Unconstrained Neutrosophic Nonlinear Programming Problems Gradient Projection Method

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Abstract:

Nonlinear programming is one of the most important methods used to obtain the optimal solution to many real-world problems. Given the importance of this method, numerous studies and research have been conducted in recent years with the aim of providing methods that help find the optimal solution. These studies and research have resulted in a basic structure used to find these solutions. This structure initially indicates that the optimal solution can be found at any boundary point in the feasible region, at a point within the feasible region, or at a discontinuity point. In this research, we present some of the important foundations and principles of nonlinear programming and the gradient projection method used in searching for the optimal solution to unrestricted nonlinear programming problems. We will reformulate these foundations and principles using neutrosophic logic concepts as a complement to our previous research, the aim of which is to provide a new vision for some operations research methods, a neutrosophic vision. Our focus will be on the improvement these concepts offer when used in the field of applied mathematics, through the more accurate and comprehensive solutions we obtain, which provide a margin of freedom commensurate with the Given the reality we live in, and the changes that can occur to the data of the actual issue under study, this requires decision makers to prepare many appropriate alternatives for each change.

Keywords: Nonlinear programming; Feasible region; Optimal solution Neutrosophic logic concepts

1. Introduction

Mathematical programming is both an art and a science. The art lies in the ability to express the concepts of efficiency and scarcity in a well-defined mathematical model for a given situation or system, while the science lies in deriving computational methods to solve this model. The system under consideration may already exist, and in this case, the goal of constructing and solving the model is to determine optimal behavior for the system to improve its performance. It may also be just an idea awaiting implementation. The goal of the model in such a situation is to identify the best structure for the system in the future. The strength of operations research methods lies in determining a solution without actually testing all possible solutions. This is done by using tests based on the basic concepts of nonlinear programming, such as the first and second derivatives of the objective function, the convexity and concavity of the function, and the Hessian matrix, through which we determine the type of optimal solution, whether it is a maximum or a minimum value. The solution we obtain after applying the appropriate method. The gradient projection method is one of the important methods used to obtain the optimal solution to unrestricted nonlinear programming problems. Given the importance of operations research methods, and after the emergence of neutrosophic logic, which introduced many important concepts that must be used in the field of applied mathematics, these concepts can be viewed through research [1]. The use of these concepts provides solutions to example problems that leave nothing to chance and are not affected by changes that may occur in the operating environment of the system under study due to the indeterminacy of these solutions, as they take into account both the worst and best conditions that the operating environment may be exposed to. This matter prompted us to reformulate many operations research methods according to the concepts of nitrosophic logic, see

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[2,3,8,9,10,11]. In continuation of what we have done previously, we present in this research a nitrosophic formulation of some of the basic foundations and principles used when searching for the optimal solution to unrestricted nonlinear programming problems, a neutrosophic vision. Our focus will be on the improvement that these concepts provide when used in the field of applied mathematics, through the more accurate and comprehensive solutions that we obtain, which give a margin of freedom that is commensurate with the reality of the situation in which we live in light of the changes that can occur to the data of the real problem under study, which requires decision makers to prepare many alternatives suitable for each change.

2. Discussion

A mathematical model is an example problem in which the objective and constraints are given in the form of mathematical functions. In other words, a mathematical model is a translation of a real problem into mathematical language. Example problems focus on finding the minimum or maximum value of a specific quantity called the objective function, which depends on a number of variables. These variables may be independent of each other or related to each other through a set of constraints. If a mathematical model contains any nonlinear expression, whether in the objective function or the constraints, the model is considered nonlinear. When studying nonlinear programming problems, we encounter two types of problems: constrained and unconstrained. In this paper, we present a study of the gradient projection method for an unconstrained nonlinear programming problem.

The study presented in this research is divided into two parts:

Part One: A classical reference study of unconstrained nonlinear programming problems and the gradient projection method used to find the optimal solution to these problems.

Part Two: A neutrosophic study of unconstrained nonlinear programming problems and the gradient projection method we will use to find the optimal solution to neutrosophic unconstrained nonlinear programming problems.

- **Part One**: [4,5,6,7]

We know that the goal of solving nonlinear programming problems without constraints is to find the general (minimum or maximum) value of a real function f that follows n real variables $x_1, x_2, \dots, x_n$ Each of these variables can take values from $-\infty$ to $+\infty$, meaning that there is no restriction on the vector $x \in R^n$

➢ **Definition of the Unrestricted Nonlinear Programming Problem:** [6]

Let us have a function $f: R^n \longrightarrow R$, defined as $f(x) = f(x_1, x_2, \dots, x_n)$, for any vector $x \in R^n$. Let the required value be:

$$Min f(x)$$
$$x \in R^n$$

1. Here, we must look for a point $\underline{x} \in R^n$ such that:

$$f(\underline{x}) \le f(x) \quad \forall x \in R^n \qquad (1)$$

If it exists, we say it is a general minimum limit point.

