On Soft Locally Closed Sets and Soft Submaximal Spaces

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Abstract

This work adds to the burgeoning knowledge of soft topology. First, we continue the study of soft locally closed sets. We present several characterizations of soft locally closed sets. Also, we investigate their behaviors using specialized soft topologies as product and subspace soft topologies. Then, we define and investigate the concept of soft dense-in-itself spaces. In particular, we characterize soft dense-in-itself subspaces in terms of locally closed sets. Given a soft topological space $(N, \rho, \mathcal{M})$, the collection of soft locally closed sets of $(N, \rho, \mathcal{M})$ forms a soft topology on N relative to $\mathcal{M}$ which is denoted by ρ_l. We obtain several symmetries between the $(N, \rho, \mathcal{M})$ and $(N, \rho_l, \mathcal{M})$. In particular, we show that $(N, \rho, \mathcal{M})$ is soft T_0 (resp. soft T_D, soft indiscrete) iff $(N, \rho_l, \mathcal{M})$ is soft T_0 (resp. soft discrete, soft connected). Moreover, we show that if $(N, \rho_l, \mathcal{M})$ is soft T_1 (resp. soft Alexandroff), then $(N, \rho_l, \mathcal{M})$ is soft discrete (resp. soft Alexandroff) but not conversely. In addition to these, we obtain several characterizations and relationships of both soft locally indiscrete spaces and soft submaximal spaces. In particular, we show that $(N, \rho, \mathcal{M})$ is soft locally indiscrete if and only if $\rho = \rho_l$. In the last section, via the soft locally closed sets, we define and investigate soft lc-regularity as a stronger form of soft regularity. Finally, the paper deals with the correspondence between some concepts in soft topology and their analog concepts in classical topology.

Keywords: Soft locally closed sets; Soft submaximal spaces; Soft Alexandroff spaces; Soft locally indiscrete spaces; Soft regular spaces

1 Introduction and Preliminaries

The area of topology that focuses on the fundamental set-theoretic concepts and techniques is known as general topology. It serves as the foundation for other branches of topology, including differential topology, algebraic topology, and geometric topology. Locally closed sets—the intersection of an open set and a closed set— as defined in [1] are essential to topology. Locally compact subspaces of a Hausdorff space are shown to be locally closed in [2]; a completely regular space is locally compact as a subspace of its Stone-Cech compactification if and only if it is locally closed in it; and a subspace of a Hausdorff locally compact space is locally compact if and only if it is locally closed. The author in [3] studied the spaces that are locally closed in every embedding. The author in [4] showed the importance of locally closed sets in the subject of simple extension. In [5, 6], the authors defined three generalizations of continuity via locally closed sets and obtained a decomposition of continuity. The authors in [7] characterized locally closed sets in the α-topology. Submaximal spaces are TSs where each subset is locally closed, defined in [1]; hence, many general topology works use this notion [8].

Scientists in fields such as economics, systems engineering, medical science, and artificial intelligence often struggle to build complicated systems that account for uncertainty. Traditional probability, fuzzy set [9], and rough set [10] theories are widely used "mathematical" strategies for dealing with such circumstances. However, due to parameter constraints, they may not always give satisfactory results. Molodtsov [11] explored soft set theory, a novel method for dealing with uncertainty that improves on previous approaches. Molodtsov's approach to object lighting uses soft sets, which are a parametrized collection of universal subsets. Unlike previous methods, soft set theory does not impose precise constraints, and parameters can be chosen in a variety of formats, such as phrases, words, integers, and mappings. This makes the theory adaptable and easy to use in real-world scenarios (see [12, 13]). Using soft sets, several mathematical structures appeared, and many research papers appeared in those structures.

Shabir and Naz [14] initiated the structure of soft topology and investigated many related topics. Then, several researchers interested in abstract structures attempted to extend topological concepts to include soft topological spaces. For instance, concepts such as soft compactness [15], soft separation axioms [16–18], soft metrics [19], and soft submaximal [20, 21] were introduced. Furthermore, some researchers have investigated the concept of generalized open sets in soft topologies, such as soft semi-open sets [22], and soft locally closed sets [23].

In this paper, we continue the study of soft locally closed sets. We provide many characterizations of each of the soft locally closed sets, soft locally indiscrete spaces, and soft submaximal spaces. Also, we examine their behaviors using specific soft topologies such as product and subspace soft topologies. Then we define and explore the concept of soft dense-in-itself spaces. Specifically, we define soft dense-in-itself subspaces in terms of locally closed sets. Moreover, we will look at the soft topology generated by soft locally closed sets, where we prove some correspondences in possessing some soft topological properties between it and the original soft topology. In addition, we study soft lc-regularity as a stronger form of soft regularity. Finally, the paper deals with the correspondence between some concepts in soft topology and their analog concepts in classical topology.

For convenience, TS will now stand for topological space.

Let $(N, \rho, \mathcal{M})$ and $(N, \Im)$ be a soft TS and a TS, respectively. Let $H \in SS(N, \mathcal{M})$ and $V \subseteq N$. Then $Int_\rho(H)$ and $Clo_\rho(H)$, $Bd_\rho(H)$, $Int_\Im(V)$, and $Clo_\Im(V)$ will denote the soft interior of H in $(N, \rho, \mathcal{M})$, the soft closure of H in $(N, \rho, \mathcal{M})$, the soft boundary of H in $(N, \rho, \mathcal{M})$, the interior of V in $(N, \Im)$, and the closure of V in $(N, \Im)$, respectively. ρ^c and $\Im^c$ will denote the collection of all soft closed sets in $(N, \rho, \mathcal{M})$ and the collection of all closed sets in $(N, \Im)$, respectively.

The sequel will utilize the following definitions:

Definition 1.1. [1] Let $(N, \Im)$ be a TS and let $V \subseteq N$. Then V is a locally closed set in $(N, \Im)$ if $V = U \cap F$ for some $U \in \Im$ and $F \in \Im^c$. $LC(\Im)$ represents the collection of all locally closed sets in $(N, \Im)$.

$LC(\Im)$ forms a soft base for a topology on N, which is denoted by $\Im_l$ [8].

Definition 1.2. A TS $(N, \Im)$ is called

(a) [2] dense-in-itself if $\{x\} \notin \Im$ for each $x \in N$;

(b) [1] submaximal if $LC(\Im) = \mathcal{P}(N)$;

(c) [8] lc-regular if for each $x \in N$ and each $W \in LC(\Im)$ such that $x \notin W$, there are $\{U, V\} \subseteq \Im$ such that $x \in U$, $W \subseteq V$, and $U \cap V = \emptyset$.

Definition 1.3. [5] A map $g : (N, \Im) \longrightarrow (S, \aleph)$ is called LC-continuous if $g^{-1}(U) \in LC(\Im)$ for each $U \in \aleph$.

Definition 1.4. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $H \in SS(N, \mathcal{M})$. Then

(a) [22] H is a soft semi-open set in $(N, \rho, \mathcal{M})$ if $H \widetilde{\subseteq} Clo_{\rho}(Int_{\rho}(H))$. $SO(\rho)$ represents the collection of all soft semi-open sets in $(N, \rho, \mathcal{M})$.

(b) [23] H is a soft locally closed set in $(N, \rho, \mathcal{M})$ if $H = K \cap G$ for some $K \in \rho$ and $G \in \rho^c$. $LC(\rho)$ represents the collection of all soft locally closed sets in $(N, \rho, \mathcal{M})$.

(c) [23] H is a soft co-locally closed set in $(N, \rho, \mathcal{M})$ if $1_{\mathcal{M}} - H \in LC(\rho)$. $COLC(\rho)$ represents the collection of all soft co-locally closed sets in $(N, \rho, \mathcal{M})$.

Definition 1.5. A soft TS $(N, \rho, \mathcal{M})$ is called

(1) [20] soft submaximal if $LC(\rho) = SS(N, \mathcal{M})$;

(2) [17] soft T_1 if $SP(N, \mathcal{M}) \subseteq \rho^c$;

(3) [17] soft T_0 if for any $a_x, b_y \in SP(N, \mathcal{M})$ with $a_x \neq b_y$, there exist $\{G, K\} \subseteq \rho$ such that $a_x \widetilde{\in} G - K$ and $b_y \widetilde{\in} K - G$;

(4) [17] soft regular if for each $a_x \in SP(N, \mathcal{M})$ and each $H \in \rho^c$ such that $a_x \widetilde{\notin} H$, there are $\{G, K\} \subseteq \rho$ such that $a_x \widetilde{\in} G$, $H \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$;

(5) [24] soft $T_{1/2}$ if $SP(N, \mathcal{M}) \subseteq \rho \cup \rho^c$;

(6) [25] soft T_D if $SP(N, \mathcal{M}) \subseteq LC(\rho)$;

(7) [26] soft connected if $\rho \cap \rho^c = \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$.

(8) [27] soft Alexandroff is for any $\upsilon \subseteq \rho$, $\widetilde{\cap}_{H \in \upsilon} H \in \rho$.

(9) [28] soft locally indiscrete (soft l.i) if $\rho = \rho^c$.

Definition 1.6. [23] A soft map $f_{qu} : (N, \rho, \mathcal{M}) \longrightarrow (S, \sigma, \mathcal{R})$ is called soft LC-continuous if $f_{qu}^{-1}(K) \in LC(\rho)$ for each $K \in \sigma$.

Definition 1.7. [14]. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $\varnothing \neq Y \subseteq N$. Then the soft topology $\{G \widetilde{\cap} C_Y : G \in \rho\}$ is denoted by ρ_Y.

Definition 1.8. [15] Let $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ be two soft TSs. Then the soft topology over $N \times R$ relative to $\mathcal{M} \times \mathcal{L}$ having $\{T \times S : T \in \rho \text{ and } S \in \chi\}$ as a soft base is denoted by $pr(\rho \times \chi)$.

2 Soft Locally Closed Sets

Theorem 2.1. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $H \in LC(\rho)$. Then $H(m) \in LC(\rho_m)$ for each $m \in \mathcal{M}$.

Proof. Let $m \in \mathcal{M}$. Since $H \in LC(\rho)$, then we find $G \in \rho$ and $K \in \rho^c$ such that $H = G \widetilde{\cap} K$. So, we have $H(m) = G(m) \cap K(m)$ with $G(m) \in \rho_m$ and $K(m) \in (\rho_m)^c$. Hence, $H(m) \in LC(\rho_m)$.

Theorem 2.2. Consider the soft TS $(N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M})$ generated by the collection $\{(N, \aleph_m) : m \in \mathcal{M}\}$ of TSs. Then $H \in LC(\oplus_{m \in \mathcal{M}} \aleph_m)$ iff $H(m) \in LC(\aleph_m)$ for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $H \in LC(\oplus_{m \in \mathcal{M}} \aleph_m)$. Let $m \in \mathcal{M}$. Then by Theorem 2.1, $H(m) \in LC((\oplus_{m \in \mathcal{M}} \aleph_m)_m)$. Conversely, since $(\oplus_{m \in \mathcal{M}} \aleph_m)_m = \aleph_m$. Therefore, $H(m) \in LC(\aleph_m)$.

Sufficiency. Let $H(m) \in LC(\aleph_m)$ for each $m \in \mathcal{M}$. Then for each $m \in \mathcal{M}$, we find $G_m \in \aleph_m$ and $K_m \in (\aleph_m)^c$ such that $H(m) = G_m \cap K_m$. Define $G, K \in SS(N, \mathcal{M})$ by $G(m) = G_m$ and $K(m) = K_m$ for each $m \in \mathcal{M}$. Then, we have $G \in \oplus_{m \in \mathcal{M}} \aleph_m$, $K \in (\oplus_{m \in \mathcal{M}} \aleph_m)^c$, and $H = G \tilde{\cap} K$. It follows that $H \in LC(\oplus_{m \in \mathcal{M}} \aleph_m)$.

Corollary 2.3. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $H \in LC(\tau(\aleph))$ iff $H(r) \in LC(\aleph)$ for each $r \in \mathcal{R}$.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r \in \mathcal{R}} \aleph_r$. Thus, we obtain the result by applying Theorem 2.2.

