eth_\Omega; \sigma\Big)$ are nonempty sub-hoops of V for all $\widetilde{\omega} = [\omega^-, \omega^+] \in \mathbb{I}_\Psi[0, 1]$ and $\sigma \in [-1, 0]$. Assume there exist $\mathfrak{r}, \mathfrak{s} \in V$ such that $\widetilde{\Psi}_\Omega(\mathfrak{r}) = \widetilde{a}$ and $\widetilde{\Psi}_\Omega(\mathfrak{s}) = \widetilde{b}$. If we take $\widetilde{\tau} = \widetilde{min}\{\widetilde{a}, \widetilde{b}\}$, then $\mathfrak{r}, \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\tau}\Big)$ and so $\mathfrak{r} \diamond \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\tau}\Big)$ and $\mathfrak{r} \hookrightarrow \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\tau}\Big)$. Hence,

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond \mathfrak{s}) &\succeq \widetilde{\tau} = \widetilde{min}\{\widetilde{a}, \widetilde{b}\} = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}, \\ \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) &\succeq \widetilde{\tau} = \widetilde{min}\{\widetilde{a}, \widetilde{b}\} = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}. \end{aligned}$$

Also, assume that there exist $\mathfrak{a}, \mathfrak{b} \in V$ such that $\mathfrak{a} \diamond \mathfrak{b} \notin \Gamma\Big(\eth_\Omega; \sigma\Big)$ or $\mathfrak{a} \hookrightarrow \mathfrak{b} \notin \Gamma\Big(\eth_\Omega; \sigma\Big)$. If $\mathfrak{a} \diamond \mathfrak{b} \notin \Gamma\Big(\eth_\Omega; \sigma\Big)$, then $\eth_\Omega(\mathfrak{a} \diamond \mathfrak{b}) > max\{\eth_\Omega(\mathfrak{a}), \eth_\Omega(\mathfrak{b})\}$. It follows that $\mathfrak{a}, \mathfrak{b} \in \Gamma\Big(\eth_\Omega; \sigma\Big)$, but $\mathfrak{a} \diamond \mathfrak{b} \notin \Gamma\Big(\eth_\Omega; \sigma\Big)$ for $\sigma = max\{\eth_\Omega(\mathfrak{a}), \eth_\Omega(\mathfrak{b})\}$. This is contradiction, and so $\eth_\Omega(\mathfrak{r} \diamond \mathfrak{s}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \forall \mathfrak{r}, \mathfrak{s} \in V$. In the same manner, the case $\mathfrak{r} \hookrightarrow \mathfrak{s} \notin \Gamma\Big(\eth_\Omega; \sigma\Big)$ performs to contradiction. Thus, $\eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \forall \mathfrak{r}, \mathfrak{s} \in V$. Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CC} - \mathcal{SH}$ of V. □

4 Crossing Cubic Filters

Here, we apply the $\mathcal{CC}s$ to filters of hoop algebras. We originate the concept of a $\mathcal{CCF}$, and investigate its characterizations.

Definition 4.1. A $\mathcal{CC}$ $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ in a hoop V is called a $\mathcal{CCF}$ of V if it satisfies (i) of Definition 3.1 and

$$(\mathbf{B}) \quad (\forall \mathfrak{r}, \mathfrak{s} \in V)(\mathfrak{r} \leq \mathfrak{s} \Rightarrow (\mathfrak{s}, \mathfrak{r}) \in \mathcal{C}(\Omega)).$$

Example 4.2. Consider the hoop $(V, \diamond, \hookrightarrow, \xi)$ which is demonstrated in Example 3.2. Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in a hoop V which is represented by the following table.

Table 4: $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is represented tabularly

V	$\widetilde{\Psi}_\Omega(\mathfrak{r})$	$\eth_\Omega(\mathfrak{r})$
0	[0.33, 0.61]	-0.36
1	[0.33, 0.61]	-0.53
ℓ	[0.39, 0.68]	-0.67
ξ	[0.41, 0.74]	-0.87

It is typical to check $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of $(V, \diamond, \hookrightarrow, \xi)$.