2. If the point $\underline{x}$ satisfies the relation:

$$f(\underline{x}) < f(x) \quad \forall x \in R^n \qquad (2)$$

we say it is a general and unique minimum limit point.

3. If there is a neighborhood $N(\underline{x})$ around the point $\underline{x}$ such that the relation:

$$f(\underline{x}) \le f(x) \quad \forall x \in N(\underline{x})$$

only in the neighborhood, then we say that $\underline{x}$ is a local minimum limit.

- **Gradient Projection Method :(As mentioned in the reference** [7])

The most commonly used component in nonlinear programming is the gradient of a function. To identify the nature of a function's gradient, we take a point x_j and search for the unknown maximum value of the function $f(x)$, which lies at point A. We assume that the only properties we can use in our search for examples are values related to the coordinates of the current solution vector. Using this local information only, we assume that we want to move from the current point x_j to a new point x_{j+1} in a way that allows us to reach the examples as quickly as possible. In particular, we assume that we want to move a distance s from point x_j to a new point x_{j+1}, which would be formed by moving a distance of one s toward the optimal solution. We write the following as a component for an i dimensional space:

$$x_{j+1}^{(i)} = x_j^{(i)} + s m_i \qquad (3)$$

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where m_i is the direction of motion of vehicle i, suppose we want to take a small step ds in such a way that the objective function $y = f(x)$ increases or decreases as much as possible. The following relation gives the distance of the movement:

$$ds = \sqrt{(dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2} \quad (4)$$

Assuming that the function y is differentiable, the change in y associated with a set of movements dx_i is given by the following relation:

$$dy = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right) dx_i$$

Or:

$$\frac{dy}{ds} = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right) \frac{dx_i}{ds} \quad (5)$$

We will make a special set of movements $\frac{dy}{ds}$ as large or small as possible. This is the direction of maximum ascent or descent. If we look at this issue as an example issue, we want to make equation (5) maximum or minimum, taking into account the constraints represented by equation (4), and we write:

Find the minimum (maximum) value of the function:

$$\frac{dy}{ds} = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right) \frac{dx_i}{ds}$$

Taking into account the following constraints:

$$ds = \sqrt{\sum_{i=1}^{n} (dx_i)^2}$$

- **Form the Lagrange function:**

Find the minimum (maximum) value of the function:

$$L = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right) \frac{dx_i}{ds} - \lambda \left[1 - \sum_{i=1}^{n} \left(\frac{dx_i}{dx}\right)^2\right]$$

- **Calculate the differential $\frac{dx_i}{ds}$ We find:**

$$\frac{\partial y}{\partial x_i} - 2\lambda \left(\frac{dx_i}{dx}\right) = 0 \quad ; \quad i = 1,2,\dots,n \quad (6)$$

- **We calculate the differential with respect to the Lagrange product:**

$$\sum_{i=1}^{n} \left(\frac{dx_i}{dx}\right)^2 = 1$$

That is:

$$\frac{1}{4\lambda^2} \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2 = 1$$

From this:

$$2\lambda = \pm \sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2}$$

- **The i component is written in parametric form as follows:**

$$x_{j+1}^{(i)} = x_j^{(i)} + \left[\frac{\partial y}{\partial x_i} \cdot \frac{1}{2\lambda}\right] s = x_j^{(i)} + m_i s$$

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Using Equation (6), we see that the motion that achieves the largest increase in y is given by the following relation:

$$m_i = \frac{\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2}}$$

The following relation gives the motion that produces the largest decrease:

$$m_i = \frac{-\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2}}$$

We note that the numerator is the gradient, while the denominator is the smoothing factor.

We illustrate the above through the following example:

Example 1:

Let the function be:

$$y = (4 - x_1)^2 + x_2^2$$

Our goal is to find the minimum point, assuming the initial solution is $(0,0)$.

