Numerical Solution of Fuzzy Second Kind Fredholm Integral Equations in Double Parametric Form of Fuzzy Number

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Abstract

In this paper, two numerical methods that are method of successive approximations and Fredholm's first fundamental theorem are developed, reformatted, and applied to solve fuzzy second kind Fredholm integral equations with a separable kernel. The fuzziness in the equations is represented utilizing convex normalized triangular fuzzy numbers, which are based on a single and double parametric form of fuzzy numbers. A comparative analysis study between the proposed schemes are discussed through numerical experiment. It was found that Fredholm's first fundamental theorem is more efficient and effective than method of successive approximations. Furthermore, the double parametric form of fuzzy number is a general and more reliable than single parametric form since it reduced the computational cost and provides more certain fuzzy cases.

Keywords: Fuzzy Fredholm integral equations; Successive approximations; Fredholm's first fundamental method; Numerical methods

1. Introduction

Integral equations and their solutions play a crucial role in scientific research and engineering applications. They are widely used to model various physical phenomena and are generally classified into two types that are fredholm and volterra integral equations. The fuzzy set theory is widely known as an effective mathematical tool for modeling uncertainty and imprecision [1−3]. It has been successfully applied in various scientific fields for providing practical solutions to real-world phenomena. In instance, fuzzy logic have been useful in communication, decision-making, and computer programming [4−6]. In integral equations, crisp quantities that involve ambiguous and uncertainty can be replaced with uncertainty fuzzy quantities, leading to fuzzy integral equations (FIEs).

The importance of FIEs is increasing in fuzzy studies, mainly due to the growing role in data analysis of high-dimension complex systems where multiple parameters are involved in observations [7-14]. Another reason for studying equations with fuzzy-valued parameters (FVPs) and fuzzy-valued variables are the challenge of accurately estimating or measuring these parameters and variables. To develop accurate models that closely represent the processes of real-world problems, the fuzzy numbers are often used to approximate precise parameters. Among the FIEs, the fuzzy fredholm integral equation (FFIE) is widely studied by many mathematicians [14-17].

In solve of FFIE is difficult to obtain since exact solutions are often unavailable for complicated integral equations. Therefore, some numerical or approximate methods are commonly utilized to approximate their

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solutions. In particular, solving FFIE is more difficult than these of the first type and often requires numerical techniques such as the Fredholm's first fundamental theorem and successive approximations method.

The essential objective of this article is to analyze, develop, and implement the Fredholm's first fundamental theorem and the method of successive approximations to solve FFIE with a separable kernel. Additionally, the study aims to investigate utilize of the parametric double form. The fuzzy integral equation in the single parametric form is transformed into two crisp equations, which must be solved separately to get the fuzzy upper and fuzzy lower bounds of the numerical solution. This approach increases computational cost. To enhance computational efficiency and improve accuracy, this study explores the potential advantages of the double parametric form of fuzzy numbers.

2. Second-Kind Fredholm Integral Equations in Fuzzy Environment

Consider FFIE of second kind, along with the fuzzy boundary and initial condition.

$$\tilde{y}(x;r) = \tilde{f}(x;r) + \tilde{\lambda}\int_a^b k\,(x,t)\tilde{y}(t;r)dt \tag{1}$$

Based on fuzzy set principle [2] and using the r-level set approach and the zadeh extension principle, the Eq. (1) is reformulated as follows

$$\tilde{f}(x) = \begin{cases} \underline{f}(x;r) \\ \overline{f}(x;r) \end{cases} \tag{2}$$

$$\tilde{y}(x) = \begin{cases} \underline{y}(x;r) \\ \overline{y}(x;r) \end{cases} \tag{3}$$

$$\tilde{y}(t) = \begin{cases} \underline{y}(t;r) \\ \overline{y}(t;r) \end{cases} \tag{4}$$

$$\tilde{\lambda} = \begin{cases} \underline{\lambda}(r) \\ \overline{\lambda}(r) \end{cases} \tag{5}$$

Equations (2) to (5) are substituted into Equation (1) to obtain the follows:

$$\begin{cases} \underline{y}(x;r) = \underline{f}(x;r) + \underline{\lambda}(r)\displaystyle\int_a^b k\,(x,t)\underline{y}(t;r)dt \\ \overline{y}(x;r) = \overline{f}(x;r) + \overline{\lambda}(r)\displaystyle\int_a^b k\,(x,t)\overline{y}(t;r)dt \end{cases} \tag{6}$$

Equation (6) provides the general form of FFIE expressed in the parametric single representation of a fuzzy uncertain number.