Next theorem debates the characterizations of a $\mathcal{CCF}$ of a hoop V.

Theorem 4.3. *A $\mathcal{CC}$ $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ in a hoop V is a $\mathcal{CCF}$ of V if and only if it satisfies the condition* $(\boldsymbol{A})$ *and*

$$(\boldsymbol{C}) \quad (\forall \mathfrak{r}, \mathfrak{s} \in V) \begin{pmatrix} \widetilde{\Psi}_\Omega(\mathfrak{s}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{s}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \end{pmatrix}.$$

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V. Since $\mathfrak{r} \leq 1$ for all $\mathfrak{r} \in V$, the condition $(\mathbf{A})$ is induced from the condition $(\mathbf{B})$. Note from (6) of Proposition 2.3 that $\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s}) \leq \mathfrak{s}$ for all $\mathfrak{r}, \mathfrak{s} \in V$, and so $(\mathfrak{s}, \mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) \in \mathcal{C}(\Omega)$ by the condition $(\mathbf{B})$. It follows from (i) of Definition 3.1 that

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{s}) &\succeq \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) \\ &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \end{aligned}$$

and

$$\begin{aligned} \eth_\Omega(\mathfrak{s}) &\leq \eth_\Omega(\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) \\ &\leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}. \end{aligned}$$

Hence, the condition $(\mathbf{C})$ is valid.

Conversely, suppose that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ satisfies the conditions $(\mathbf{A})$ and $(\mathbf{C})$. The combination of (V_1) and (V_3) induces $\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow (\mathfrak{r} \diamond \mathfrak{s})) = 1$ for all $\mathfrak{r}, \mathfrak{s} \in V$. It follows from the conditions $(\mathbf{A})$ and $(\mathbf{C})$ that

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond \mathfrak{s}) &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{\Psi}_\Omega(\mathfrak{s} \hookrightarrow (\mathfrak{r} \diamond \mathfrak{s}))\} \\ &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow (\mathfrak{r} \diamond \mathfrak{s})))\}\} \\ &= \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(1)\}\} \\ &= \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{\Psi}_\Omega(\mathfrak{r})\} \end{aligned}$$

and

$$\begin{aligned} \eth_\Omega(\mathfrak{r} \diamond \mathfrak{s}) &\leq max\{\eth_\Omega(\mathfrak{s}), \eth_\Omega(\mathfrak{s} \hookrightarrow (\mathfrak{r} \diamond \mathfrak{s}))\} \\ &\leq max\{\eth_\Omega(\mathfrak{s}), max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow (\mathfrak{r} \diamond \mathfrak{s})))\}\} \\ &= max\{\eth_\Omega(\mathfrak{s}), max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(1)\}\} \\ &= max\{\eth_\Omega(\mathfrak{s}), \eth_\Omega(\mathfrak{r})\}. \end{aligned}$$

Now, let $\mathfrak{r}, \mathfrak{s} \in V$ be such that $\mathfrak{r} \leq \mathfrak{s}$. Then, $\mathfrak{r} \hookrightarrow \mathfrak{s} = 1$, and so

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{s}) &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \\ &= \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(1)\} = \widetilde{\Psi}_\Omega(\mathfrak{r}) \end{aligned}$$

and

$$\begin{aligned} \eth_\Omega(\mathfrak{s}) &\leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \\ &= max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(1)\} = \eth_\Omega(\mathfrak{r}), \end{aligned}$$

that is, $(\mathfrak{s}, \mathfrak{r}) \in \mathcal{C}(\Omega)$. Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V. □

Theorem 4.4. *Every $\mathcal{CCF}$ is a $\mathcal{CCS}$.*

Proof. Straightforward. □

The opposite of Theorem 4.4 is not generally true, as shown in the following example.

Example 4.5. Consider the following tables that illustrate the binary operations "$\diamond$" and "$\hookrightarrow$" on the set $V = \{0, 1, \ell, \xi\}$.