Theorem 2.4. Let $\{(N, \aleph_m) : m \in \mathcal{M}\}$ and $\{(S, \Im_r) : r \in \mathcal{R}\}$ be two collections of TSs. Let $q : N \longrightarrow S$ and $u : \mathcal{M} \longrightarrow \mathcal{R}$ be maps with u being injective. Then $f_{qu} : (N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M}) \longrightarrow (S, \oplus_{r \in \mathcal{R}} \Im_r, \mathcal{R})$ is soft LC-continuous iff $q : (N, \aleph_m) \longrightarrow \left(S, \Im_{u(m)}\right)$ is LC-continuous for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $f_{qu} : (N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M}) \longrightarrow (S, \oplus_{r \in \mathcal{R}} \Im_r, \mathcal{R})$ be soft LC-continuous. Let $m \in \mathcal{M}$. Let $V \in \Im_{u(m)}$. Then $(u(m))_V \in \oplus_{r \in \mathcal{R}} \Im_r$. So, $f_{qu}^{-1}\left((u(m))_V\right) \in LC(\oplus_{m \in \mathcal{M}} \aleph_m)$ and by Theorem 2.2, $\left(f_{qu}^{-1}\left((u(m))_V\right)\right)(m) \in LC(\aleph_m)$. Since $u : \mathcal{M} \longrightarrow \mathcal{R}$, $f_{qu}^{-1}\left((u(m))_V\right) = m_{q^{-1}(V)}$ and so, $\left(f_{qu}^{-1}\left((u(m))_V\right)\right)(m) = \left(m_{q^{-1}(V)}\right)(m) = q^{-1}(V) \in LC(\aleph_m)$. This shows that $q : (N, \aleph_m) \longrightarrow \left(S, \Im_{u(m)}\right)$ is LC-continuous.

Sufficiency. Let $q : (N, \aleph_m) \longrightarrow \left(S, \Im_{u(m)}\right)$ be LC-continuous for each $m \in \mathcal{M}$. Let $K \in \oplus_{r \in \mathcal{R}} \Im_r$. Then, $K(r) \in \Im_r$ for each $r \in \mathcal{R}$. For each $r \in \mathcal{R}$, $q : \left(N, \aleph_{u^{-1}(r)}\right) \longrightarrow (S, \Im_r)$ is LC-continuous and so, $q^{-1}(K(r)) \in LC\left(\aleph_{u^{-1}(r)}\right)$. Thus, for each $m \in \mathcal{M}$, $\left(f_{qu}^{-1}(K)\right)(m) = q^{-1}(K(u(m)) \in LC\left(\aleph_{u^{-1}(u(m))}\right) = LC(\aleph_m)$. Therefore, by Theorem 2.2, $f_{qu}^{-1}(K) \in \oplus_{m \in \mathcal{M}} \aleph_m$. This shows that $f_{qu} : (N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M}) \longrightarrow (S, \oplus_{r \in \mathcal{R}} \Im_r, \mathcal{R})$ is soft LC-continuous.

Corollary 2.5. Let $q : (N, \aleph) \longrightarrow (S, \Im)$ and $u : \mathcal{M} \longrightarrow \mathcal{R}$ be two maps where u is injective. Then $q : (N, \aleph) \longrightarrow (S, \Im)$ is LC-continuous iff $f_{qu} : (N, \tau(\aleph), \mathcal{M}) \longrightarrow (S, \tau(\Im), \mathcal{R})$ is soft LC-continuous.

Proof. For each $m \in \mathcal{M}$ and $r \in \mathcal{R}$, let $\aleph_m = \aleph$ and $\Im_r = \Im$. Then $\tau(\aleph) = \oplus_{m \in \mathcal{M}} \aleph_m$ and $\tau(\Im) = \oplus_{r \in \mathcal{R}} \Im_r$. Theorem 2.4 ends the proof.

Theorem 2.6. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $H \in SS(N, \mathcal{M})$. Then T.F.A.E:

(a) $H \in LC(\rho)$.

(b) We find $K \in \rho$ such that $H = K \tilde{\cap} Clo_\rho(H)$.

(c) $Clo_\rho(H) - H \in \rho^c$.

(d) $H \tilde{\cup} (1_{\mathcal{M}} - Clo_\rho(H)) \in \rho$.

(e) $H \tilde{\subseteq} Int_\rho (H \tilde{\cup} (1_{\mathcal{M}} - Clo_\rho(H)))$.

(f) We find $\{G, T\} \subseteq \rho^c$ such that $H = G - T$.

(g) We find $\{U, V\} \subseteq \rho$ such that $H = U - V$.

Proof. (a) $\longrightarrow$ (b): By (a), we find $K \in \rho$ and $G \in \rho^c$ such that $H = K \tilde{\cap} G$. Since we have $H \tilde{\subseteq} G \in \rho^c$, then $Clo_\rho(H) \tilde{\subseteq} G$. Thus, $H \tilde{\subseteq} K \tilde{\cap} Clo_\rho(H) \tilde{\subseteq} K \tilde{\cap} G = H$. Hence, $H = K \tilde{\cap} Clo_\rho(H)$.

(b) $\longrightarrow$ (c): By (b),

$$\begin{aligned} Clo_\rho(H) - H &= Clo_\rho(H) - (K\tilde{\cap}Clo_\rho(H)) \\ &= Clo_\rho(H)\tilde{\cap}(1_{\mathcal{M}} - (K\tilde{\cap}Clo_\rho(H))) \\ &= Clo_\rho(H)\tilde{\cap}((1_{\mathcal{M}} - K)\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H))) \\ &= (Clo_\rho(H)\tilde{\cap}(1_{\mathcal{M}} - K))\tilde{\cup}(Clo_\rho(H)\tilde{\cap}(1_{\mathcal{M}} - Clo_\rho(H))) \\ &= (Clo_\rho(H)\tilde{\cap}(1_{\mathcal{M}} - K))\tilde{\cup}0_{\mathcal{M}} \\ &= Clo_\rho(H)\tilde{\cap}(1_{\mathcal{M}} - K) \in \rho^c. \end{aligned}$$

(c) $\longrightarrow$ (d): Since by (c), $1_{\mathcal{M}} - (H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H))) = (1_{\mathcal{M}} - H)\tilde{\cap}Clo_\rho(H) = Clo_\rho(H) - H \in \rho^c$, then $H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)) \in \rho$.

(d) $\longrightarrow$ (e): Since $H\tilde{\subseteq}(H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)))$ and by (d), $H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)) \in \rho$, then $H\tilde{\subseteq}Int_\rho(H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)))$.

(e) $\longrightarrow$ (a): Let $K = Int_\rho(H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)))$. Then $K \in \rho$ and by (e), $H\tilde{\subseteq}K$. Thus, we have

$$\begin{aligned} H &\tilde{\subseteq} K\tilde{\cap}Clo_\rho(H) \\ &= Int_\rho(H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)))\tilde{\cap}Clo_\rho(H) \\ &\tilde{\subseteq} (H\tilde{\cup}(1_{\mathcal{M}} - Clo_\rho(H)))\tilde{\cap}Clo_\rho(H) \\ &= H. \end{aligned}$$

Hence, $H = K\tilde{\cap}Clo_\rho(H)$. Therefore, $H \in LC(\rho)$.

(a) $\longrightarrow$ (f): By (a), we find $R \in \rho$ and $G \in \rho^c$ such that $H = R\tilde{\cap}G$. Let $T = 1_{\mathcal{M}} - R$. Then, we have $\{G, T\} \subseteq \rho^c$ and $G - T = G - (1_{\mathcal{M}} - R) = R\tilde{\cap}G = H$.

(f) $\longrightarrow$ (g): By (f), we find $\{G, T\} \subseteq \rho^c$ such that $H = G - T$. Let $U = 1_{\mathcal{M}} - T$ and $V = 1_{\mathcal{M}} - G$. Then $\{U, V\} \subseteq \rho$ and $U - V = (1_{\mathcal{M}} - T) - (1_{\mathcal{M}} - G) = (1_{\mathcal{M}} - T)\tilde{\cap}G = G - T = H$.

(g) $\longrightarrow$ (a): By (g), we find $\{U, V\} \subseteq \rho$ such that $H = U - V$. Then, $H = U\tilde{\cap}(1_{\mathcal{M}} - V)$ with $U \in \rho$ and $1_{\mathcal{M}} - V \in \rho^c$. Hence, $H \in LC(\rho)$.

Theorem 2.7. If $(N, \rho, \mathcal{M})$ is soft T_1 and $\varnothing \neq Y \subseteq N$ such that $\rho_Y = SS(Y, \mathcal{M})$, then $C_Y \in LC(\rho)$.

Proof. Since $\rho_Y = SS(Y, \mathcal{M})$, then for each $m_y \in SP(Y, \mathcal{M})$, we find $G_{m_y} \in \rho$ such that $m_y = G_{m_y}\tilde{\cap}C_Y$. Let $G = \tilde{\cup}_{m_y \in SP(Y,\mathcal{M})} G_{m_y}$. Then $G \in \rho$.

Claim. $C_Y = G\tilde{\cap}Clo_\rho(C_Y)$.

Proof of Claim.

$$\begin{aligned} C_Y &= \tilde{\cup}_{m_y \in SP(Y,\mathcal{M})} m_y \\ &= \tilde{\cup}_{m_y \in SP(Y,\mathcal{M})} \left(G_{m_y}\tilde{\cap}C_Y\right) \\ &= G\tilde{\cap}C_Y \\ &\tilde{\subseteq} G\tilde{\cap}Clo_\rho(C_Y). \end{aligned}$$

To see that $G\tilde{\cap}Clo_\rho(C_Y)\tilde{\subseteq}C_Y$, let us assume the contrary that there is an $a_x\tilde{\in}(G\tilde{\cap}Clo_\rho(C_Y)) - C_Y$. Since $a_x\tilde{\in}G$, then we find $m_y \in SP(Y, \mathcal{M})$ such that $a_x\tilde{\in}G_{m_y}$. Since $a_x\tilde{\notin}C_Y$, then $a_x \neq m_y$. Since $(N, \rho, \mathcal{M})$ is soft T_1, then we find $\{U, V\} \subseteq \rho$ such that $a_x\tilde{\in}U - V$ and $m_y\tilde{\in}V - U$. Since $a_x\tilde{\in}U\tilde{\cap}G_{m_y} \in \rho$ and $a_x\tilde{\in}Clo_\rho(C_Y)$, then $\left(U\tilde{\cap}G_{m_y}\right)\tilde{\cap}C_Y \neq 0_{\mathcal{M}}$. But $\left(U\tilde{\cap}G_{m_y}\right)\tilde{\cap}C_Y\tilde{\subseteq}G_{m_y}\tilde{\cap}C_Y = m_y$. Therefore, $\left(U\tilde{\cap}G_{m_y}\right)\tilde{\cap}C_Y = m_y$ and hence, $m_y\tilde{\in}U$, a contradiction.

By the above claim and Theorem 2.6 (d), $H \in LC(\rho)$.

Definition 2.8. A soft TS $(N, \rho, \mathcal{M})$ is said to be soft dense-in-itself if $\rho \cap SP(N, \mathcal{M}) = \varnothing$.

Theorem 2.9. Let $(N, \rho, \mathcal{M})$ be a soft TS such that (N, ρ_m) is dense-in-itself for each $m \in \mathcal{M}$, then $(N, \rho, \mathcal{M})$ is soft dense-in-itself.

Proof. Let us assume the contrary, that we find $m_x \in \rho \cap SP(N, \mathcal{M})$. Then $m_x(m) = \{x\} \in \rho_m$ and so, (N, ρ_m) is not dense-in-itself, which is a contradiction.

Theorem 2.9's opposite need not be true:

Example 2.10. Let $N = \{1, 2\}$, $\mathcal{M} = \{a, b\}$, and $\rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, a_1 \tilde{\cup} b_2\}$. Then the soft TS $(N, \rho, \mathcal{M})$ is soft dense-in-itself. Conversely, since $\rho_a = \{\emptyset, N, \{1\}\}$ and $\rho_b = \{\emptyset, N, \{2\}\}$, then neither (N, ρ_a) is dense-in-itself nor (N, ρ_b) is dense-in-itself.