In its single representation parametric form in Equation (1) defined as following:

$$\left[\underline{y}(x,r),\overline{y}(x,r)\right] = \left[\underline{f}(x,r),\overline{f}(x,r)\right] + [\underline{\lambda},\overline{\lambda}]\int_a^b [k(x,t)]\left[\underline{y}(t,r),\overline{y}(t,r)\right]dt$$

Based on use the parametric double form defined in [15], Equation (2) expressed as follwoing:

$$\begin{aligned} \beta\left[\overline{y}(x;r) - \underline{y}(x;r)\right] + \underline{y}(x;r) \\ = \beta\left[\overline{f}(x;r) - \underline{f}(x;r)\right] + \underline{f}(x;r) + [\beta[\overline{\lambda} - \underline{\lambda}] \\ + \underline{\lambda}]\int_a^b [k(x,t)]\beta\left[\overline{y}(t;r) - \underline{y}(t;r)\right] + \overline{y}(t;r)dt \end{aligned}$$

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Note that:

- $\tilde{f}(x;r,\beta) = \beta\left[\overline{f}(x;r) - \underline{f}(x;r)\right] + \underline{f}(x;r)$
- $\tilde{y}(t;r,\beta) = \beta\left[\overline{y}(t;r) - \underline{y}(t;r)\right] + \overline{y}(t;r)$
- $\tilde{y}(x;r,\beta) = \beta\left[\overline{y}(x;r) - \underline{y}(x;r)\right] + \underline{y}(x;r)$
- $\tilde{\lambda}(r,\beta) = \beta\left[\overline{\lambda}(r) - \underline{\lambda}(r)\right] + \underline{\lambda}(r)$

Replacing the above equations into Equation (1) yields:

$$\tilde{y}(x;r,\beta) = \tilde{f}(x;r,\beta) + \tilde{\lambda}(r,\beta)\int_a^b k(x,t)\tilde{y}(t;r,\beta)\,dt \qquad (7)$$

Eq. (7) is the general form of the FFIE under double-parametric form. To get the fuzzy upper and fuzzy lower borders for numerical results. We assigned $\beta = 1$ and $\beta = 0$, respectively which can may be represented as $\tilde{y}(x,r,1) = \overline{y}(x,r)$ and $\tilde{y}(x,r,0) = \underline{y}(x,r)$.

3. Fuzzy successive approximation method

In this section, we develop, reformat, and implement the method of successive approximation for solving the FFIE in both fuzzy double and single parametric forms.

3.1 Solving FFIE in single parametric formula.

An iterative method will explore in this section using the same approach for second-order integral equation. We will let $k(x,t)$ and $f(x)$ including in the integral equation.

Consider the FFIE presented as follows:

$$\begin{cases} \underline{y}(x;r) = \underline{f}(x;r) + \underline{\lambda}(r)\int_a^b k(x,t)\underline{y}(t;r)dt \\ \overline{y}(x;r) = \overline{f}(x;r) + \overline{\lambda}(r)\int_a^b k(x,t)\overline{y}(t;r)dt \end{cases} \qquad (8)$$

So, $k_n(x,t), n = 1,2\ldots$ is assigned as following:

$$k(x,t) = k_1(x,t) \qquad (9)$$

$$k_n(x,t) = \int_a^b k(x,z)k_{n-1}(z,t)dz \quad n = 1,2,3\ldots \qquad (10)$$

Then, $\tilde{R}(x,t;\lambda)$ is finds utilizing the iterated kernel as the following approach:

Assume $k_n(x,t)$ and assume $\tilde{R}(x,t;\lambda)$ be the resolving kernel of Eq.1, so we have the following

$$\tilde{R}(x,t;\lambda;r) = \begin{cases} \underline{R}(x,t;\lambda;r) = \sum_{n=1}^{\infty} \underline{\lambda}^{n-1}(r)k_n(x,t;r) \\ \overline{R}(x,t;\lambda;r) = \sum_{n=1}^{\infty} \overline{\lambda}^{n-1}(r)\,k_n(x,t;r) \end{cases} \qquad (11)$$

Assume the sum of infinite series in Eq. (2) is exists and then $\tilde{R}(x,t,\lambda;r)$ presented in the closed formula. Therefore, the demand solution for Eq.(1) is presented by

$$\tilde{y}(x;r) = \tilde{f}(x;r) + \tilde{\lambda}(r)\int_a^b \tilde{R}(x,t;\lambda;r)\tilde{f}(t;r)\,dt \qquad (12)$$

3.2 Solving FFIE in double parametric form.

Consider the FFIE is presented in the Equation (1) as follows:

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$$\beta\left[\overline{y}(x;r)-\underline{y}(x;r)\right]+\underline{y}(x;r)$$
$$=\beta\left[\overline{f}(x;r)-\underline{f}(x;r)\right]+\underline{f}(x;r)+[\beta[\overline{\lambda}(r)-\underline{\lambda}(r)]$$
$$+\underline{\lambda}(r)]\int_a^b[k(x,t)]\beta\left[\overline{y}(t;r)-\underline{y}(t;r)\right]+\overline{y}(t;r)dt$$

(13)