That is:

$$x^{(1)} = (0,0)$$

We calculate the value of the function y:

$$y = (4 - x_1)^2 + x_2^2$$

$$y|_{x^{(1)}} = 16$$

The direction of greatest descent for component i is given by the following relation:

$$m_i = \frac{-\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2}}$$

For the given problem, we write the following:

$$\frac{\partial y}{\partial x_1} = -8 + 2x_1 \qquad \frac{\partial y}{\partial x_2} = 2x_2$$

$$\frac{\partial y}{\partial x_1}|_{(0,0)} = -8 \qquad \frac{\partial y}{\partial x_2}|_{(0,0)} = 0$$

$$m_1 = \frac{8}{\sqrt{64}} = 1 \qquad m_2 = \frac{0}{\sqrt{64}} = 0$$

Substitute into the following relation:

$$x_{j+1}^{(i)} = x_j^{(i)} + \left[\frac{\partial y}{\partial x_i} . \frac{1}{2\lambda}\right] s = x_j^{(i)} + m_i s$$

$$x_1^{(2)} = 0 + s(1) = s$$

$$x_2^{(2)} = 0 + s(0) = 0$$

We need to minimize the following expression with respect to s:

$$y = (4 - s\ )^2$$

$$\frac{\partial y}{\partial s} = -8 + 2s = 0 \Longrightarrow s = 4$$

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$$x_1^{(2)} = 4 \qquad x_2^{(2)} = 0$$

$$x^{(2)} = (4,0)$$

$$y|_{x^{(2)}} = 0$$

So $x^{(2)}$ is a stable point to determine whether it is a minimum value. We know that:

- **Necessary and sufficient conditions for the existence of general and local minimums**: [6,7]

Assume that the function $f(x)$ is continuous and that its first partial derivatives $\frac{\partial f}{\partial x_n}$ and second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are also continuous for any point $x \in R^n$, then we have:

Theorem: If the point $\underline{x} \in R^n$ is a local minimum limit of the function $f(x)$, then:

a. $\nabla f(\underline{x}) = 0$

b. The Hessian matrix of the function f:

$$\nabla^2 f(\underline{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

is a positive near-perfect matrix.

Theorem: With the same assumptions as the previous theorem, we say that the point x is a local minimum limit of the function $f(x)$ over R^n if:

$$\nabla f(\underline{x}) = 0$$

The Hessian matrix of the function f:

$$\nabla^2 f(\underline{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

is a positive near-perfect matrix.

- **Convex Functions:** [6,7]

When the function f is convex and defined on R^n, we can obtain necessary and sufficient conditions for a point to be a general minimum limit point. This can be summarized in the following theorem:

Theorem: If f is a convex, continuously differentiable function, then the necessary and sufficient condition for the point x to be a general minimum limit point of the function f on R^n is that

$\nabla f(\underline{x}) = 0$

The Hessian matrix for the given function is:

$$H = [2\ 0\ 0\ 2\ ]$$

The elements of the main diagonal are positive and the principal minor determinants are positive because:

$$|2| > 0$$

$$|2\ 0\ 0\ 2\ | = 4 > 0$$

If the Hessian matrix is defined positive for the point where the partial derivatives are zero, then the point $(4,0)$ is a local minimum limit point of the function $y = (4 - x_1)^2 + x_2^2$

The optimal solution point represents the center of a set of circles whose points on their circumferences are gradients to the optimal solution (gradient projection with circular circumferences).

- **Part Two**: A Neutrosophic Study of Unconstrained Nonlinear Programming Problems and the Gradient Projection Method, which we will use to find the optimal solution to neutrosophic nonlinear programming problems. Nonlinear programming constitutes an important and fundamental part of operations research and is more comprehensive than linear programming. Its applications have spread across all branches of science, including engineering, physics, chemistry, management, economics, and military fields, among others. A mathematical model is considered a nonlinear model if any component of the objective function or constraints is a nonlinear expression and nonlinear expressions may be in both. In previous research, we reformulated some concepts of nonlinear programming using neutrosophic logic concepts and some methods used to find the optimal solution $[8 - 11]$. In this research, we will use the gradient projection method presented in the first section of this research to find the optimal **solution to the neutrosophic nonlinear model, which is defined as follows:** [9]**.**

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The Neutrosophic Mathematical Model:

In an example problem where the objective and constraints are in the form of neutrosophic mathematical functions, the neutrosophic mathematical model is written in the following form:

$$f_N = f_N(x_1, x_2, \dots, x_n) \to (Max\ ) or\ (Min)$$

Subject to the following constraints:

$$g_{N_i}(x_1, x_2, \dots, x_n)(\le \ge =) b_{N_i}\ ; i = 1,2, \dots, m$$

$$x_1, x_2, \dots, x_n \ge 0$$

As previously mentioned, this model is nonlinear if any component of the objective function or constraints is a nonlinear expression, and nonlinear expressions may be in both. A neutrosophic nonlinear model is also a nonlinear model if one or all of the values of the variables in the objective function and constraints are neutrosophic. For the values on the other side of the relations that represent constraints, neutrosophic values are values that are not fully defined. See [1].