Theorem 2.11. Let $\{(N, \aleph_m) : m \in \mathcal{M}\}$ be a collection of TSs. Then $(N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M})$ is soft dense-in-itself iff $(N, \aleph_m)$ is dense-in-itself for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $(N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M})$ be soft dense-in-itself. Let $k \in \mathcal{M}$. Let us assume the contrary, that there is $x \in N$ such that $\{x\} \in \aleph_k$. Then $k_x \in SP(N, \mathcal{M}) \cap \oplus_{m \in \mathcal{M}} \aleph_m$. Thus, $(N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M})$ is not soft dense-in-itself, a contradiction.

Sufficiency. Let $(N, \aleph_m)$ be dense-in-itself for each $m \in \mathcal{M}$. Let $\rho = \oplus_{m \in \mathcal{M}} \aleph_m$. Then, $\rho_m = \aleph_m$ for each $m \in \mathcal{M}$. Thus, by Theorem 2.9, $(N, \rho, \mathcal{M}) = (N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M})$ is soft dense-in-itself.

Corollary 2.12. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $(N, \tau(\aleph), \mathcal{R})$ is soft dense-in-itself iff $(N, \aleph)$ is dense-in-itself.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r \in \mathcal{R}} \aleph_r$. Thus, we obtain the result by applying Theorem 2.11.

Theorem 2.13. Let $(N, \rho, \mathcal{M})$ be soft dense-in-itself and let $\emptyset \neq Y \subseteq N$ such that $\rho_Y = SS(Y, \mathcal{M})$. Then $Int_\rho(C_Y) = 0_{\mathcal{M}}$.

Proof. Let us assume the contrary, that we find $m_y \tilde{\in} Int_\rho(C_Y) \tilde{\subseteq} C_Y$. Then, $m_y \in SS(Y, \mathcal{M}) = \rho_Y$. So, there is $G \in \rho$ such that $m_y = G \tilde{\cap} C_Y$. Since $m_y \tilde{\in} Int_\rho(C_Y) \tilde{\subseteq} C_Y$, then $m_y = G \tilde{\cap} Int_\rho(C_Y) \in \rho$. Thus, $(N, \rho, \mathcal{M})$ is not soft dense-in-itself, a contradiction.

Theorem 2.14. Let $(N, \rho, \mathcal{M})$ be soft dense-in-itself and let $\emptyset \neq Y \subseteq N$ such that $\rho_Y = SS(Y, \mathcal{M})$. Then $1_{\mathcal{M}} - C_Y \in LC(\rho)$ iff $C_Y \in \rho^c$.

Proof. *Necessity.* Let $1_{\mathcal{M}} - C_Y \in LC(\rho)$. Then we find $G \in \rho$ such that $1_{\mathcal{M}} - C_Y = G \tilde{\cap} Clo_\rho(1_{\mathcal{M}} - C_Y) = G \tilde{\cap} (1_{\mathcal{M}} - Int_\rho(C_Y))$. By Theorem 2.13, $Int_\rho(C_Y) = 0_{\mathcal{M}}$ and so, $1_{\mathcal{M}} - C_Y = G \tilde{\cap} (1_{\mathcal{M}} - 0_{\mathcal{M}}) = G$. Therefore, $1_{\mathcal{M}} - C_Y \in \rho$ and hence, $C_Y \in \rho^c$.

Sufficiency. If $C_Y \in \rho^c$, then $1_{\mathcal{M}} - C_Y \in \rho \subseteq LC(\rho)$.

Lemma 2.15. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $\emptyset \neq Y \subseteq N$. Then for each $m \in \mathcal{M}$, $(\rho_Y)_m = (\rho_m)_Y$.

Proof. To show that $(\rho_Y)_m \subseteq (\rho_m)_Y$, let $V \in (\rho_Y)_m$. Then we find $K \in \rho_Y$ such that $K(m) = V$. Pick $G \in \rho$ such that $K = G \tilde{\cap} C_Y$. Then $K(m) = (G \tilde{\cap} C_Y)(m) = G(m) \cap Y$. Therefore, we have $G(m) \in \rho_m$ and $V = G(m) \cap Y$; hence, $V \in (\rho_m)_Y$. To show that $(\rho_m)_Y \subseteq (\rho_Y)_m$, let $V \in (\rho_m)_Y$. Then there $U \in \rho_m$ such that $V = U \cap Y$. Pick $G \in \rho$ such that $U = G(m)$. Then $G \tilde{\cap} C_Y \in \rho_Y$ and so, $(G \tilde{\cap} C_Y)(m) = G(m) \cap Y = U \cap Y = V \in (\rho_Y)_m$.

Theorem 2.16. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $\emptyset \neq Y \subseteq N$ such that $\rho_Y = SS(Y, \mathcal{M})$. Then for each $m \in \mathcal{M}$, $(\rho_m)_Y = \mathcal{P}(Y)$.

Proof. Let $m \in \mathcal{M}$. Let $y \in Y$. Then $m_y \in SS(Y, \mathcal{M}) = \rho_Y$ and so, $\{y\} = (m_y)(m) \in (\rho_Y)_m$. Thus, by Lemma 2.15, $\{y\} \in (\rho_m)_Y$. This shows that $(\rho_m)_Y = \mathcal{P}(Y)$.

Theorem 2.16's opposite need not be true:

Example 2.17. Let $N = \{1,2\}$, $\mathcal{M} = \{a,b\}$. Let $K, G \in SS(N, \mathcal{M})$, where $K(a) = \{1\}$, $K(b) = \{2\}$, $G(a) = \{2\}$, and $G(b) = \{1\}$. Let $\rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, K, G\}$ .Consider the soft TS $(N, \rho, \mathcal{M})$. Let $Y = N$. Then $(\rho_a)_Y = (\rho_b)_Y = \mathcal{P}(Y)$ while $\rho_Y \neq SS(Y, \mathcal{M})$.

Theorem 2.18. Let $\{(N, \aleph_m) : m \in \mathcal{M}\}$ be a collection of TSs and let $\emptyset \neq Y \subseteq N$. Then $(\oplus_{m\in\mathcal{M}}\aleph_m)_Y = SS(Y, \mathcal{M})$ iff $\aleph_m = \mathcal{P}(Y)$ for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $(\oplus_{m\in\mathcal{M}}\aleph_m)_Y = SS(Y, \mathcal{M})$. Let $m \in \mathcal{M}$. Then by Theorem 2.16, $((\oplus_{m\in\mathcal{M}}\aleph_m)_m)_Y = \mathcal{P}(Y)$. But $(\oplus_{m\in\mathcal{M}}\aleph_m)_m = \aleph_m$. Hence, $\aleph_m = \mathcal{P}(Y)$.

Sufficiency. Let $\aleph_m = \mathcal{P}(Y)$ for each $m \in \mathcal{M}$. Let $m_y \in SS(Y, \mathcal{M})$. Then by assumption, $\{y\} \in \mathcal{P}(Y) = \aleph_m$ and so, $m_y \in SS(Y, \mathcal{M}) \oplus_{m\in\mathcal{M}} (\aleph_m)_Y = (\oplus_{m\in\mathcal{M}}\aleph_m)_Y$. This shows that $(\oplus_{m\in\mathcal{M}}\aleph_m)_Y = SS(Y, \mathcal{M})$.

Corollary 2.19. Let $(N, \aleph)$ be a TS, $\mathcal{R}$ be a set, and $\emptyset \neq Y \subseteq N$. Then $(\tau(\aleph))_Y = SS(Y, \mathcal{M})$ iff $\aleph_Y = \mathcal{P}(Y)$.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r\in\mathcal{R}}\aleph_r$. Thus, we obtain the result by applying Theorem 2.18.

Example 2.20. Let $N = \mathbb{R}$ and let $\aleph$ be the usual topology on $\mathbb{R}$. Let $\mathcal{M} = \{a,b\}$ and $Y = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then $(N, \tau(\aleph), \mathcal{M})$ is soft T_1. Since $\aleph_Y = \mathcal{P}(Y)$, then by Corollary 2.19, $(\tau(\aleph))_Y = SS(Y, \mathcal{M})$. Thus, by Theorem 2.7, $C_Y \in LC(\tau(\aleph))$.

Theorem 2.21. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $\emptyset \neq Y \subseteq N$ such that $C_Y \in LC(\rho)$. Then $LC(\rho_Y) \subseteq LC(\rho)$.

Proof. Let $H \in LC(\rho_Y)$. Then we find $K \in \rho$ such that $H = K\tilde{\cap}Clo_{\rho_Y}(H) = K\tilde{\cap}(Clo_\rho(H)\tilde{\cap}C_Y) = (K\tilde{\cap}C_Y)\tilde{\cap}Clo_\rho(H)$. Since $C_Y \in LC(\rho)$, then we find $T \in \rho$ such that $C_Y = T\tilde{\cap}Clo_\rho(C_Y)$. Therefore, $H = (K\tilde{\cap}(T\tilde{\cap}Clo_\rho(C_Y)))\tilde{\cap}Clo_\rho(H) = (K\tilde{\cap}T)\tilde{\cap}(Clo_\rho(C_Y)\tilde{\cap}Clo_\rho(H))$ with $K\tilde{\cap}T \in \rho$ and $Clo_\rho(C_Y)\tilde{\cap}Clo_\rho(H) \in \rho^c$. Hence, $H \in LC(\rho)$.

Theorem 2.22. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $\mathcal{Y} \subseteq \mathcal{P}(N) - \{\emptyset\}$ such that $\{C_Y : Y \in \mathcal{Y}\} \subseteq \rho$. Let $H \in SS(N, \mathcal{M})$. If $H\tilde{\cap}C_Y \in LC(\rho_Y)$ for each $Y \in \mathcal{Y}$, then $H \in LC(\rho)$.

Proof. Since $H\tilde{\cap}C_Y \in LC(\rho_Y)$ for each $Y \in \mathcal{Y}$, then for each $Y \in \mathcal{Y}$, we find $G(Y) \in \rho_Y$ such that $H\tilde{\cap}C_Y = G(Y)\tilde{\cap}Clo_{\rho_Y}(H\tilde{\cap}C_Y) = G(Y)\tilde{\cap}Clo_\rho(H\tilde{\cap}C_Y)\tilde{\cap}C_Y = G(Y)\tilde{\cap}Clo_\rho(H\tilde{\cap}C_Y)$. Since $\{C_Y : Y \in \mathcal{Y}\} \subseteq \rho$, then $\{G(Y) : Y \in \mathcal{Y}\} \subseteq \rho$. Now, for each

$Y \in \mathcal{Y}$, $G(Y)\tilde{\cap}Clo_\rho(H) = G(Y)\tilde{\cap}C_Y\tilde{\cap}Clo_\rho(H)\tilde{\subseteq}G(Y)\tilde{\cap}Clo_\rho(H\tilde{\cap}C_Y) = H\tilde{\cap}C_Y$. Let $G = \tilde{\cup}_{Y\in\mathcal{Y}}G(Y)$. Then $G \in \rho$ and $H = G\tilde{\cap}Clo_\rho(H)$. It follows that $H \in LC(\rho)$.

Theorem 2.23. Let $(N, \rho, \mathcal{M})$ be a soft TS and let $\mathcal{Y} \subseteq \mathcal{P}(N) - \{\emptyset\}$ such that $\{C_Y : Y \in \mathcal{Y}\} \subseteq \rho^c$ and $\{C_Y : Y \in \mathcal{Y}\}$ is soft locally finite. Let $H \in SS(N, \mathcal{M})$. If $H\tilde{\cap}C_Y \in LC(\rho_Y)$ for each $Y \in \mathcal{Y}$, then $H \in LC(\rho)$.