So, the iterated kernels $k_n(x,t), n=1,2,3,\ldots$ are presented as following:

$$k(x,t)=k_1(x,t)$$

Then, $\tilde{R}(x,t;\lambda,\beta)$ is finds based on the $k_n(x,t)$ in following approach:

Assume $k_n(x,t)$ and assume $\tilde{R}(x,t;\lambda,\beta)$ of Eq. (1) in the following form:

$$\tilde{R}(x,t;\lambda,r,\beta)=\sum_{n=1}^{\infty}\tilde{\lambda}^{n-1}(r,\beta)\,k_n(x,t;r,\beta)$$

Assume the sum of infinite series in Eq. (2) is exists and then $\widetilde{R}(x,t,\lambda)$ can be presented in the closed form. Therefore, the demand solution of Eq.(1) is presented:

$$\tilde{y}(x;r,\beta)=\tilde{f}(x;r,\beta)+\tilde{\lambda}(r,\beta)\int_a^b\tilde{R}(x,t;\lambda,\beta)\tilde{f}(t;r,\beta)\,dt$$

4. Fuzzy Fredholm's First Fundamental Theorem

In this section, we develop, reformat, and implement the Fredholm's first fundamental theorem to solve the FFIE in both fuzzy single and fuzzy parametric double formula.

Consider the FFIE that is presented in the Equation (6):

$$\begin{cases}\underline{y}(x;r)=\underline{f}(x;r)+\underline{\lambda}(r)\displaystyle\int_a^b k(x,t)\underline{y}(t;r)dt\\ \overline{y}(x;r)=\overline{f}(x;r)+\overline{\lambda}(r)\displaystyle\int_a^b k(x,t)\overline{y}(t;r)dt\end{cases}$$

As a uniformly convergent power series in the parameter λ for $|\lambda|$suitably small. Fredholm derived the solution of FFIE in general form which is valid for all values of the parameter λ. He gave three important results which are known as Fredholm's first, second and third fundamental theorems. In the current study we propose to discuss Fredholm's first theorem.

4.1 Solving FFIE in single parametric form

Consider the FFIE presented in the Equation (6):

$$\begin{cases}\underline{y}(x;r)=\underline{f}(x;r)+\underline{\lambda}(r)\int_a^b K(x,t)\underline{y}(t;r)dt\\ \overline{y}(x;r)=\overline{f}(x,r)+\overline{\lambda}(r)\int_a^b K(x,t)\overline{y}(t;r)dt\end{cases}$$

where the functions $\underline{f}(x;r)$ and $\underline{y}(t;r)$ are integrable, has unique solution. This solution is defined using the resolvent kernel $\underline{R}(x,t\!:\!\lambda)$ which is given by:

$$\begin{cases}\underline{R}(x;t\!:\!\lambda;r)=\underline{D}(x,t\!:\!\lambda;r)/\underline{D}(\lambda)\\ \overline{R}(x,t\!:\!\lambda;r)=\overline{D}(x,t\!:\!\lambda;r)/\overline{D}(\lambda)\end{cases}$$

(14)

The function $\widetilde{D}(x,t\!:\!\lambda)$ is called the Fredholm minor and $\widetilde{D}(\lambda)$ is called the Fredholm determinant with $\widetilde{D}(\lambda)\neq 0$, is a mermorphic function of the complex variable λ, being the ratio of two entire functions defined by the series

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$$\begin{cases} \underline{D}(x,t,\lambda;r) = \underline{K}(x,t;r) + \sum_{p=1}^{\infty} \frac{(-\underline{\lambda})^p}{p!}\, \underline{B}_p\,(x,t;r) \\ \overline{D}(x,t,\lambda;r) = \overline{K}(x,t;r) + \sum_{p=1}^{\infty} \frac{(-\overline{\lambda}\,)^p}{p!}\, \overline{B}_p\,(x,t;r) \end{cases} \quad (15)$$

and

$$\begin{cases} \underline{D}(\lambda;r) = 1 + \sum_{p=1}^{\infty} \frac{(-\underline{\lambda})^p}{p!}\ \underline{C}_p \\ \overline{D}(\lambda;r) = 1 + \sum_{p=1}^{\infty} \frac{(-\overline{\lambda})^p}{p!}\ \overline{C}_p \end{cases} \quad (16)$$

where $\underline{B}_p(x,t;r)$ and $\underline{C}_p$ are defined as follows:

$$\begin{cases} \underline{B}_p(x,t;r) = \int_a^b \dots \int_a^b \begin{vmatrix} K(x,t;r) & K(x,z_1;r) & \cdots & K(x,z_n;r) \\ K(z_1,t;r) & K(z_1,z_1;r) & \cdots & K(z_1,z_n;r) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,t;r) & K(z_n,z_1;r) & \cdots & K(z_n,z_n;r) \end{vmatrix} dz_1 dz_2 \cdots dz_n \\ \overline{B}_p(x,t;r) = \int_a^b \dots \int_a^b \begin{vmatrix} K(x,t;r) & K(x,z_1;r) & \cdots & K(x,z_n;r) \\ K(z_1,t;r) & K(z_1,z_1;r) & \cdots & K(z_1,z_n;r) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,t;r) & K(z_n,z_1;r) & \cdots & K(z_n,z_n;r) \end{vmatrix} dz_1 dz_2 \cdots dz_n \end{cases} \quad (17)$$