Table 5: Cayley tables for the binary operations "$\diamond$" and "$\hookrightarrow$"

$\diamond$	0	1	ℓ	ξ
0	0	0	0	0
1	0	1	1	1
ℓ	0	1	1	ℓ
ξ	0	1	ℓ	ξ

$\hookrightarrow$	0	1	ℓ	ξ
0	ξ	ξ	ξ	ξ
1	0	ξ	ξ	ξ
ℓ	0	ℓ	ξ	ξ
ξ	0	1	ℓ	ξ

Consequently, $(V, \diamond, \hookrightarrow, \xi)$ is a hoop. Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V, which is illustrated in the table below.

Table 6: $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is represented tabularly

V	$\widetilde{\Psi}_\Omega(\mathfrak{r})$	$\eth_\Omega(\mathfrak{r})$
0	[0.43, 0.71]	-0.70
1	[0.36, 0.62]	-0.75
ℓ	[0.29, 0.58]	-0.55
ξ	[0.56, 0.84]	-0.85

It is typical to check that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CC} - \mathcal{SH}$ of $(V, \diamond, \hookrightarrow, \xi)$. Note that

$$\begin{aligned} \widetilde{\Psi}_\Omega(\ell) &= [0.29, 0.58] \\ &\prec \widetilde{min}\{\widetilde{\Psi}_\Omega(1), \widetilde{\Psi}_\Omega(1 \hookrightarrow \ell)\} \\ &= \widetilde{min}\{\widetilde{\Psi}_\Omega(1), \widetilde{\Psi}_\Omega(\xi)\} \\ &= \widetilde{min}\{[0.36, 0.62], [0.56, 0.84]\} = [0.36, 0.62] \end{aligned}$$

and

$$\begin{aligned} \eth_\Omega(\ell) &= -0.55 \\ &> max\{\eth_\Omega(1), \eth_\Omega(1 \hookrightarrow \ell)\} \\ &= max\{\eth_\Omega(1), \eth_\Omega(\xi)\} \\ &= max\{-0.75, -0.85\} = -0.75. \end{aligned}$$

Hence, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is not a $\mathcal{CCF}$ of $(V, \diamond, \hookrightarrow, \xi)$.

Proposition 4.6. *Every $\mathcal{CCF}$ $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ of V satisfies:*

$$(\boldsymbol{D}) \quad (\forall \mathfrak{r}, \mathfrak{s} \in V) \left(\begin{array}{c} \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond \mathfrak{s}) = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{r} \diamond \mathfrak{s}) = max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \end{array} \right).$$

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V. Since $\mathfrak{r} \diamond \mathfrak{s} \le \mathfrak{r}$ and $\mathfrak{r} \diamond \mathfrak{s} \le \mathfrak{s}$ for all $\mathfrak{r}, \mathfrak{s} \in V$, we have $(\mathfrak{r}, \mathfrak{r} \diamond \mathfrak{s}) \in \mathcal{C}(\Omega)$ and $(\mathfrak{s}, \mathfrak{r} \diamond \mathfrak{s}) \in \mathcal{C}(\Omega)$ by the condition (**B**). Hence, $\widetilde{\Psi}_\Omega(\mathfrak{r} \diamond \mathfrak{s}) \preceq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}$ and $\eth_\Omega(\mathfrak{r} \diamond \mathfrak{s}) \ge max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\}$. The combination of these and (i) of Definition 3.1 induces the condition (**D**). □

Theorem 4.7. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V. Then, it is a $\mathcal{CCF}$ of V if and only if it satisfies the conditions (**B**) and (**D**).*

Proof. The necessity is from Definition 4.1 and Proposition 4.6. Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in a hoop V that satisfies the conditions (**B**) and (**D**). Since $\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s}) \leq \mathfrak{s}$ for all $\mathfrak{r}, \mathfrak{s} \in V$, it follows from the conditions (**B**) and (**D**) that

$$\widetilde{\Psi}_\Omega(\mathfrak{s}) \quad \succeq \quad \widetilde{\Psi}_\Omega(\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}$$

and

$$\eth_\Omega(\mathfrak{s}) \quad \leq \quad \eth_\Omega(\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) = max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}.$$