Proof. Since $H\tilde{\cap}C_Y \in LC(\rho_Y)$ for each $Y \in \mathcal{Y}$, then for each $Y \in \mathcal{Y}$, we find $G(Y) \in \rho_Y$ such that $H\tilde{\cap}C_Y = G(Y)\tilde{\cap}Clo_{\rho_Y}(H\tilde{\cap}C_Y) = G(Y)\tilde{\cap}Clo_\rho(H\tilde{\cap}C_Y)\tilde{\cap}C_Y = G(Y)\tilde{\cap}Clo_\rho(H\tilde{\cap}C_Y)$. For each $Y \in$

$\mathcal{Y}$, we find $K(Y) \in \rho$ such that $G(Y) = K(Y) \tilde{\cap} C_Y$ and hence $H \tilde{\cap} C_Y = G(Y) = (K(Y) \tilde{\cap} C_Y) \tilde{\cap} Clo_\rho(H \tilde{\cap} C_Y)$. Since $\{C_Y : Y \in \mathcal{Y}\} \subseteq \rho^c$, then for each $Y \in \mathcal{Y}$, $Clo_\rho(H \tilde{\cap} C_Y) \tilde{\subseteq} Clo_\rho(H) \tilde{\cap} Clo_\rho(C_Y) = Clo_\rho(H) \tilde{\cap} C_Y$ and so, $H \tilde{\cap} C_Y = (K(Y) \tilde{\cap} C_Y) \tilde{\cap} Clo_\rho(H \tilde{\cap} C_Y) = K(Y) \tilde{\cap} Clo_\rho(H \tilde{\cap} C_Y)$. Let $m_x \tilde{\in} H$. Since $\{C_Y : Y \in \mathcal{Y}\}$ is soft locally finite, then $\{C_Y : Y \in \mathcal{Y}\}$ is soft point finite, and so we find a finite sub-collection $\mathcal{Y}_1 \subseteq \mathcal{Y}$ such that $m_x \tilde{\in} C_Y$ for each $Y \in \mathcal{Y}_1$ and $m_x \tilde{\notin} C_Y$ for each $Y \in \mathcal{Y} - \mathcal{Y}_1$. Since $\{C_Y : Y \in \mathcal{Y}\} \subseteq \rho^c$ and $\{C_Y : Y \in \mathcal{Y}\}$ is soft locally finite, then $\{C_Y : Y \in \mathcal{Y} - \mathcal{Y}_1\} \subseteq \rho^c$ and, by Proposition 5.2 of [29], $\tilde{\cup}\{C_Y : Y \in \mathcal{Y} - \mathcal{Y}_1\} \in \rho^c$. Let $U_{m_x} = (1_\mathcal{M} - \tilde{\cup}\{C_Y : Y \in \mathcal{Y} - \mathcal{Y}_1\}) \tilde{\cap} (\tilde{\cap}\{K(Y) : Y \in \mathcal{Y}_1\})$. Then $m_x \tilde{\in} U_{m_x} \in \rho$. Let $U = \tilde{\cup}\{U_{m_x} : m_x \tilde{\in} H\}$. Then $U \in \rho$ and $H \tilde{\subseteq} U \tilde{\cap} Clo_\rho(H)$. Let $a_y \tilde{\in} U \tilde{\cap} Clo_\rho(H)$. Since $a_y \tilde{\in} U$, then $a_y \tilde{\in} U_{m_x}$ for some $m_x \tilde{\in} H$. Since $a_y \tilde{\in} Clo_\rho(H) = \tilde{\cup}\{Clo_\rho(H \tilde{\cap} C_Y) : Y \in \mathcal{Y}\}$, then there is $Y_0 \in \mathcal{Y}$ such that $a_y \tilde{\in} Clo_\rho(H \tilde{\cap} C_Y)$; hence $Y_0 \in \mathcal{Y}_1$ and $U_{m_x} \tilde{\subseteq} K(Y_0)$. Thus, $a_y \tilde{\in} K(Y_0) \tilde{\cap} Clo_\rho(H \tilde{\cap} C_{Y_0}) = H \tilde{\cap} C_{Y_0} \tilde{\subseteq} H$. It follows that $H = U \tilde{\cap} Clo_\rho(H)$. Therefore, $H \in LC(\rho)$.

Theorem 2.24. Let $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ be two soft TSs. If $H \in LC(\rho)$ and $K \in LC(\chi)$, then $H \times K \in LC(pr(\rho \times \chi))$

Proof. Since $H \in LC(\rho)$ and $K \in LC(\chi)$, then we find $G \in \rho$ and $W \in \chi$ such that $H = G \tilde{\cap} Clo_\rho(H)$ and $K = W \tilde{\cap} Cl_\chi(K)$. Thus, we have $G \times W \in pr(\rho \times \chi)$ and

$$\begin{aligned} H \times K &= (G \tilde{\cap} Clo_\rho(H)) \times (W \tilde{\cap} Cl_\chi(K)) \\ &= (G \times W) \tilde{\cap} (Clo_\rho(H) \times Cl_\chi(K)) \\ &= (G \times W) \tilde{\cap} Cl_{pr(\rho \times \chi)}(H \times K). \end{aligned}$$

Therefore, $H \times K \in LC(pr(\rho \times \chi))$.

Theorem 2.25. For a soft T_1 space $(N, \rho, \mathcal{M})$, T.F.A.E:

(a) $H \in LC(\rho)$ iff $1_\mathcal{M} - H \in LC(\rho)$.

(b) $LC(\rho)$ is closed under finite soft union.

(c) For each $G \in \rho$ and each $m_x \tilde{\in} Bd_\rho(G)$, we find $U \in \rho$ such that $m_x = U \tilde{\cap} Bd_\rho(G)$.

(d) For each $T \in SO(\rho)$ and each $m_x \tilde{\in} Bd_\rho(T)$, we find $U \in \rho$ such that $m_x = U \tilde{\cap} Bd_\rho(T)$.

(e) $SO(\rho) \subseteq LC(\rho)$.

Proof. (a) $\longrightarrow$ (b): Let $\{H, K\} \subseteq LC(\rho)$. Then by (a), $\{1_\mathcal{M} - H, 1_\mathcal{M} - K\} \subseteq LC(\rho)$. So,

$(1_\mathcal{M} - H) \tilde{\cap} (1_\mathcal{M} - K) = 1_\mathcal{M} - (H \tilde{\cup} K) \in LC(\rho)$. Thus, again by (a), $H \tilde{\cup} K \in LC(\rho)$.

(b) $\longrightarrow$ (a): It is sufficient to show that $H \in LC(\rho)$ implies $1_\mathcal{M} - H \in LC(\rho)$. Let $H \in LC(\rho)$. Then we find $U \in \rho$ and $T \in \rho^c$ such that $H = U \tilde{\cap} T$. Then $1_\mathcal{M} - H = (1_\mathcal{M} - U) \tilde{\cup} (1_\mathcal{M} - T)$ with $1_\mathcal{M} - U \in \rho^c \subseteq LC(\rho)$ and $1_\mathcal{M} - T \in \rho \subseteq LC(\rho)$. Thus, by (a), $1_\mathcal{M} - H \in LC(\rho)$.

(b) $\longrightarrow$ (c): Let $G \in \rho$ and let $m_x \tilde{\in} Bd_\rho(G) = Clo_\rho(G) \tilde{\cap} Clo_\rho(1_\mathcal{M} - G) = Clo_\rho(G) \tilde{\cap} (1_\mathcal{M} - G)$. Since $(N, \rho, \mathcal{M})$ is soft T_1, then $m_x \in \rho^c \subseteq LC(\rho)$. Moreover, we have $G \in \rho \subseteq LC(\rho)$. Thus, by (c), $G \tilde{\cup} m_x \in LC(\rho)$. So, we find $K \in \rho$ such that

$$\begin{aligned} G \tilde{\cup} m_x &= K \tilde{\cap} Clo_\rho(G \tilde{\cup} m_x) \\ &= K \tilde{\cap} (Clo_\rho(G) \tilde{\cup} m_x) \\ &= (K \tilde{\cap} Clo_\rho(G)) \tilde{\cup} (K \tilde{\cap} m_x) \\ &= (K \tilde{\cap} Clo_\rho(G)) \tilde{\cup} m_x. \end{aligned}$$

Claim. $m_x = Bd_\rho(G) \tilde{\cap} K$.

Proof of Claim. We have $m_x \tilde{\in} Bd_\rho(G)$. Moreover, since $G \tilde{\cup} m_x = K \tilde{\cap} Clo_\rho(G \tilde{\cup} m_x)$, then $m_x \tilde{\in} K$. Therefore, $m_x \tilde{\subseteq} Bd_\rho(G) \tilde{\cap} K$. Conversely, let $a_y \tilde{\in} (Bd_\rho(G) \tilde{\cap} K)$. Since

$$\begin{array}{ll} Bd_{\rho}(G)\widetilde{\cap}K & (Clo_{\rho}(G)\widetilde{\cap}(1_{\mathcal{M}}-G))\widetilde{\cap}K \\ & (K\widetilde{\cap}Clo_{\rho}(G))\widetilde{\cap}(1_{\mathcal{M}}-G), \end{array}$$

then $a_y\widetilde{\in}K\widetilde{\cap}Clo_{\rho}(G)$ and $a_y\widetilde{\in}1_{\mathcal{M}}-G$. Thus, $a_y\widetilde{\in}(K\widetilde{\cap}Clo_{\rho}(G))\widetilde{\cup}m_x=G\widetilde{\cup}m_x$. Since $a_y\widetilde{\in}1_{\mathcal{M}}-G$, then $a_y=m_y$. Therefore, $Bd_{\rho}(G)\widetilde{\cap}K\widetilde{\subseteq}m_x$.

(c) $\longrightarrow$ (d): Let $T\in SO(\rho)$ and let $m_x\widetilde{\in}Bd_{\rho}(T)$. Since $T\in SO(\rho)$, then $T\widetilde{\subseteq}Clo_{\rho}(Int_{\rho}(T))$ and so $Clo_{\rho}(T)\widetilde{\subseteq}Clo_{\rho}(Int_{\rho}(T))$; hence $Clo_{\rho}(T)=Clo_{\rho}(Int_{\rho}(T))$. Thus,

$$\begin{array}{rcl} Bd_{\rho}(T) & = & Clo_{\rho}(T)\widetilde{\cap}Clo_{\rho}(1_{\mathcal{M}}-T) \\ & = & Clo_{\rho}(Int_{\rho}(T))\widetilde{\cap}Clo_{\rho}(1_{\mathcal{M}}-T) \\ & \widetilde{\subseteq} & Clo_{\rho}(Int_{\rho}(T))\widetilde{\cap}Clo_{\rho}(1_{\mathcal{M}}-Int_{\rho}(T)) \\ & = & Bd_{\rho}(Int_{\rho}(T)). \end{array}$$

Since $Int_{\rho}(T)\in\rho$ and $m_x\widetilde{\in}Bd_{\rho}(Int_{\rho}(T))$, then by (c), we find $U\in\rho$ such that $m_x=U\widetilde{\cap}Bd_{\rho}(Int_{\rho}(T))$. Since $m_x\widetilde{\in}U\widetilde{\cap}Bd_{\rho}(T)\widetilde{\subseteq}U\widetilde{\cap}Clo_{\rho}(Int_{\rho}(T))$, then $m_x=U\widetilde{\cap}Bd_{\rho}(T)$.

(d) $\longrightarrow$ (e): Let $T\in SO(\rho)$. For each $a_x\widetilde{\in}T\widetilde{\cap}Bd_{\rho}(T)$, by (d), we find $U_{a_x}\in\rho$ such that $a_x=U_{a_x}\widetilde{\cap}Bd_{\rho}(T)$. Let $U=Int_{\rho}(T)\widetilde{\cup}(\widetilde{\cup}\{U_{a_x}:a_x\widetilde{\in}T\widetilde{\cap}Bd_{\rho}(T)\})$. Then $U\in\rho$.

Claim. $T=U\widetilde{\cap}Clo_{\rho}(T)$.