Using these values, the model in Example (1) is written as follows:

Find the minimum value of:

$$y_N = \left((4 \mp \varepsilon) - x_1\right)^2 + \left((1 \mp \delta) - x_2\right)^2$$

where $\varepsilon \in \{\lambda_1, \lambda_2\}$ or $\varepsilon \in [\lambda_1, \lambda_2\ ]$ and $\delta \in \{\mu_1, \mu_2\}$ or $\delta \in [\mu_1, \mu_2\ ]$.

These values $\left((4 \mp \varepsilon), (1 \mp \delta)\right)$ are centers of circles whose equations are

$$y_N = \left((4 \mp \varepsilon) - x_1\right)^2 + \left((1 \mp \delta) - x_2\right)^2$$

The objective function values are the radii of these circles, with $(4 \mp \varepsilon)$ representing the horizontal component and $(1 \mp \delta)$ representing the vertical component.

Example 2:

Let's take $(1 \mp \delta) = 0$ and $(4 \mp \varepsilon) \in [3,5]$,The problem becomes:

Using the gradient projection method, find the minimum of

$$y_N = ([3,5] - x_1)^2 + x_2^2$$

Assume the initial solution is $(0,0)$.

That is:

$$x^{(1)} = (0,0)$$

$$y|_{x^{(1)}} = [9,25]$$

The direction of greatest descent for vehicle i is given by the following relation:

$$m_i = \frac{-\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2}}$$

For the given problem, we write the following:

$$\frac{\partial y}{\partial x_1} \in [-10, -6] + 2x_1 \qquad \frac{\partial y}{\partial x_2} = 2x_2$$

$$\frac{\partial y}{\partial x_1}|_{(0,0)} \in [-10, -6] \qquad \frac{\partial y}{\partial x_2}|_{(0,0)} = 0$$

$$m_1 = \frac{[6,10]}{\sqrt{[36,100]}} = 1 \qquad m_2 = \frac{0}{\sqrt{[36,100]}} = 0$$

Substitute into the following equation:

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$$x_{j+1}^{(i)} = x_j^{(i)} + \left[\frac{\partial y}{\partial x_i} \cdot \frac{1}{2\lambda}\right] s = x_j^{(i)} + m_i s$$

$$x_1^{(2)} = 0 + s(1) = s$$

$$x_2^{(2)} = 0 + s(0) = 0$$

We need to minimize the following expression with respect to s:

$$y = ([3,5] - s)^2$$

$$\frac{\partial y}{\partial s} = -[6,10] + 2s = 0 \Longrightarrow s \in [3,5]$$

$$x_1^{(2)} \in [3,5] \qquad x_2^{(2)} = 0$$

$$x_N^{(2)} \in ([3,5], 0)$$

$$\frac{\partial y}{\partial x_1} = -[6,10] + 2[3,5] = 0 \qquad \frac{\partial y}{\partial x_2} = 2x_2 = 0$$

Therefore, the points $x_N^{(2)} \in ([3,5], 0)$ are stable.

Substituting the objective function expression, we find:

$$y|_{x^{(2)}} = 0$$

To determine whether the values ??are minimum and satisfy the requirement, we perform the Hessian matrix test.

The Hessian matrix for the given function is:

$$H = [[3,5]\ 0\ 0\ 2\ ]$$

The elements of the main diagonal are positive, and the main principal minor determinants are positive because:

$$|[3,5]| > 0$$

$$|[3,5]\ 0\ 0\ 2\ | = [6,10] > 0$$

The Hessian matrix is ??defined positive for the points where the partial derivatives are zero. Therefore, the points are optimal solutions and give minimum values ??of the objective function

$y_N = ([3,5] - x_1)^2 + x_2^2$

Each point of the optimal solution represents the center of a set of circles whose points on its circumference are gradients to the optimal solution (gradient projection with circular circumferences).

3. Conclusion and Results

One of the important methods used to find the optimal solution for unconstrained nonlinear models is the gradient projection method. In this research, we presented the method as it appears in references. We then used this method to find the optimal solution for neutrosophic nonlinear models, and we found the following:

In the traditional study, the optimal solution was the point $x^{(2)} = (4,0)$. This point represents the center of a set of circles whose points on their circumferences are graded toward the optimal solution (a gradient projection with circular contours). In the neutrosophic study, the optimal solution was a set of points $x_N^{(2)} \in ([3,5], 0)$, each of which represents the center of a set of circles whose points on their circumferences are graded toward the optimal solution (a gradient projection with circular contours). Comparing the two solutions, we find that the optimal solution point in the traditional study is one of the points in the set of optimal solutions we obtained in the neutrosophic study.

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