Since $\mathfrak{r} \leq 1$ for all $\mathfrak{r} \in V$, the condition (**A**) is induced from the condition (**B**). Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V by Theorem 4.3. □

Theorem 4.8. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V. Then it is a $\mathcal{CCF}$ of V if and only if it satisfies the conditions (**B**) and*

$$(\boldsymbol{E}) \quad (\forall \mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V) \left(\begin{array}{c} \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}), \widetilde{\Psi}_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\}, \\ \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) \leq max\{\eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}), \eth_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\} \end{array} \right).$$

Proof. Suppose that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V and let $\mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V$. Since $(\mathfrak{r} \hookrightarrow \mathfrak{s}) \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u}) \leq \mathfrak{r} \hookrightarrow \mathfrak{u}$ by (10) of Proposition 2.3, it follows from the conditions (**B**) and (**D**) that

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) &\succeq \widetilde{\Psi}_\Omega((\mathfrak{r} \hookrightarrow \mathfrak{s}) \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u})) \\ &= \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}), \widetilde{\Psi}_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\} \end{aligned}$$

and

$$\begin{aligned} \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) &\leq \eth_\Omega((\mathfrak{r} \hookrightarrow \mathfrak{s}) \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u})) \\ &= max\{\eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}), \eth_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\}. \end{aligned}$$

This proves the condition (**E**).

Conversely, let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in a hoop V that satisfies the conditions (**B**) and (**E**). Since $\mathfrak{r} \leq 1$ for all $\mathfrak{r} \in V$, the condition (**A**) is induced from the condition (**B**). If we take $\mathfrak{r} = 1$ in the condition (**E**) and use (5) of Proposition 2.3, then we have the condition (**C**). Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V by Theorem 4.3. □

Theorem 4.9. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V. Then it is a $\mathcal{CCF}$ of V if and only if it satisfies the condition (**B**) and*

$$(\boldsymbol{F}) \quad (\forall \mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V) \left(\begin{array}{c} \widetilde{\Psi}_\Omega(\mathfrak{u} \diamond \mathfrak{s}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{u} \diamond \mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{u} \diamond \mathfrak{s}) \leq max\{\eth_\Omega(\mathfrak{u} \diamond \mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \end{array} \right).$$

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V and let $\mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V$. Since $(V, \diamond, 1)$ is a commutative monoid, and by (6) and (8) of Proposition 2.3, we can write

$$(\mathfrak{u} \diamond \mathfrak{r}) \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s}) = \mathfrak{u} \diamond (\mathfrak{r} \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) \leq \mathfrak{u} \diamond \mathfrak{s}.$$

It follows from the conditions (**B**) and (**D**) that

$$\widetilde{\Psi}_\Omega(\mathfrak{u} \diamond \mathfrak{s}) \quad \succeq \quad \widetilde{\Psi}_\Omega((\mathfrak{u} \diamond \mathfrak{r}) \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{u} \diamond \mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}$$

and

$$\eth_\Omega(\mathfrak{u} \diamond \mathfrak{s}) \quad \leq \quad \eth_\Omega((\mathfrak{u} \diamond \mathfrak{r}) \diamond (\mathfrak{r} \hookrightarrow \mathfrak{s})) = max\{\eth_\Omega(\mathfrak{u} \diamond \mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}.$$

This proves the condition (**F**).