Proof of Claim. To show that $T\widetilde{\subseteq}U$, let $a_x\widetilde{\in}T-Int_{\rho}(T)\widetilde{\subseteq}Clo_{\rho}(T)-Int_{\rho}(T)=Bd_{\rho}(T)$. Then $a_x\widetilde{\in}T\widetilde{\cap}Bd_{\rho}(T)$ and so, $a_x\widetilde{\in}U_{a_x}\widetilde{\subseteq}\widetilde{\cup}\{U_{a_x}:a_x\widetilde{\in}T\widetilde{\cap}Bd_{\rho}(T)\}$. Therefore, $T\widetilde{\subseteq}U$. Hence, $T\widetilde{\subseteq}U\widetilde{\cap}Clo_{\rho}(T)$. To show that $U\widetilde{\cap}Clo_{\rho}(T)\widetilde{\subseteq}T$, let us assume the contrary that there is $m_y\widetilde{\in}(U\widetilde{\cap}Clo_{\rho}(T))-T$. Since $m_y\widetilde{\notin}T$, then $m_y\widetilde{\notin}Int_{\rho}(T)$. Since $m_y\widetilde{\in}U$, then $m_y\widetilde{\in}\{U_{a_x}:a_x\widetilde{\in}T\widetilde{\cap}Bd_{\rho}(T)\}$, then we find $a_x\widetilde{\in}T\widetilde{\cap}Bd_{\rho}(T)$ such that $m_y\widetilde{\in}U_{a_x}$. Since $m_y\widetilde{\in}Clo_{\rho}(T)-Int_{\rho}(T)=Bd_{\rho}(T)$, then $m_y\widetilde{\in}U_{a_x}\widetilde{\cap}Bd_{\rho}(T)=a_x$. Thus, $m_y=a_x$. But $a_x\widetilde{\in}T$ while $m_y\widetilde{\notin}T$, a contradiction.

This claim shows that $T\in LC(\rho)$.

(e) $\longrightarrow$ (a): We will show that $U\widetilde{\cup}D\in LC(\rho)$ for each $U\in\rho$ and $D\in\rho^c$. Let $H=U\widetilde{\cup}D$ with $U\in\rho$ and $D\in\rho^c$. Let $V=U-D$. Then $V\in\rho$, $H=V\widetilde{\cup}D$, and $V\widetilde{\cap}D=0_{\mathcal{M}}$. Let $T=V\widetilde{\cup}(Clo_{\rho}(V)\widetilde{\cap}D)$. Since $V\in\rho$ and

$$\begin{array}{rcl} V & \widetilde{\subseteq} & T \\ & \widetilde{\subseteq} & Clo_{\rho}(T) \\ & = & Clo_{\rho}(V\widetilde{\cup}(Clo_{\rho}(V)\widetilde{\cap}D)) \\ & = & Clo_{\rho}(V)\widetilde{\cup}(Clo_{\rho}(Clo_{\rho}(V)\widetilde{\cap}D)) \\ & = & Clo_{\rho}(V)\widetilde{\cup}(Clo_{\rho}(V)\widetilde{\cap}D) \\ & = & Clo_{\rho}(V), \end{array}$$

then $T\in SO(\rho)$.

Thus, by (e), $T\in LC(\rho)$ and hence, we find $G\in\rho$ such that $T=G\widetilde{\cap}Clo_{\rho}(T)$.

Since $Clo_{\rho}(T)=Clo_{\rho}(V)$, then $T=G\widetilde{\cap}Clo_{\rho}(V)$.

Let $R=G\widetilde{\cup}(1_{\mathcal{M}}-Clo_{\rho}(V))$. Then $R\in\rho$ and

$$\begin{array}{rcl} R\widetilde{\cap}Clo_{\rho}(H) & = & R\widetilde{\cap}Clo_{\rho}(V\widetilde{\cup}D) \\ & = & R\widetilde{\cap}(Clo_{\rho}(V)\widetilde{\cup}D) \\ & = & (R\widetilde{\cap}Clo_{\rho}(V))\widetilde{\cup}(R\widetilde{\cap}D) \\ & = & (G\widetilde{\cap}Clo_{\rho}(V))\widetilde{\cup}(R\widetilde{\cap}D) \\ & = & (V\widetilde{\cup}(Clo_{\rho}(V)\widetilde{\cap}D))\widetilde{\cup}((G\widetilde{\cup}(1_{\mathcal{M}}-Clo_{\rho}(V)))\widetilde{\cap}D) \\ & = & (V\widetilde{\cup}(Clo_{\rho}(V)\widetilde{\cap}D))\widetilde{\cup}(((G\widetilde{\cap}D)\widetilde{\cup}((1_{\mathcal{M}}-Clo_{\rho}(V)))\widetilde{\cap}D) \\ & = & (V\widetilde{\cup}(G\widetilde{\cap}D))\widetilde{\cup}((Clo_{\rho}(V)\widetilde{\cap}D)\widetilde{\cup}(1_{\mathcal{M}}-Clo_{\rho}(V)\widetilde{\cap}D) \\ & = & (V\widetilde{\cup}(G\widetilde{\cap}D))\widetilde{\cup}(D\widetilde{\cap}(Clo_{\rho}(V)\widetilde{\cup}(1_{\mathcal{M}}-Clo_{\rho}(V)))) \\ & = & (V\widetilde{\cup}(G\widetilde{\cap}D))\widetilde{\cup}(D\widetilde{\cap}1_{\mathcal{M}}) \\ & = & (V\widetilde{\cup}(G\widetilde{\cap}D))\widetilde{\cup}D \\ & = & V\widetilde{\cup}D \\ & = & H. \end{array}$$

Therefore, $H \in LC(\rho)$.

3 The Soft Topology ρ_l

Given a soft TS $(N, \rho, \mathcal{M})$, $LC(\rho)$ forms a soft base for a soft topology on N relative to $\mathcal{M}$, which is denoted by ρ_l.

Theorem 3.1. Let $(N, \rho, \mathcal{M})$ be a soft TS. Then $\rho \cup \rho^c$ is a soft subbase for $(N, \rho_l, \mathcal{M})$.

Proof. Let $G \in \rho_l - \{0_{\mathcal{M}}\}$ and let $m_x \widetilde{\in} G$. Since $LC(\rho)$ is a soft base for $(N, \rho_l, \mathcal{M})$, there is $H \in LC(\rho)$ such that $m_x \widetilde{\in} H \widetilde{\subseteq} G$. Pick $K \in \rho$ and $G \in \rho^c$ such that $H = K \widetilde{\cap} G$. This ends the proof.

Theorem 3.2. Let $(N, \rho, \mathcal{M})$ be a soft TS. Then for any $m \in \mathcal{M}$, $(\rho_m)_l = (\rho_l)_m$.

Proof. To show that $(\rho_m)_l \subseteq (\rho_l)_m$, let $U \in (\rho_m)_l - \{\varnothing\}$. Let $x \in U$. Then there is $V \in LC(\rho_m)$ such that $x \in V \subseteq U$. Pick $S \in \rho_m$ and $T \in (\rho_m)^c$ such that $V = S \cap T$. Pick $K \in \rho$ and $G \in \rho^c$ such that $K(m) = S$ and $G(m) = T$. Thus, we have $K \widetilde{\cap} G \in LC(\rho) \subseteq \rho_l$ and so, $(K \widetilde{\cap} G)(m) = K(m) \cap G(m) = S \cap T = V$. Therefore, $V \in (\rho_l)_m$. This shows that $U \in (\rho_l)_m$. To show that $(\rho_l)_m \subseteq (\rho_m)_l$, let $U \in (\rho_l)_m - \{\varnothing\}$. Let $x \in U$. Pick $F \in \rho_l$ such that $U = F(m)$. Since $m_x \widetilde{\in} F \in \rho_l$, then there is $H \in LC(\rho)$ such that $m_x \widetilde{\in} H \widetilde{\subseteq} F$; hence, $x \in H(m) \subseteq F(m) = U$. By Theorem 2.1, $H(m) \in LC(\rho_m)$. It follows that $U \in (\rho_m)_l$.

Theorem 3.3. Consider the soft TS $(N, \oplus_{m \in \mathcal{M}} \aleph_m, \mathcal{M})$ generated by the collection $\{(N, \aleph_m) : m \in \mathcal{M}\}$ of TSs. Then $(\oplus_{m \in \mathcal{M}} \aleph_m)_l = \oplus_{m \in \mathcal{M}} (\aleph_m)_l$.

Proof. To show that $(\oplus_{m \in \mathcal{M}} \aleph_m)_l \subseteq \oplus_{m \in \mathcal{M}} (\aleph_m)_l$, let $G \in (\oplus_{m \in \mathcal{M}} \aleph_m)_l - \{0_{\mathcal{M}}\}$. Let $a \in \mathcal{M}$. Then $G(a) \in ((\oplus_{m \in \mathcal{M}} \aleph_m)_l)_a$. By Theorem 3.2, $((\oplus_{m \in \mathcal{M}} \aleph_m)_l)_a = ((\oplus_{m \in \mathcal{M}} \aleph_m)_a)_l$. Moreover, we have, $(\oplus_{m \in \mathcal{M}} \aleph_m)_a = \aleph_a$. Therefore, we have $G(a) \in (\aleph_a)_l$. It follows that $G \in \oplus_{m \in \mathcal{M}} (\aleph_m)_l$. To show that $\oplus_{m \in \mathcal{M}} (\aleph_m)_l \subseteq (\oplus_{m \in \mathcal{M}} \aleph_m)_l$, let $G \in (\oplus_{m \in \mathcal{M}} (\aleph_m)_l) - \{0_{\mathcal{M}}\}$. Let $a_x \widetilde{\in} G$. Then $x \in G(a) \in (\aleph_a)_l$. Pick $V \in LC(\aleph_a)$ such that $x \in V \subseteq G(a)$; hence, $a_x \widetilde{\in} a_V \widetilde{\subseteq} G$. Moreover, by Theorem 2.2, $a_V \in LC(\oplus_{m \in \mathcal{M}} \aleph_m)$. It follows that $G \in (\oplus_{m \in \mathcal{M}} \aleph_m)_l$.

Corollary 3.4. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $(\tau(\aleph))_l$ iff $\tau(\aleph_l)$.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r \in \mathcal{R}} \aleph_r$. Thus, we obtain the result by applying Theorem 3.3.

Theorem 3.5. For any soft TS $(N, \rho, \mathcal{M})$, $\rho \subseteq \rho_l$.

Proof. Obvious.

Equality cannot take the place of inclusion in Theorem 3.5:

Example 3.6. Let $N = \mathbb{R}$, $\mathcal{M} = \mathbb{Z}$, and $\rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, C_{\mathbb{Q}}\}$. Then $\rho_l = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, C_{\mathbb{Q}}, C_{\mathbb{R}-\mathbb{Q}}\}$. So, $\rho \neq \rho_l$.

Theorem 3.7. A soft TS $(N, \rho, \mathcal{M})$ is soft T_0 iff $(N, \rho_l, \mathcal{M})$ is soft T_0.

Proof. *Necessity.* Obvious.

Sufficiency. Let $(N, \rho_l, \mathcal{M})$ be soft T_0 and let $a_x, b_y \in SP(N, \mathcal{M})$ such that $a_x \neq b_y$. Then there is $T \in \rho_l$, say $a_x \widetilde{\in} T$ and $b_y \widetilde{\notin} T$. Pick $H \in LC(\rho_l)$ such that $a_x \widetilde{\in} H \widetilde{\subseteq} T$. Pick $G \in \rho$ and $K \in \rho^c$ such that $H = G \widetilde{\cap} K$. If $b_y \widetilde{\notin} G$, then we are done; otherwise $b_y \widetilde{\in} 1_{\mathcal{M}} - K \in \rho$ and $a_x \widetilde{\notin} 1_{\mathcal{M}} - K$.

Theorem 3.8. A soft TS $(N, \rho, \mathcal{M})$ is soft T_D iff ρ_l is soft discrete.

Proof. *Necessity.* Let $(N, \rho, \mathcal{M})$ be soft T_D. Then $SP(N, \mathcal{M}) \subseteq LC(\rho) \subseteq \rho_l$. Hence, ρ_l is soft discrete.

Sufficiency. Let ρ_l be soft discrete. Let $m_x \in SP(N, \mathcal{M})$. Since ρ_l is soft discrete, then $m_x \in \rho_l$. So, there is $H \in LC(\rho)$ such that $m_x \widetilde{\in} H \widetilde{\subseteq} m_x$. Thus, $m_x = H \in LC(\rho)$. Therefore, $(N, \rho, \mathcal{M})$ is soft T_D.

Corollary 3.9. If $(N, \rho, \mathcal{M})$ is soft T_1, then ρ_l is soft discrete.