and

$$\begin{cases} \underline{C}_p(r) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(z_1,z_1;r) & K(z_1,z_2;r) & \cdots & K(z_1,z_n;r) \\ K(z_2,z_1;r) & K(z_2,z_2;r) & \cdots & K(z_2,z_n;r) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,z_1;r) & K(z_n,z_2;r) & \cdots & K(z_n,z_n;r) \end{vmatrix} dz_1 dz_2 \cdots dz_n \\ \overline{C}_p(r) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(z_1,z_1;r) & K(z_1,z_2;r) & \cdots & K(z_1,z_n;r) \\ K(z_2,z_1;r) & K(z_2,z_2;r) & \cdots & K(z_2,z_n;r) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,z_1;r) & K(z_n,z_2;r) & \cdots & K(z_n,z_n;r) \end{vmatrix} dz_1 dz_2 \cdots dz_n \end{cases} . \quad (18)$$

To calculate the $\tilde{B}_p(x,t;r)$ and $\tilde{C}_p(r)$, the Alternative procedure approach is used as define in [18] as follows .

$$\begin{cases} \underline{C}_p(r) = \int_a^b \underline{B}_{p-1}(s,s;r)ds\,, & p \geq 1 \\ \overline{C}_p(r) = \int_a^b \overline{B}_{p-1}(s,s;r)ds\,, & p \geq 1 \end{cases} \quad (19)$$

with assume $\underline{C}_0$, $\overline{C}_0 = 1$

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$$\begin{cases}\underline{B}_0(x,t;r)=\underline{K}(x,t;r)\\ \overline{B}_0(x,t;r)=\overline{K}(x,t;r)\end{cases} \tag{20}$$

$$\begin{cases}\underline{B}_p(r)=\underline{C}_p(r)K(x,t)-p\int_a^b K(x,z;r)\underline{B}_{p-1}(z,t;r)dz\,, & p\geq 1\\ \overline{B}_p(r)=\overline{C}_p(r)K(x,t)-p\int_a^b K(x,z;r)\,\overline{B}_{p-1}(z,t;r)dz\,, & p\geq 1\end{cases} \tag{21}$$

After getting $\tilde{R}(x,t;\lambda)$, the required solution is defined as following:

$$\tilde{y}(x;r,\beta)=\tilde{f}(x;t,\beta)+\overline{\lambda}(r,\beta)\int_a^b \tilde{R}(x,t;\lambda)\,\tilde{f}(t;r,\beta)\,dt \tag{22}$$

4.2 Solving FFIE in double parametric form

Consider the FFIE that is prsented in the Eq. (6)

$$\tilde{y}(x;r,\beta)=\tilde{f}(x;r,\beta)+\tilde{\lambda}(r,\beta)\int_a^b \tilde{K}(x,t)\,\tilde{y}(t;r\beta)dt$$

where the functions $\tilde{f}(x;r,\beta)$ and $\tilde{y}(x;r,\beta)$ are integrable, has unique solution. This solution is defined using the resolvent kernel $\tilde{R}(x,t\colon\lambda,\beta)$ which is given by:

$$\tilde{R}(x,t\colon\lambda,\beta)=\widetilde{D}(x,t\colon\lambda,\beta)/\widetilde{D}(\lambda;r,\beta) \tag{23}$$

The function $\widetilde{D}(x,t\colon\lambda,\beta)$ is called the Fredholm minor and $\widetilde{D}(\lambda;r,\beta)$ is called the Fredholm determinant with $\widetilde{D}(\lambda)\neq 0$, is a mermorphic function of the complex variable λ, being the ratio of two entire functions defined by the series

$$\widetilde{D}(x,t,\lambda;r,\beta)=\widetilde{K}(x,t;r,\beta)+\sum_{p=1}^{\infty}\frac{(-\underline{\lambda})^p}{p!}\,\widetilde{B}_p\,(x,t;r,\beta) \tag{24}$$

and

$$\widetilde{D}(\lambda;r,\beta)=1+\sum_{p=1}^{\infty}\frac{(-\underline{\lambda})^p}{p!}\;\tilde{C}_p \tag{25}$$

where $\tilde{B}_p(x,t;r,\beta)$and $\underline{C}_p$ are defined in the following:

$$\tilde{B}_p(x,t;r,\beta)=\int_a^b\dots\int_a^b\begin{vmatrix}K(x,t;r,\beta) & K(x,z_1;r,\beta) & \cdots & K(x,z_n;r,\beta)\\ K(z_1,t;r,\beta) & K(z_1,z_1;r,\beta) & \cdots & K(z_1,z_n;r,\beta)\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ K(z_n,t;r,\beta) & K(z_n,z_1;r,\beta) & \cdots & K(z_n,z_n;r,\beta)\end{vmatrix}dz_1dz_2\cdots dz_n \tag{26}$$

and

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$$\underline{C}_p(r) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(z_1,z_1;r,\beta) & K(z_1,z_2;r,\beta) & \cdots & K(z_1,z_n;r,\beta) \\ K(z_2,z_1;r,\beta) & K(z_2,z_2;r,\beta) & \cdots & K(z_2,z_n;r,\beta) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,z_1;r,\beta) & K(z_n,z_2;r,\beta) & \cdots & K(z_n,z_n;r,\beta) \end{vmatrix} dz_1 dz_2 \cdots dz_n \quad (27)$$

To calculate the $\tilde{B}_p(x,t;r,\beta)$ and $\tilde{C}_p(r,\beta)$, the Alternative procedure approach is used as define in [18] as follows .

$$\tilde{C}_p(r,\beta) = \int_a^b \tilde{B}_{p-1}(s,s;r,\beta)ds, \qquad p \geq 1 \quad (28)$$

with assume $\tilde{C}_0(r,\beta) = 1$

$$\tilde{B}_0(x,t;r,\beta) = \tilde{K}(x,t;r,\beta) \quad (29)$$

$$\tilde{B}_p(r,\beta) = \tilde{C}_p(r,\beta)K(x,t) - p\int_a^b K(x,z;r,\beta)\tilde{B}_{p-1}(z,t;r,\beta)dz, \qquad p \geq 1 \quad (30)$$

After getting $\tilde{R}(x,t;\lambda,r,\beta)$, the demand solution is given by.

$$\tilde{y}(x;r,\beta) = \tilde{f}(x;r,\beta) + \tilde{\lambda}(r,\beta)\int_a^b \tilde{R}(x,t;\lambda;r,\beta)\,\tilde{f}(t;r,\beta)dt \quad (31)$$

where:

$\tilde{y}(x;r,\beta) = \beta\left[\overline{y}(x;r) - \underline{y}(x;r)\right] + \underline{y}(x;r)$

$\tilde{\lambda}(r,\beta) = \beta\left[\overline{\lambda}(r) + \underline{\lambda}(r)\right] + \underline{\lambda}(r)$

$\tilde{f}(x;r,\beta) = \beta\left[\overline{f}(x;r) - \underline{f}(x;r)\right] + \underline{f}(x;r)$

$\tilde{R}(x,t;\lambda) = \beta\left[\overline{R}(x,t;\lambda;r) + \underline{R}(x,t;\lambda;r)\right] + \underline{R}(x,t;\lambda;r)$

5. Numerical Example

In this section, a numerical example is presented of FFIE.

Example (1): Consider the FFIE in single parametric form:

$$\tilde{y}(x;r) = [0.75 + 0.25r, 1.25 - 0.25r]\frac{5}{6}\tilde{x} + \frac{\tilde{1}}{2}\int_0^1 xt\,\tilde{y}(t;r)\,dt$$

With $\tilde{y}_0(x;r) = 1$

Solution

$$\begin{cases} \underline{y}(x;r) = [0.75 + 0.25r]\frac{5}{6}\underline{x} + \frac{1}{2}\int_0^1 xt\underline{y}(t;r)\,dt \\ \overline{y}(x;r) = [1.25 - 0.25r]\frac{5}{6}\overline{x} + \frac{1}{2}\int_0^1 xt\overline{y}(t;r)\,dt \end{cases}$$

Where $K(x,t) = xt$ and $\tilde{\lambda} = \frac{1}{2}$

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For $\underline{y}(x;r) = [0.75 + 0.25r]\frac{5}{6}\underline{x} + \frac{1}{2}\int_0^1 xt\underline{y}(t;r)\,dt$

First, using Fredholm's first fundamental theorem

$$\underline{B}_0(x,t) = K(x,t) = xt$$
$$\underline{C}_0 = 1$$

$$\underline{C}_1 = \int_a^b \underline{B}_{p-1}(s,s)\,ds$$

$$\underline{C}_1 = \int_0^1 s^2\,ds = \frac{1}{3}$$

$$\underline{B}_p(x,t) = \underline{C}_p K(x,t) - p\int_a^b K(x,z)\underline{B}_{p-1}(z,t)\,dz$$

$$\underline{B}_1(x,t) = \frac{1}{3}xt - \frac{1}{3}xt = 0$$

$\underline{C}_2 = \int_0^1 0\,ds$ =0

$$\underline{B}_2(x,t) = 0 - \int_0^1 0\,dz = 0$$

$$\vdots$$

Now,

$$\underline{D}(x,t;\lambda) = K(x,t) + \sum_{p=1}^{\infty}\frac{(-\underline{\lambda})^p}{p!}\underline{B}_p(x,t) = xt$$