Conversely, assume that a $\mathcal{CC}$ $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ in a hoop V satisfies the conditions (**B**) and (**F**). Since $\mathfrak{r} \leq 1$ for all $\mathfrak{r} \in V$, the condition (**A**) is induced from the condition (**B**). Also the assertion of the condition (**C**) is induced by taking $\mathfrak{u} = 1$ in the condition (**F**). Hence, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V by Theorem 4.3. □

Theorem 4.10. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V. Then it is a $\mathcal{CCF}$ of V if and only if it satisfies:*

$$(\boldsymbol{G}) \quad (\forall \mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V) \left(\mathfrak{r} \leq \mathfrak{s} \hookrightarrow \mathfrak{u} \Rightarrow \begin{cases} \widetilde{\Psi}_\Omega(\mathfrak{u}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{u}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\} \end{cases} \right).$$

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V and let $\mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V$ be such that $\mathfrak{r} \leq \mathfrak{s} \hookrightarrow \mathfrak{u}$. Then,

$$\widetilde{\Psi}_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow \mathfrak{u})\} = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(1)\} = \widetilde{\Psi}_\Omega(\mathfrak{r}),$$

and

$$\eth_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u}) \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow (\mathfrak{s} \hookrightarrow \mathfrak{u})\} = max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(1)\} = \eth_\Omega(\mathfrak{r}).$$

It follows that

$$\widetilde{\Psi}_\Omega(\mathfrak{u}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{\Psi}_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\} \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{s})\},$$

and

$$\eth_\Omega(\mathfrak{u}) \leq max\{\eth_\Omega(\mathfrak{s}), \eth_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\} \leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{s})\}.$$

This proves the condition (**G**).

Conversely, let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in a hoop V that satisfies the condition (**G**). Since $\mathfrak{r} \leq \mathfrak{r} \hookrightarrow 1$ for all $\mathfrak{r} \in V$, it is clear that $(1, \mathfrak{r}) \in V$ for all $\mathfrak{r} \in V$ by the conditions (**G**). The combination of (11) of Proposition 2.3 and the condition (**G**) induces the condition (**C**). Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V by Theorem 4.3. □

Theorem 4.11. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V. Then, it is a $\mathcal{CCF}$ of V if and only if it satisfies the conditions (**B**) and*

$$(\boldsymbol{H}) \quad (\forall \mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V) \left(\begin{array}{c} \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) \succeq \widetilde{min}\{\widetilde{\Psi}_\Omega((\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{u}), \widetilde{\Psi}_\Omega(\mathfrak{s})\}, \\ \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) \leq max\{\eth_\Omega((\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{u}), \eth_\Omega(\mathfrak{s})\} \end{array} \right).$$

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V and let $\mathfrak{r}, \mathfrak{s}, \mathfrak{u} \in V$. Using (4), (6) and (8) of Proposition 2.3, we have $(\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{u} \leq \mathfrak{s} \hookrightarrow \mathfrak{u}$ and $\mathfrak{s} \diamond ((\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{u}) \leq \mathfrak{s} \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u}) \leq \mathfrak{u} \leq \mathfrak{r} \hookrightarrow \mathfrak{u}$. From the conditions (**B**) and (**D**), we imply that

$$\begin{aligned} \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) &\succeq \widetilde{\Psi}_\Omega(\mathfrak{u}) \succeq \widetilde{\Psi}_\Omega(\mathfrak{s} \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u})) \\ &= \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{\Psi}_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\} \\ &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{s}), \widetilde{\Psi}_\Omega((\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{u})\} \end{aligned}$$

and

$$\begin{aligned} \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{u}) &\leq \eth_\Omega(\mathfrak{u}) \leq \eth_\Omega(\mathfrak{s} \diamond (\mathfrak{s} \hookrightarrow \mathfrak{u})) \\ &= max\{\eth_\Omega(\mathfrak{s}), \eth_\Omega(\mathfrak{s} \hookrightarrow \mathfrak{u})\} \\ &\leq max\{\eth_\Omega(\mathfrak{s}), \eth_\Omega((\mathfrak{r} \hookrightarrow \mathfrak{s}) \hookrightarrow \mathfrak{u})\} \end{aligned}$$

This proves the condition (**H**).