Remark 3.10. The implication in Corollary 3.9 may not be valid in reverse, as demonstrated by Example 4.3 of [25]. Additionally, it demonstrates that we cannot substitute soft T_1 for soft T_0 in Theorem 3.7.

Theorem 3.11. For a soft TS $(N, \rho, \mathcal{M})$, T.F.A.E:

(a) $(N, \rho, \mathcal{M})$ is soft indiscrete.

(b) $(N, \rho_l, \mathcal{M})$ is soft indiscrete.

(c) $(N, \rho_l, \mathcal{M})$ is soft connected.

Proof. (a) $\longrightarrow$ (b): Suppose, on the contrary, that there is $G \in \rho_l - \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$. Pick $H \in LC(\rho) - \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$ such that $H \widetilde{\subseteq} G$. Pick $G \in \rho$ and $K \in \rho^c$ such that $H = G \widetilde{\cap} K$. Since $H \neq 0_{\mathcal{M}}$, then $G \neq 0_{\mathcal{M}}$ and $K \neq 0_{\mathcal{M}}$. Since $G \in \rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$ and $K \in \rho^c = \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$, then $G = K = 1_{\mathcal{M}}$. Thus, $H = 1_{\mathcal{M}} \widetilde{\cap} 1_{\mathcal{M}} = 1_{\mathcal{M}}$, which is a contradiction.

(b) $\longrightarrow$ (c): Obvious.

(c) $\longrightarrow$ (a): Suppose on the contrary that there is $G \in \rho - \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$. By Theorem 3.1, we have $\{G, 1_{\mathcal{M}} - G\} \subseteq \rho_l - \{0_{\mathcal{M}}, 1_{\mathcal{M}}\}$. Therefore, $(N, \rho_l, \mathcal{M})$ is not soft connected, a contradiction.

Theorem 3.12. Every soft Alexandroff soft T_0 space is soft T_D.

Proof. Let $(N, \rho, \mathcal{M})$ be soft Alexandroff and soft T_0. Let $m_x \in SP(N, \mathcal{M})$. Let $H = \widetilde{\cap} \{G \in \rho : m_x \in G\}$. Since $(N, \rho, \mathcal{M})$ is soft Alexandroff, then $H \in \rho$.

Claim. $m_x = H \widetilde{\cap} Clo_{\rho}(m_x)$.

Proof of Claim. Clearly that $m_x \widetilde{\in} H \widetilde{\cap} Clo_{\rho}(m_x)$. To end the proof, suppose on the contrary that there is $b_y \widetilde{\in} H \widetilde{\cap} Clo_{\rho}(m_x)$ such that $m_x \neq b_y$. Since $(N, \rho, \mathcal{M})$ is soft T_0, there is $K \in \rho$ such that ($m_x \widetilde{\in} K$ and $b_y \widetilde{\notin} K$) or ($m_x \widetilde{\notin} K$ and $b_y \widetilde{\in} K$). Let $m_x \widetilde{\in} K$ and $b_y \widetilde{\notin} K$. Since $m_x \widetilde{\in} K \in \rho$, then $H \widetilde{\subseteq} K$. Since $b_y \widetilde{\in} H$, then $b_y \widetilde{\in} K$. But $b_y \widetilde{\notin} K$, a contradiction. Let $m_x \widetilde{\notin} K$ and $b_y \widetilde{\in} K$. Since $b_y \widetilde{\in} K \widetilde{\cap} H \in \rho$ and $b_y \widetilde{\in} Clo_{\rho}(m_x)$, then $m_x \widetilde{\cap} (K \widetilde{\cap} H) \neq 0_{\mathcal{M}}$; hence, $m_x \widetilde{\in} K$. But $m_x \widetilde{\notin} K$, a contradiction.

Theorem 3.13. If $(N, \rho, \mathcal{M})$ is soft Alexandroff, then $(N, \rho_l, \mathcal{M})$ is soft Alexandroff.

Proof. Let $\ominus \subseteq \rho_l$ with $\widetilde{\cap}_{T \in \ominus} T \neq 0_{\mathcal{M}}$. Let $m_x \widetilde{\in} \widetilde{\cap}_{T \in \ominus} T$. Then $m_x \widetilde{\in} T$ for each $T \in \ominus$. Thus, for each $T \in \ominus$, there are $G_T \in \rho$ and $K_T \in \rho^c$ such that $m_x \widetilde{\in} G_T \widetilde{\cap} K_T \widetilde{\subseteq} T$. Since $(N, \rho, \mathcal{M})$ is soft Alexandroff, then $\widetilde{\cap}_{T\in\ominus}G_T \in \rho$. Also, we have $\widetilde{\cap}_{T\in\ominus}K_T \in \rho^c$. Therefore, we have $m_x\widetilde{\in}\left(\widetilde{\cap}_{T\in\ominus}G_T\right)\widetilde{\cap}\left(\widetilde{\cap}_{T\in\ominus}K_T\right) \in LC\left(\rho\right)$ and $\left(\widetilde{\cap}_{T\in\ominus}G_T\right)\widetilde{\cap}\left(\widetilde{\cap}_{T\in\ominus}K_T\right) = \widetilde{\cap}_{T\in\ominus}\left(G_T\widetilde{\cap}K_T\right)\widetilde{\subseteq}\widetilde{\cap}_{T\in\ominus}T$; hence $\widetilde{\cap}_{T\in\ominus}T \in \rho_l$. This shows that $(N, \rho_l, \mathcal{M})$ is soft Alexandroff.

Theorem 3.14. If $(N, \rho, \mathcal{M})$ is soft T_D, then (N, ρ_m) is T_D for each $m \in \mathcal{M}$.

Proof. Assume that $(N, \rho, \mathcal{M})$ is soft T_D. Let $m \in \mathcal{M}$. Let $x \in N$. Then $m_x \in SP\left(N, \mathcal{M}\right)$. Since $(N, \rho, \mathcal{M})$ is soft T_D, then $m_x \in LC\left(\rho\right)$. Thus, by Theorem 2.1, $m_x\left(m\right) = \{x\} \in LC\left(\rho_m\right)$. Therefore, (N, ρ_m) is T_D.

The implication in Theorem 3.14 may not be valid in reverse, as demonstrated by the following example:

Example 3.15. Let $N = \{1, 2\}$, $\mathcal{M} = \{a, b\}$. Let $K, G \in SS\left(N, \mathcal{M}\right)$, where $K\left(a\right) = \{1\}$, $K\left(b\right) = \{2\}$, $G\left(a\right) = \{2\}$, and $G\left(b\right) = \{1\}$. Let $\rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, K, G\}$ .Consider the soft TS $(N, \rho, \mathcal{M})$. Then, (N, ρ_a) and (N, ρ_b) are discrete TSs, and so they are T_D spaces. On the other hand, since $a_1 \notin LC\left(\rho\right)$, then $(N, \rho, \mathcal{M})$ is not soft T_D.

Theorem 3.16. Consider the soft TS $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ generated by the collection $\{(N, \aleph_m) : m \in \mathcal{M}\}$ of TSs. Then $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft T_D iff $(N, \aleph_m)$ is T_D for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ be soft T_D. Let $a \in \mathcal{M}$. Then by Theorem 3.14, $\left(N, \left(\oplus_{m\in\mathcal{M}}\aleph_m\right)_a\right)$ is T_D. Since $\left(\oplus_{m\in\mathcal{M}}\aleph_m\right)_a = \aleph_a$, then $(N, \aleph_a)$ is T_D.

Sufficiency. Let $(N, \aleph_m)$ be T_D for each $m \in \mathcal{M}$. Let $a_x \in SP\left(N, \mathcal{M}\right)$. Since $(N, \aleph_a)$ is T_D, then $\{x\} \in LC\left(\aleph_a\right)$. Thus, by Theorem 2.2, $a_x \in LC\left(\oplus_{m\in\mathcal{M}}\aleph_m\right)$. Therefore, $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft T_D.

Corollary 3.17. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $(N, \tau\left(\aleph\right), \mathcal{R})$ is soft T_D iff $(N, \aleph)$ is T_D.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau\left(\aleph\right) = \oplus_{r\in\mathcal{R}}\aleph_r$. Thus, we obtain the result by applying Theorem 3.16.

Theorem 3.18. If $(N, \rho, \mathcal{M})$ is soft Alexandroff, then (N, ρ_m) is Alexandroff for each $m \in \mathcal{M}$.

Proof. Assume that $(N, \rho, \mathcal{M})$ is soft Alexandroff. Let $m \in \mathcal{M}$. Let $\Phi \subseteq \rho_m$ such that $\cap_{U\in\Phi}U \neq \emptyset$. For each $U \in \Phi$, Pick $G_U \in \rho$ such that $G_U\left(m\right) = U$. Since $(N, \rho, \mathcal{M})$ is soft Alexandroff, then $\widetilde{\cap}_{U\in\Phi}G_U \in \rho$; hence, $\left(\widetilde{\cap}_{U\in\Phi}G_U\right)\left(m\right) = \cap_{U\in\Phi}U \in \rho_m$. It follows that (N, ρ_m) is Alexandroff.

Question 3.19. Let $(N, \rho, \mathcal{M})$ be a soft TS such that (N, ρ_m) is Alexandroff for each $m \in \mathcal{M}$. Is it true that $(N, \rho, \mathcal{M})$ is soft Alexandroff?

Theorem 3.20. Consider the soft TS $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ generated by the collection $\{(N, \aleph_m) : m \in \mathcal{M}\}$ of TSs. Then $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft Alexandroff iff $(N, \aleph_m)$ is Alexandroff for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ be soft Alexandroff. Let $a \in \mathcal{M}$. Then by Theorem 3.18, $\left(N, \left(\oplus_{m\in\mathcal{M}}\aleph_m\right)_a\right)$ is Alexandroff. Since $\left(\oplus_{m\in\mathcal{M}}\aleph_m\right)_a = \aleph_a$, then $(N, \aleph_a)$ is Alexandroff.

Sufficiency. Let $(N, \aleph_m)$ be Alexandroff for each $m \in \mathcal{M}$. Let $\ominus \subseteq \oplus_{m\in\mathcal{M}}\aleph_m$ with $\widetilde{\cap}_{T\in\ominus}T \neq 0_{\mathcal{M}}$. For each $m \in \mathcal{M}$, $\{T\left(m\right) : T \in \ominus\} \subseteq \aleph_m$ and so, $\cap_{T\in\ominus}T\left(m\right) = \left(\widetilde{\cap}_{T\in\ominus}T\right)\left(m\right) \in \aleph_m$. Hence, $\widetilde{\cap}_{T\in\ominus}T \in \oplus_{m\in\mathcal{M}}\aleph_m$.

Corollary 3.21. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $(N, \tau\left(\aleph\right), \mathcal{R})$ is soft Alexandroff iff $(N, \aleph)$ is Alexandroff.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r\in\mathcal{R}}\aleph_r$. Thus, we obtain the result by applying Theorem 3.20.

The implication in Theorem 3.13 may not be valid in reverse, as demonstrated by the following example:

Example 3.22. Let $N = \mathbb{R}$ and let $\aleph$ be the usual topology on $\mathbb{R}$. Let $\mathcal{M} = \{a, b\}$. Since $(N, \aleph)$ is T_D, then by Corollary 3.17, $(N, \tau(\aleph), \mathcal{M})$ is soft T_D. So, by Theorem 3.8, ρ_l is soft discrete. Thus, $(N, (\tau(\aleph))_l, \mathcal{M})$ is soft Alexandroff. Conversely, since $(N, \aleph)$ is not Alexandroff, then by Corollary 3.21, $(N, \tau(\aleph), \mathcal{M})$ is not soft Alexandroff.

Theorem 3.23. Let $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ be two soft TSs. Then $(N \times R, pr(\rho \times \chi), \mathcal{M})$ is soft T_D iff $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ are soft T_D.