$$\underline{D}(\lambda) = 1 + \sum_{p=1}^{\infty}\frac{(-\underline{\lambda})^p}{p!}\underline{C}_p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\underline{R}(x,t;\lambda) = \frac{\underline{D}(x,t,\lambda)}{\underline{D}(\lambda)} = \frac{xt}{5/6} = \frac{6xt}{5}$$

$$\underline{y}(x;t) = \underline{f}(x;t) + \underline{\lambda}\int_a^b \underline{R}(x,t;\lambda)\underline{f}(t;r)\,dt$$

$$\underline{y}(x;t) = [0.75 + 0.25r]\frac{5}{6}x + \frac{1}{2}\int_0^1 \frac{6xt}{5}[0.75 + 0.25r]\frac{5}{6}tdt = [0.75 + 0.25r]\frac{5}{6}x + [0.75 + 0.25r]\frac{1}{6}x$$
$$= [0.75 + 0.25]x$$

Similarly, for $\overline{y}(x;r)$ we get

$$\overline{y}(x;r) = [1.25 - 0.25]x$$

Now, using method of successive approximation

$$\underline{y}_n(x,r) = \underline{f}(x;r) + \underline{\lambda}\int_0^1 K(x,t)\underline{y}_{n-1}(t;r)$$

$$\underline{y}_1(x;r) = [0.75 + 0.25r]\frac{5}{6}x + \frac{1}{2}\int_0^1 xt\underline{y}_0(t;r)\,dt$$

$$\vdots$$

Similarly, for $\overline{y}(x;r)$.

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Table 1: Lower Fuzzy Approximation of FFIE for Various Values of x at $r = 0$

		MSA				**FFFT**
	X	**n=1**	**n=2**	**…**	**n=8**	
$\boldsymbol{r = 0}$	0.1	0.0875	0.0770833	…	0.075	0.075
	0.2	0.175	0.154167	…	0.15	0.15
	0.3	0.2625	0.23125	…	0.225	0.225
	0.4	0.35	0.308333	…	0.3	0.3
	0.5	0.4375	0.385417	…	0.375	0.375
	0.6	0.525	0.4625	…	0.45	0.45
	0.7	0.6125	0.539583	…	0.525	0.525
	0.8	0.7	0.616667	…	0.6	0.6
	0.9	0.7875	0.69375	…	0.675	0.675

Table 2: Lower Fuzzy Approximation of FFIE at Various Values of x at $r = 0.3$

		MSA				**FFFT**
	X	**n=1**	**n=2**	**…**	**n=8**	
$\boldsymbol{r = 0.3}$	0.1	0.09375	0.084375	…	0.0825	0.0825
	0.2	0.1875	0.16875	…	0.165	0.165
	0.3	0.28125	0.253125	…	0.2475	0.2475
	0.4	0.375	0.3375	…	0.33	0.33
	0.5	0.46875	0.421875	…	0.4125	0.4125
	0.6	0.5625	0.50625	…	0.495	0.495
	0.7	0.65625	0.590625	…	0.5775	0.5775
	0.8	0.75	0.675	…	0.66	0.66
	0.9	0.84375	0.759375	…	0.7425	0.7425

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Table 3: Upper Fuzzy Approximation of FFIE at Various Values of x at $r = 0$

		MSA				**FFFT**
	X	**n=1**	**n=2**	**…**	**n=8**	
$\boldsymbol{r = 0}$	0.1	0.129167	0.125694	…	0.125	0.125
	0.2	0.258333	0.251389	…	0.25	0.25
	0.3	0.3875	0.377083	…	0.375	0.375
	0.4	0.516667	0.502778	…	0.5	0.5
	0.5	0.645833	0.628472	…	0.625	0.625
	0.6	0.775	0.754167	…	0.75	0.75
	0.7	0.904167	0.879861	…	0.875	0.875
	0.8	1.03333	1.00556	…	1	1
	0.9	1.1625	1.13125	…	1.125	1.125

Table 4: Upper Fuzzy Approximation of FFIE at Various Values of x at $r = 0.3$

		MSA				**FFFT**
	X	**n=1**	**n=2**	**…**	**n=8**	
$\boldsymbol{r = 0.3}$	0.1	0.122917	0.118403	…	0.122917	0.122917
	0.2	0.245833	0.236806	…	0.245833	0.245833
	0.3	0.36875	0.355208	…	0.36875	0.36875
	0.4	0.491667	0.473611	…	0.491667	0.491667
	0.5	0.614583	0.592014	…	0.614583	0.614583
	0.6	0.7375	0.710417	…	0.7375	0.7375
	0.7	0.860417	0.828819	…	0.860417	0.860417
	0.8	0.983333	0.947222	…	0.983333	0.983333
	0.9	1.10625	1.06563	…	1.10625	1.10625