Conversely, suppose that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ satisfies the conditions (**B**) and (**H**). The condition (**A**) is induced from (5) of Proposition 2.3 and the condition (**B**). If we put $\mathfrak{r} = 1$ in the condition (**H**) and use (5) of Proposition 2.3, we can obtain the condition (**C**). Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V by Theorem 4.3. □

Theorem 4.12. *Let $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ be a $\mathcal{CC}$ in V. Then, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of V if and only if the nonempty sets $\Gamma\left(\widetilde{\Psi}_\Omega; \widetilde{\omega}\right)$ and $\Gamma\left(\eth_\Omega; \sigma\right)$ are filters of V for all $\widetilde{\omega} = [\omega^-, \omega^+] \in \mathbb{I}_\Psi[0,1]$ and $\sigma \in [-1, 0]$.*

Proof. Assume that $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of *V*. Then, it is a $\mathcal{CC}-\mathcal{SH}$ of *V*, and so $(1,\mathfrak{r}) \in \mathcal{C}(\Omega)$ for all $\mathfrak{r} \in V$ by Proposition 3.3. Let $\widetilde{\omega} = [\omega^-, \omega^+]$ and $\sigma \in [-1,0]$ be such that $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\Gamma\Big(\eth_\Omega; \sigma\Big)$ are nonempty. Then, there exist $\mathfrak{r} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\mathfrak{s} \in \Gamma\Big(\eth_\Omega; \sigma\Big)$ which imply that $\widetilde{\Psi}_\Omega(1) \succeq \widetilde{\Psi}_\Omega(\mathfrak{r}) \succeq \widetilde{\omega}$ and $\eth_\Omega(1) \leq \eth_\Omega(\mathfrak{s}) \leq \sigma$. Hence $1 \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$. Let $\mathfrak{r}, \mathfrak{s} \in V$ be such that $\mathfrak{r} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$ and $\mathfrak{r} \hookrightarrow \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$. Then, $\widetilde{\Psi}_\Omega(\mathfrak{r}) \succeq \widetilde{\omega}, \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) \succeq \widetilde{\omega}, \eth_\Omega(\mathfrak{r}) \leq \sigma$ and $\eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) \leq \sigma$. It follows that

$$\begin{aligned}
\widetilde{\Psi}_\Omega(\mathfrak{s}) &\succeq \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \succeq \widetilde{\omega}, \\
\eth_\Omega(\mathfrak{s}) &\leq max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\} \leq \sigma,
\end{aligned}$$

that is $\mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$. Therefore, $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\Gamma\Big(\eth_\Omega; \sigma\Big)$ are filters of *V* by Lemma 2.5.

Conversely, suppose that $\Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ and $\Gamma\Big(\eth_\Omega; \sigma\Big)$ are nonempty filters of *V* for all $\widetilde{\omega} = [\omega^-, \omega^+] \in \mathbb{I}_\Psi[0,1]$ and $\sigma \in [-1,0]$. Assume that $(1,\mathfrak{a}) \notin \mathcal{C}(\Omega)$ for some $\mathfrak{a} \in V$. Then $\widetilde{\Psi}_\Omega(1) \not\succeq \widetilde{\Psi}_\Omega(\mathfrak{a})$ or $\eth_\Omega(1) \not\leq \partial_\Omega(\mathfrak{a})$. If $\widetilde{\Psi}_\Omega(1) \not\succeq \widetilde{\Psi}_\Omega(\mathfrak{a})$, then $1 \notin \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big)$ for $\widetilde{\omega} = \widetilde{\Psi}_\Omega(\mathfrak{a})$. The case $\eth_\Omega(1) \not\leq \eth_\Omega(\mathfrak{a})$ induces $\eth_\Omega(1) > \eth_\Omega(\mathfrak{a})$ and so $1 \notin \Gamma\Big(\eth_\Omega; \sigma\Big)$ for $\sigma = \eth_\Omega(\mathfrak{a})$. This is a contradiction, and thus $(1,\mathfrak{r}) \in \mathcal{C}(\Omega)$ for all $\mathfrak{r} \in V$. For every $\mathfrak{r}, \mathfrak{s} \in V$, let $\widetilde{\omega}_\mathfrak{r}, \widetilde{\omega}_\mathfrak{s} \in \mathbb{I}_\Psi[0,1]$ and $\sigma_\mathfrak{r}, \sigma_\mathfrak{s} \in [-1,0]$ be such that $\widetilde{\Psi}_\Omega(\mathfrak{r}) = \widetilde{\omega}_\mathfrak{r}$, $\widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) = \widetilde{\omega}_\mathfrak{s}$, $\eth_\Omega(\mathfrak{r}) = \sigma_\mathfrak{r}$, $\eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s}) = \sigma_\mathfrak{s}$. If we take $\widetilde{\omega} = \widetilde{min}\{\widetilde{\omega}_\mathfrak{r}, \widetilde{\omega}_{\mathfrak{r}\hookrightarrow\mathfrak{s}}\}$ and $\sigma = max\{\sigma_\mathfrak{r}, \sigma_{\mathfrak{r}\hookrightarrow\mathfrak{s}}\}$, then $\mathfrak{r}, \mathfrak{r} \hookrightarrow \mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$. Thus, $\mathfrak{s} \in \Gamma\Big(\widetilde{\Psi}_\Omega; \widetilde{\omega}\Big) \cap \Gamma\Big(\eth_\Omega; \sigma\Big)$ and so