Proof. *Necessity.* Let $(N \times R, pr(\rho \times \chi), \mathcal{M} \times \mathcal{L})$ be soft T_D. Let $a_x \in SP(N, \mathcal{M})$ and $b_y \in SP(R, \mathcal{L})$. Then $(a, b)_{(x,y)} \in SP(N \times R, \mathcal{M} \times \mathcal{L})$. So, $(a, b)_{(x,y)} \in LC(pr(\rho \times \chi))$. Then there is $G \in pr(\rho \times \chi)$ such that

$$(a, b)_{(x,y)} = G\tilde{\cap}Clo_{pr(\rho\times\chi)}\left((a, b)_{(x,y)}\right) = G\tilde{\cap}\left(Clo_\rho(a_x) \times Clo_\chi(b_y)\right).$$

Since $(a, b)_{(x,y)} \tilde{\in} G \in pr(\rho \times \chi)$, then there are $U \in \rho$ and $V \in \chi$ such that $(a, b)_{(x,y)} \tilde{\in} U \times V \tilde{\subseteq} G$. This implies that

$$(a, b)_{(x,y)} = (U \times V)\tilde{\cap}\left(Clo_\rho(a_x) \times Clo_\chi(b_y)\right) = \left(U\tilde{\cap}Clo_\rho(a_x)\right) \times \left(V\tilde{\cap}Clo_\chi(b_y)\right).$$

Hence, $a_x = U\tilde{\cap}Clo_\rho(a_x)$ and $b_y = V\tilde{\cap}Clo_\chi(b_y)$. It follows that $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ are soft T_D.

Sufficiency. Let $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ be soft T_D. Let $(a, b)_{(x,y)} \in SP(N \times R, \mathcal{M} \times \mathcal{L})$. Then $a_x \in SP(N, \mathcal{M})$ and $b_y \in SP(R, \mathcal{L})$. So, $a_x \in LC(\rho)$ and $b_y \in LC(\chi)$. Thus, by Theorem 2.24, $a_x \times b_y = (a, b)_{(x,y)} \in LC(pr(\rho \times \chi))$. Therefore, $(N \times R, pr(\rho \times \chi), \mathcal{M})$ is soft T_D.

Theorem 3.24. Every soft $T_{1/2}$ space is soft T_D.

Proof. Let $(N, \rho, \mathcal{M})$ be soft $T_{1/2}$ and let $a_x \in SP(N, \mathcal{M})$. Then, $a_x \in \rho \cup \rho^c \subseteq LC(\rho)$. Hence, $(N, \rho, \mathcal{M})$ is soft T_D.

The implication in Theorem 3.24 may not be valid in reverse, as demonstrated by the following example:

Example 3.25. For each $n \in \mathbb{N}$, let $A_n = \{n, n+1, ...\}$ and let $\aleph = \{\varnothing\} \cup \{A_n : n \in \mathbb{N}\}$. Since for each $n \in \mathbb{N}$, we have $A_n \in \aleph$, $\mathbb{N} - A_{n+1} \in \aleph^c$, and $\{n\} = A_n \cap (\mathbb{N} - A_{n+1}) \in LC(\aleph)$, then $(N, \aleph)$ is T_D. On the other hand, it is clear that $\{2\} \notin \aleph \cup \aleph^c$. Let $\mathcal{M} = \{a, b\}$. Then, $(N, \tau(\aleph), \mathcal{M})$ is soft T_D but not soft $T_{1/2}$.

4 Soft Locally Indiscreteness and Soft Submaximality

Theorem 4.1. For a soft TS $(N, \rho, \mathcal{M})$, T.F.A.E:

(a) $(N, \rho, \mathcal{M})$ is soft l.i.

(b) $\rho = \rho_l$.

(c) $LC(\rho) \subseteq \rho$.

(d) $LC(\rho) \subseteq \rho^c$.

(e) $\{Clo_\rho(F) : F \in SS(N, \mathcal{M})\} \subseteq \rho$.

Proof. (a) $\longrightarrow$ (b): Let $(N, \rho, \mathcal{M})$ be soft l.i. We will show that $\rho_l \subseteq \rho$. Let $H \in \rho_l - \{0_\mathcal{M}\}$ and let $m_x \widetilde{\in} H$. Pick $G \in \rho$ and $K \in \rho^c$ such that $H = G \widetilde{\cap} K$. Since $(N, \rho, \mathcal{M})$ is soft l.i, then $K \in \rho$ and so, $H = G \widetilde{\cap} K \in \rho$. Therefore, $H \in \rho$.

(b) $\longrightarrow$ (c): Since $LC(\rho) \subseteq \rho_l$ and by (b), $\rho = \rho_l$, then $LC(\rho) \subseteq \rho$.

(c) $\longrightarrow$ (d): Always we have $\rho^c \subseteq LC(\rho)$. So, by (c), $\rho^c \subseteq \rho$; hence, $\rho = \rho^c$. Therefore, by (c), $LC(\rho) \subseteq \rho^c$.

(d) $\longrightarrow$ (e): Let $F \in SS(N, \mathcal{M})$. Since always we have $\rho \subseteq LC(\rho)$, then by (d), $\rho \subseteq \rho^c$; hence, $\rho = \rho^c$. Since $Clo_\rho(F) \in \rho^c$, then $Clo_\rho(F) \in \rho$.

(e) $\longrightarrow$ (a): We will show that $\rho^c \subseteq \rho$. Let $G \in \rho^c$. Then by (e), $Clo_\rho(G) = G \in \rho$.

Theorem 4.2. Every soft l.i space is soft Alexandroff.

Proof. Let $(N, \rho, \mathcal{M})$ be soft l.i. Let $\Psi \subseteq \rho$. Since $(N, \rho, \mathcal{M})$ is soft l.i, then $\rho = \rho^c$ and so, $\Psi \subseteq \rho^c$. Thus, $\widetilde{\cap}_{G \in \Psi} G \in \rho^c = \rho$.

The soft TS in Example 3.6 is soft Alexandroff but not soft l.i.

Theorem 4.3. For a soft T_1 space $(N, \rho, \mathcal{M})$, T.F.A.E:

(a) $(N, \rho, \mathcal{M})$ is soft discrete.

(b) $(N, \rho, \mathcal{M})$ is soft l.i.

(c) $(N, \rho, \mathcal{M})$ is soft Alexandroff.

Proof. (a) $\longrightarrow$ (b): Obvious.

(b) $\longrightarrow$ (c): Follows from Theorem 4.2.

(c) $\longrightarrow$ (a): Let $m_x \in SP(N, \mathcal{M})$. Since $(N, \rho, \mathcal{M})$ is soft T_1, then $m_x = \widetilde{\cap}\{G \in \rho : m_x \widetilde{\in} G\}$. Since $(N, \rho, \mathcal{M})$ is soft Alexandroff, then $\widetilde{\cap}\{G \in \rho : m_x \widetilde{\in} G\} \in \rho$; hence $m_x \in \rho$. Therefore, $(N, \rho, \mathcal{M})$ is soft discrete.

Theorem 4.4. Every soft T_0 soft l.i space is soft T_1.

Proof. Let $(N, \rho, \mathcal{M})$ be soft T_0 and soft l.i. Let $m_x \in SP(N, \mathcal{M})$. Suppose on the contrary that there is $b_y \widetilde{\in} Clo_\rho(m_x)$ such that $m_x \neq b_y$. Since $(N, \rho, \mathcal{M})$ is soft T_0, then there is $K \in \rho$ such that ($m_x \widetilde{\in} K$ and $b_y \widetilde{\notin} K$) or ($m_x \widetilde{\notin} K$ and $b_y \widetilde{\in} K$). Let $m_x \widetilde{\in} K$ and $b_y \widetilde{\notin} K$. Since $(N, \rho, \mathcal{M})$ is soft l.i, then $1_\mathcal{M} - K \in \rho$. Since $b_y \widetilde{\in} 1_\mathcal{M} - K$ and $b_y \widetilde{\in} Clo_\rho(m_x)$, then $(1_\mathcal{M} - K) \widetilde{\cap} m_x \neq 0_\mathcal{M}$; hence, $m_x \widetilde{\in} 1_\mathcal{M} - K$. But $b_y \widetilde{\notin} K$, a contradiction. Let $m_x \widetilde{\notin} K$ and $b_y \widetilde{\in} K$. Since $b_y \widetilde{\in} K \in \rho$ and $b_y \widetilde{\in} Clo_\rho(m_x)$, then $m_x \widetilde{\cap} K \neq 0_\mathcal{M}$; hence, $m_x \widetilde{\in} K$. But $m_x \widetilde{\notin} K$, a contradiction.

Theorem 4.5. For a soft T_0 space $(N, \rho, \mathcal{M})$, T.F.A.E:

(a) $(N, \rho, \mathcal{M})$ is soft discrete.

(b) $(N, \rho, \mathcal{M})$ is soft l.i.

Proof. (a) $\longrightarrow$ (b): Obvious.

(b) $\longrightarrow$ (a): Follows from Theorems 4.3 and 4.4.

Theorem 4.6. If $(N, \rho, \mathcal{M})$ is soft submaximal, then (N, ρ_m) is submaximal for each $m \in \mathcal{M}$.

Proof. Assume that $(N, \rho, \mathcal{M})$ is soft submaximal. Let $m \in \mathcal{M}$. Let $V \subseteq N$. Then $m_V \in SS(N, \mathcal{M})$. Since $(N, \rho, \mathcal{M})$ is soft submaximal, then $m_V \in LC(\rho)$. Thus, by Theorem 2.1, $m_V(m) = V \in LC(\rho_m)$. Therefore, (N, ρ_m) is submaximal.

The implication in Theorem 4.6 may not be valid in reverse, as demonstrated by the following example:

Example 4.7. Let $N = \{1, 2\}$ and $\mathcal{M} = \{a, b\}$. Let $K, G \in SS(N, \mathcal{M})$, where $K(a) = \{1\}$, $K(b) = \{2\}$, $G(a) = \{2\}$, and $G(b) = \{1\}$. Let $\rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, K, G\}$ .Consider the soft TS $(N, \rho, \mathcal{M})$. Then, (N, ρ_a) and (N, ρ_b) are discrete TSs, and so they are submaximal spaces. On the other hand, since $a_1 \notin LC(\rho)$, then $(N, \rho, \mathcal{M})$ is not soft submaximal.

Theorem 4.8. Consider the soft TS $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ generated by the collection $\{(N, \aleph_m) : m \in \mathcal{M}\}$ of TSs. Then $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft submaximal iff $(N, \aleph_m)$ is submaximal for each $m \in \mathcal{M}$.

Proof. *Necessity.* Let $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ be soft submaximal. Let $a \in \mathcal{M}$. Then by Theorem 4.6, $(N, (\oplus_{m\in\mathcal{M}}\aleph_m)_a)$ is submaximal. Since $(\oplus_{m\in\mathcal{M}}\aleph_m)_a = \aleph_a$, then $(N, \aleph_a)$ is submaximal.

Sufficiency. Let $(N, \aleph_m)$ be submaximal for each $m \in \mathcal{M}$. Let $K \in SS(N, \mathcal{M})$. Then $K(m) \in LC(\aleph_m)$ for each $m \in \mathcal{M}$. Thus, by Theorem 2.2, $K \in LC(\oplus_{m\in\mathcal{M}}\aleph_m)$. Therefore, $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft submaximal.

Corollary 4.9. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $(N, \tau(\aleph), \mathcal{R})$ is soft submaximal iff $(N, \aleph)$ is submaximal.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r\in\mathcal{R}}\aleph_r$. Thus, we obtain the result by applying Theorem 4.8.

The following two examples show that ”soft submaximality” and ”soft local indiscreteness” are independent concepts:

Example 4.10. Let $N = \{1, 2, 3\}$ and $\aleph = \{\varnothing, N, \{1\}, \{2, 3\}\}$. Then $(N, \aleph)$ is locally indiscrete but not submaximal. Let $\mathcal{M} = \mathbb{Q}$. Then by Corollary 1 of [18] and Corollary 4.9, $(N, \tau(\aleph), \mathcal{M})$ is soft l.i but not soft submaximal.

Example 4.11. Let $N = \{1, 2, 3, 4\}$ and $\aleph = \{\varnothing, N, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Then $(N, \aleph)$ is submaximal but not locally indiscrete. Let $\mathcal{M} = \mathbb{Q}$. Then by Corollary 4.9 and Corollary 1 of [18], $(N, \tau(\aleph), \mathcal{M})$ is soft submaximal but not soft l.i.