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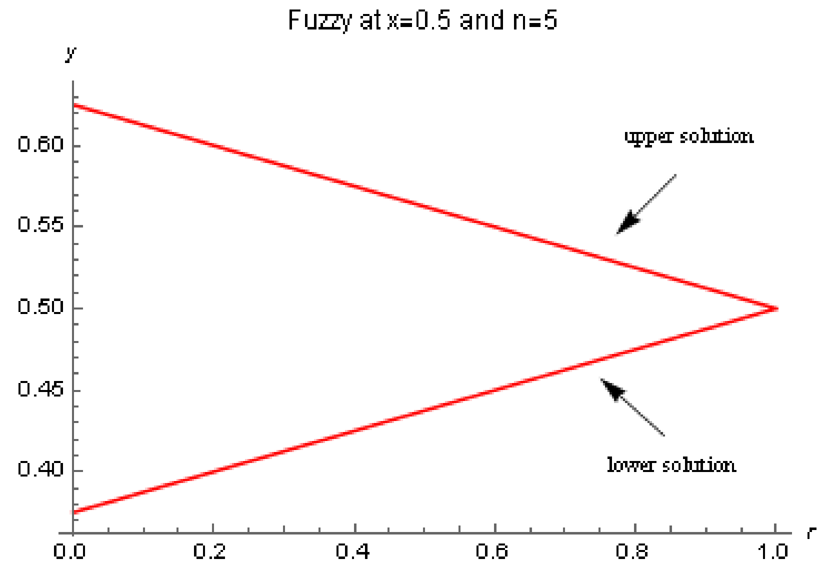

Figure 1. Fuzzy Upper and Lower Solution for FFIE at $n = 5$ and $x = 0.5$ for all $r \in [0,1]$

Tables 1-4 and Figure 1 demonstrate the fuzzy approximate solutions obtained utilizing the proposed methods verify the properties and aspects of fuzzy numbers, displaying a fuzzy number triangular shape, as shown in Figure 1 at $n = 5$ and $x = 0.5$. Additionally, Fredholm's first fundamental theorem proves to be more efficient and effective than the method of successive approximations, as it only requires one iteration to achieve the same results that the successive approximations method takes eight iterations to obtain.

Example 2: Consider the FFIE in double parametric form:

$$\tilde{y}(x;r) = [0.75 + 0.25r, 1.25 - 0.25r]\frac{5}{6}\tilde{x} + \frac{1}{2}\int_0^1 xt\,\tilde{y}(t;r)dt$$

With $\tilde{y}_0(x,r) = 1$

Solution

$$\beta\big((1.25 - 0.25r) - (0.75 + 0.25r)\big) + (0.75 + 0.25r)$$
$$= \beta(1.25 - 0.25r - 0.75 - 0.25r) + 0.75 + 0.25r = \beta(0.5 - 0.5r) + (0.75 + 0.25r)$$

As we said previously, when substituting $\beta = 0$ we have $(0.75 + 0.25r)$ (lower approximation) and when $\beta = 1$ we have the upper approximation$(1.25 - 0.25r)$.

We get that:

$$\tilde{y}(x,\beta;r) = \big(\beta(0.5 - 0.5r) + (0.75 + 0.25r)\big)\frac{5}{6}\tilde{x} + \frac{1}{2}\int_0^1 xt\tilde{y}(t,\beta;r)\,dt$$

First, using Fredholm's first fundamental theorem

$$\tilde{B}_0(x,t) = K(x,t) = xt$$

$$\tilde{C}_0 = 1$$

$$\tilde{C}_1 = \int_0^1 s^2\,ds = \frac{1}{3}$$

$$\tilde{B}_1(x,t) = \frac{1}{3}xt - \frac{1}{3}xt = 0$$

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$$\tilde{C}_2 = 0$$

$$\tilde{B}_2(x,t) = 0$$

$$\vdots$$

$$\widetilde{D}(x,t;\lambda) = K(x,t) + \sum_{p=1}^{\infty} \frac{(-\tilde{\lambda})^p}{p!} \tilde{B}_p(x,t) = xt$$

$$\widetilde{D}(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\tilde{\lambda})^p}{p!} \tilde{C}_p = 1$$

$$\tilde{R}(x,t;\lambda) = \frac{\widetilde{D}(x,t;\lambda)}{\widetilde{D}(\lambda)} = \frac{6xt}{5}$$

$$\tilde{y}(x,\beta;r) = [\beta(0.5-0.5r) + (0.75+0.25r)]\frac{5}{6}x + \frac{1}{2}\int_0^1 \frac{6xt}{5}[\beta(0.5-0.5r) + (0.75+0.25r)]\frac{5}{6}t\,dt$$
$$= [\beta(0.5-0.5r) + (0.75+0.25r)]x$$