$$\begin{aligned}
\widetilde{\Psi}_\Omega(\mathfrak{s}) &\succeq \widetilde{\omega} = \widetilde{min}\{\widetilde{\omega}_\mathfrak{r}, \widetilde{\omega}_\mathfrak{s}\} = \widetilde{min}\{\widetilde{\Psi}_\Omega(\mathfrak{r}), \widetilde{\Psi}_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}, \\
\eth_\Omega(\mathfrak{s}) &\leq \sigma = max\{\sigma_\mathfrak{r}, \sigma_\mathfrak{s}\} = max\{\eth_\Omega(\mathfrak{r}), \eth_\Omega(\mathfrak{r} \hookrightarrow \mathfrak{s})\}.
\end{aligned}$$

Therefore, $\Omega = (\widetilde{\Psi}_\Omega, \eth_\Omega)$ is a $\mathcal{CCF}$ of *V* by Theorem 4.3. □

5 Conclusion

The idea of $\mathcal{CC}s$ permeates the fabric of modern mathematical evolution, directly affecting the interpretation of complex phenomena, innovative solutions to complex problems, and improving computing efficiency. This makes it even more important to study this concept. For this purpose, by using the concept of interval-valued fuzzy sets with N-structures, we decided to search and investigate this notion on hoop algebras, as sources of inspiration and ideas for this paper. We scrutinized the concepts of $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$, and analyzed their properties. We revealed the effect of the relationship between $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$. We found many theories on $\mathcal{CCF}s$ and proved them. We analyzed the characterizations of $\mathcal{CC}-\mathcal{SH}s$ and $\mathcal{CCF}s$.

This research offers the enhancement of current scientific knowledge in the world of $\mathcal{CC}s$, stimulating the development of all-new models and systems. It paves the way for promising future in-depth researches in this area, see ([26]). This opens new horizons for practical applications and contributes to the advancement of human knowledge and meeting the needs of the future. Further, the results of this study can be circulated to include a set of algebraic structures as an illustration of multi-hoop algebras, pseudo-hoop algebras, Kleene algebra, and EQ/BL/BCC/BG-algebras.

References

[1] B. Bosbach. Komplementäre halbgruppen, axiomatik und arithmetik. *Fund. Math.*, 64(3): 257—287, 1969.

[2] B. Bosbach. Komplementäre halbgruppen, kongruenzen und quotienten. *Fund. Math.*, 69(1): 1–14, 1970.

[3] J. R. Buchi, T. M.Owens. Complemented monoids and hoops. Unpublished manuscript, 1975.

[4] W.J. Blok, D. Pigozzi. On the structure of varieties with equationally definable principal congruences, III. *Algebra Universalis*, 32(4): 545–608, 1994.

[5] I. M. A. Ferreirim. On varieties and quasi varieties of hoops and their reducts. PhD thesis. University of Illinois at Chicago, 1992.

[6] P. Aglian, I. M. Ferreirim, F. Montagna. Basic hoops: An algebraic study of continuous t-norms. Studia Logica, 87(1): 73-98, 2007.

[7] F. Esteva, L. Godo. Monoidal t-norm based logic, towards a logic for left- continuous t-norms. *Fuzzy Sets Syst.*,124(3): 271–288, 2001.

[8] T. Kowalski, H. Ono. Residuated lattices: An algebraic glimpse at logic without contraction. *Japan Advanced Institute of Science and Technology*, 2001.

[9] G. Georgescu, L. Leustean, V. Preoteasa. Pseudo-hoops. *J. Mult.-Valued Log. Soft Comput.*, 11(1-2): 153–184, 2005.

[10] L. A. Zadeh. Fuzzy sets. *Inf. Control*, 8(3): 338–353, 1965.

[11] A. Rosenfeld. Fuzzy groups. *J. Math. Anal. Appl.*, 35(3): 512–517, 1971.

[12] T. Head. A metatheorem for deriving fuzzy theorems from crisp versions. *Fuzzy Sets. Syst.*, 73(3): 349–358, 1995.

[13] I. Jahan. The lattice of L-ideals of a ring is modular. *Fuzzy Sets. Syst.*, 199: 121–129, 2012.

[14] K. T. Atanassov. Intuitionistic fuzzy sets. *Int. J. Bioautomation.*, 20: 1–6, 2016.

[15] S. Tan, P. Wang, E. Lee. Fuzzy set operations based on the theory of falling shadows. *J. Math. Anal. Appl.* , 174(1): 242–255, 1993.

[16] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning-i. *Inf. Sci.*, 8(3): 199–249, 1975.

[17] Y. B. Jun, K. Hur, J. G. Lee, J. and Kim. Crossing cubic structures as an extension of bipolar fuzzy sets. *Ann. Fuzzy Math. Inform.*, 22(1): 1–15, 2021.

[18] M. A. Oztürk, D. Yılmaz, Y. B. Jun. Semigroup structures and commutative ideals of BCK-algebras based on crossing cubic set structures. *Axioms*, 11(1): 25, 2022.

[19] A. Al-Masarwah, N. Kdaisat, M. Abuqamar and K. Alsager. Crossing cubic Lie algebras. *AIMS Math.*, 9(8): 22112–22129, 2024.

[20] N. B. A. Al-Rahman, A. Al-Masarwah, A. U. Alkouri. Algebraic aspects of crossing cubic BE-algebras. *Fuzzy Inf. Eng.*, 17(1): 20–38, 2025.

[21] R. A. Borzooei, M. A. Kologani. On fuzzy filters of hoop-algebras. *J. Fuzzy. Math.*, 25(1): 177–195, 2017.

[22] A. Namdar. Fuzzy n-fold fantastic filters on hoop algebras. *J. algebr. hyperstrucres log. algebr.*, 4(2): 189–203, 2023.

[23] M. Aaly Kologani, R. A. Borzooei, H. S. Kim, Y. B. Jun, S. S. Ahn. Construction of some algebras of logics by using intuitionistic fuzzy filters on hoops. *AIMS Math.*, 6(11): 11950–11973, 2021.

[24] R. A. Borzooei, G. R. Rezaei, M. A. Kologhani, Y. B. Jun. Falling shadow theory with applications in hoops. *Bull. Sect. Log.*, 50(3): 337–353, 2021.

[25] M. M. Takallo, M. A. Kologani, Y. B. Jun, R. A. Borzooei. Lukasiewicz fuzzy filters in hoops. *J. Algebr. Syst.*, 12(1): 1–20, 2024.

[26] M. Abuqamar, A.G. Ahmad, N. Hassan. The algebraic structure of normal groups associated with Q-neutrosophic soft sets. *Neutrosophic Sets Syst.*, 48(1): 328–338, 2022.