Now, using method of successive approximation we get

$$\tilde{y}_n(x;\beta,r) = \tilde{f}(x;\beta,r) + \tilde{\lambda}\int_a^b K(x,t)\tilde{y}_{n-1}(t;\beta,r)$$

$$\tilde{y}_1(x;\beta,r) = [\beta(0.5-0.5r) + (0.75+0.25)]\frac{5}{6}\tilde{x} + \frac{\tilde{1}}{2}\int_0^1 xt\,dt$$

$$\vdots$$

Table 5: Numerical Results of Fuzzy FFIE at Various Values of x at $r = 0$ and $\beta = 0.2$

			MSA				**FFFT**
		X	**n=1**	**n=2**	**…**	**n=8**	
$r=0$	$\beta=0.2$	0.1	0.0958333	0.0868056	…	0.085	0.085
		0.2	0.191667	0.173611	…	0.17	0.17
		0.3	0.2875	0.260417	…	0.255	0.255
		0.4	0.383333	0.347222	…	0.34	0.34
		0.5	0.479167	0.434028	…	0.425	0.425
		0.6	0.575	0.520833	…	0.51	0.51
		0.7	0.670833	0.607639	…	0.595	0.595
		0.8	0.766667	0.694444	…	0.68	0.68
		0.9	0.8625	0.78125	…	0.765	0.765

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Table 6: Numerical Results of Fuzzy FFIE at Various Values of x at $r = 0.3$ And $\beta = 0.4$

			MSA				**FFFT**
		X	**"n=1"**	**"n=2"**	**"…"**	**"n=8"**	
$r = 0.7$	$\beta = 0.6$	0.1	0.109583	0.102847	…	0.1015	0.1015
		0.2	0.219167	0.205694	…	0.203	0.203
		0.3	0.32875	0.308542	…	0.3045	0.3045
		0.4	0.438333	0.411389	…	0.406	0.406
		0.5	0.547917	0.514236	…	0.5075	0.5075
		0.6	0.6575	0.617083	…	0.609	0.609
		0.7	0.767083	0.719931	…	0.7105	0.7105
		0.8	0.876667	0.822778	…	0.812	0.812
		0.9	0.98625	0.925625	…	0.9135	0.9135

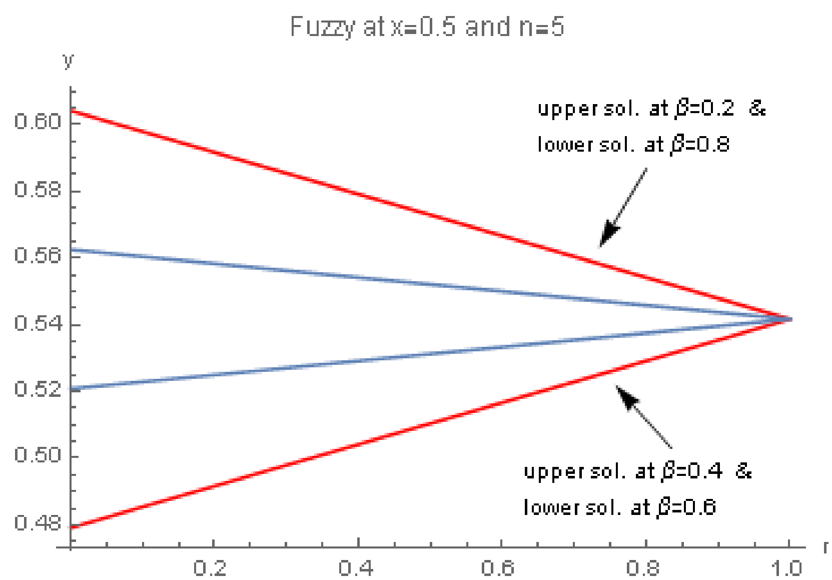

Figure 2. Fuzzy Upper, Lower, and Solution for FFIE at $n = 5$ and $x = 0.5$ for all $r, \beta \in [0,1]$

Tables 5-6 and Figure 2 demonstrate the fuzzy approximate solutions obtained utilizing the proposed methods accept the properties and aspects of fuzzy numbers, displaying a fuzzy number triangular shape, as shown in Figure 2 at $x = 0.5$ and $n = 5$. Additionally, Fredholm's first fundamental theorem proves to be more efficient and effective than the method of successive approximations, as it only requires one iteration to achieve the same results that the successive approximations method takes eight iterations to obtain. Furthermore, the double parametric form is found to be straightforward, and efficient computationally as it transforms the standard equation from fuzzy form to a crisp form.

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5. Conclusion

In this paper, the Fredholm's first fundamental theorem and the method of successive approximations are developed, reformatted, and applied for solving FFIE based on single and double parametric form of fuzzy numbers. Through a comparative analysis, it was demonstrated that Fredholm's first fundamental theorem outperforms the successive approximations method in terms of efficiency and effectiveness. Moreover, the double parametric form of fuzzy numbers proved to be a more general and reliable approach since it reducing computational costs while providing more accurate and reliable fuzzy solutions.

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