Theorem 4.12. For a soft l.i space $(N, \rho, \mathcal{M})$, T.F.A.E:

(a) $(N, \rho, \mathcal{M})$ is soft T_1.

(b) $(N, \rho, \mathcal{M})$ is soft $T_{1/2}$.

(c) $(N, \rho, \mathcal{M})$ is soft T_D.

(d) $(N, \rho, \mathcal{M})$ is soft T_0.

(e) $(N, \rho, \mathcal{M})$ is soft discrete.

(f) $(N, \rho, \mathcal{M})$ is soft submaximal.

Proof. (a) $\longrightarrow$ (b), (b) $\longrightarrow$ (c), (c) $\longrightarrow$ (d), (e) $\longrightarrow$ (f), and (f) $\longrightarrow$ (c) are obvious.

(d) $\longrightarrow$ (a): Follows from Theorem 4.4.

(a) $\longrightarrow$ (e): Follows from Theorem 4.3.

5 Soft lc-Regularity

Definition 5.1. A soft TS $(N, \rho, \mathcal{M})$ is called soft lc-regular if for each $a_x \in SP(N, \mathcal{M})$ and each $H \in LC(\rho)$ such that $a_x \widetilde{\notin} H$, there are $\{G, K\} \subseteq \rho$ such that $a_x \widetilde{\in} G$, $H \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$.

Theorem 5.2. A soft TS $(N, \rho, \mathcal{M})$ is soft lc-regular iff for each $a_x \in SP(N, \mathcal{M})$ and each $T \in COLC(\rho)$ such that $a_x \widetilde{\in} T$, there is $G \in \rho$ such that $a_x \widetilde{\in} G \widetilde{\subseteq} Clo_\rho(G) \widetilde{\subseteq} T$.

Proof. *Necessity.* Assume that $(N, \rho, \mathcal{M})$ is soft lc-regular. Let $a_x \in SP(N, \mathcal{M})$ and let $T \in COLC(\rho)$ such that $a_x \widetilde{\in} T$. Then, we have $a_x \widetilde{\notin} 1_{\mathcal{M}} - T \in LC(\rho)$ and by soft lc-regularity of $(N, \rho, \mathcal{M})$, there are $\{G, K\} \subseteq \rho$ such that $a_x \widetilde{\in} G$, $1_{\mathcal{M}} - T \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$. Thus, we have $a_x \widetilde{\in} G \widetilde{\subseteq} 1_{\mathcal{M}} - K \widetilde{\subseteq} T$. Since $G \widetilde{\subseteq} 1_{\mathcal{M}} - K \in \rho^c$, then $Clo_\rho(G) \widetilde{\subseteq} 1_{\mathcal{M}} - K \widetilde{\subseteq} T$. This ends the proof.

Sufficiency. Let $a_x \in SP(N, \mathcal{M})$ and let $H \in LC(\rho)$ such that $a_x \widetilde{\notin} H$. Then $a_x \widetilde{\in} 1_{\mathcal{M}} - H \in COLC(\rho)$ and by assumption, there is $G \in \rho$ such that $a_x \widetilde{\in} G \widetilde{\subseteq} Clo_\rho(G) \widetilde{\subseteq} 1_{\mathcal{M}} - H$. Let $K = 1_{\mathcal{M}} - Clo_\rho(G)$. Therefore, we have $\{G, K\} \subseteq \rho$ such that $a_x \widetilde{\in} G$, $H \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$. Hence, $(N, \rho, \mathcal{M})$ is soft lc-regular.

Theorem 5.3. Every soft lc-regular space is soft regular.

Proof. Let $(N, \rho, \mathcal{M})$ be soft lc-regular. Let $a_x \in SP(N, \mathcal{M})$ and let $T \in \rho$ such that $a_x \widetilde{\in} T$. Since $1_{\mathcal{M}} - T \in \rho^c \subseteq LC(\rho)$, then $T \in COLC(\rho)$. Thus, by Theorem 5.3, there is $G \in \rho$ such that $a_x \widetilde{\in} G \widetilde{\subseteq} Clo_\rho(G) \widetilde{\subseteq} T$. This shows that $(N, \rho, \mathcal{M})$ is soft regular.

Theorem 5.4. If $(N, \rho, \mathcal{M})$ is soft lc-regular, then (N, ρ_m) is lc-regular for each $m \in \mathcal{M}$.

Proof. Assume that is $(N, \rho, \mathcal{M})$ is soft lc-regular. Let $m \in \mathcal{M}$. Let $x \in N$ and $V \in LC(\rho_m)$ such that $x \notin V$. Pick $U \in \rho_m$ and $W \in (\rho_m)^c$ such that $V = U \cap W$. Pick $T \in \rho$ and $S \in \rho^c$ such that $U = T(m)$ and $W = S(m)$. Then, we have $T \widetilde{\cap} S \in LC(\rho)$ and $m_x \widetilde{\notin} T \widetilde{\cap} S$. Thus, by soft lc-regularity of $(N, \rho, \mathcal{M})$, there are $\{G, K\} \subseteq \rho$ such that $m_x \widetilde{\in} G$, $T \widetilde{\cap} S \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$. Therefore, we have $\{G(m), K(m)\} \subseteq \rho_m$, $x \in G(m)$, $V = U \cap W = T(m) \cap S(m) \subseteq K(m)$, and $G(m) \cap K(m) = \emptyset$. It follows that (N, ρ_m) is lc-regular.

The following example shows that the inverse of the implication in Theorem 5.4 may not be true:

Example 5.5. Let $N = \{1, 2\}$, $\mathcal{M} = \{a, b\}$. Let $K, G \in SS(N, \mathcal{M})$, where $K(a) = \{1\}$, $K(b) = \{2\}$, $G(a) = \{2\}$, and $G(b) = \{1\}$. Let $\rho = \{0_{\mathcal{M}}, 1_{\mathcal{M}}, K, G, a_1, G \widetilde{\cup} a_1\}$. Consider the soft TS $(N, \rho, \mathcal{M})$. Then, (N, ρ_a) and (N, ρ_b) are discrete TSs, and so they are lc-regular spaces. Suppose that $(N, \rho, \mathcal{M})$ is soft regular. Since $a_1 \widetilde{\in} a_1 \in \rho$, then there is $G \in \rho$ such that $a_1 \widetilde{\in} G \widetilde{\subseteq} Clo_\rho(G) \widetilde{\subseteq} a_1$. Thus, $G = Clo_\rho(G) = a_1$. But $Clo_\rho(a_1) = K \neq a_1$. Therefore, $(N, \rho, \mathcal{M})$ is not soft regular. Hence, $(N, \rho, \mathcal{M})$ is not soft lc-regular.

Theorem 5.6. Let $\{(N, \aleph_m) : m \in \mathcal{M}\}$ be a collection of TSs. Then $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft lc-regular iff $(N, \aleph_m)$ is lc-regular for each $m \in \mathcal{M}$.

Proof. *Necessity.* Assume that $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft lc-regular. Then by Theorem 5.4, $(N, (\oplus_{m\in\mathcal{M}}\aleph_m)_m)$ is lc-regular for each $m \in \mathcal{M}$. But $(\oplus_{m\in\mathcal{M}}\aleph_m)_m = \aleph_m$ for each $m \in \mathcal{M}$. This completes the proof.

Sufficiency. Assume that $(N, \aleph_m)$ is lc-regular for each $m \in \mathcal{M}$. Let $a_x \in SP(N, \mathcal{M})$ and let $H \in LC(\oplus_{m\in\mathcal{M}}\aleph_m)$ such that $a_x \widetilde{\notin} H$. Since $H \in LC(\oplus_{m\in\mathcal{M}}\aleph_m)$, then by Theorem 2.2, $H(a) \in LC(\aleph_a)$. Since $a_x \widetilde{\notin} H$, then $x \notin H(a)$. Since $(N, \aleph_a)$ is lc-regular, then there are $\{U, V\} \subseteq \aleph_a$ such that $x \in U$, $H(a) \subseteq V$, and $U \cap V = \varnothing$. Define $G, K \in SS(N, \mathcal{M})$ by

$$G(m) = \begin{cases} U & \text{if} \quad m = a \\ \varnothing & \text{if} \quad m \neq a \end{cases} \text{ and } K(m) = \begin{cases} V & \text{if} \quad m = a \\ N & \text{if} \quad m \neq a \end{cases}.$$

Then, we have $\{G, K\} \subseteq \oplus_{m\in\mathcal{M}}\aleph_m$ such that $a_x \widetilde{\in} G$, $H \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$. It follows that $(N, \oplus_{m\in\mathcal{M}}\aleph_m, \mathcal{M})$ is soft lc-regular.

Corollary 5.7. For a given TS $(N, \aleph)$ and a set $\mathcal{R}$, $(N, \tau(\aleph), \mathcal{R})$ is soft lc-regular iff $(N, \aleph)$ is lc-regular.

Proof. For each $r \in \mathcal{R}$, let $\aleph_r = \aleph$. Then $\tau(\aleph) = \oplus_{r\in\mathcal{R}}\aleph_r$. Thus, we obtain the result by applying Theorem 5.6.

The following example shows that the inverse of the implication in Theorem 5.3 may not true:

Example 5.8. Let $\aleph$ be the usual topology on $\mathbb{R}$ and let $\mathcal{M} = \{a, b\}$. It is proved in Example 4.6 of [8] that $(N, \aleph)$ is regular but not lc-regular. Thus, by Corollary 4 of [18] and Corollary 5.7, $(N, \tau(\aleph), \mathcal{M})$ is soft regular but not soft lc-regular.

Theorem 5.9. Let $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ be two soft TSs. If $(N \times R, pr(\rho \times \chi), \mathcal{M} \times \mathcal{L})$ is soft lc-regular, then $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ are soft lc-regular spaces.

Proof. Assume that $(N \times R, pr(\rho \times \chi), \mathcal{M} \times \mathcal{L})$ is soft lc-regular. To show that $(N, \rho, \mathcal{M})$ is a soft lc-regular space, let $a_x \widetilde{\in} H \in COLC(\rho)$. Pick $A \in \rho$ and $B \in \rho^c$ such that $H = A \widetilde{\cup} B$. Then $A \times 1_{\mathcal{L}} \in pr(\rho \times \chi)$ and $B \times 1_{\mathcal{L}} \in (pr(\rho \times \chi))^c$; hence, $H \times 1_{\mathcal{L}} = (A \widetilde{\cup} B) \times 1_{\mathcal{L}} = (A \times 1_{\mathcal{L}}) \widetilde{\cup} (B \times 1_{\mathcal{L}}) \in COLC(pr(\rho \times \chi))$. Pick $b_y \widetilde{\in} 1_{\mathcal{L}}$. Then we have $(a, b)_{(x,y)} \widetilde{\in} H \times 1_{\mathcal{L}} \in COLC(pr(\rho \times \chi))$. So, by Theorem 5.2, there is $T \in pr(\rho \times \chi)$ such that $(a, b)_{(x,y)} \widetilde{\in} T \widetilde{\subseteq} Clo_{pr(\rho\times\chi)}(T) \widetilde{\subseteq} H \times 1_{\mathcal{L}}$. Pick $U \in \rho$ and $V \in \chi$ such that $(a, b)_{(x,y)} \widetilde{\in} U \times V \widetilde{\subseteq} T$. Therefore, $(a, b)_{(x,y)} \widetilde{\in} U \times V \widetilde{\subseteq} Clo_\rho(U) \times Clo_\chi(V) = Clo_{pr(\rho\times\chi)}(U \times V) \widetilde{\subseteq} Clo_{pr(\rho\times\chi)}(T) \widetilde{\subseteq} H \times 1_{\mathcal{L}}$. Thus, we have $a_x \widetilde{\in} U \widetilde{\subseteq} Clo_\rho(U) \widetilde{\subseteq} H$. Hence, again by Theorem 5.2, $(N, \rho, \mathcal{M})$ is soft lc-regular. Similarly, we can show that $(R, \chi, \mathcal{L})$ is soft lc-regular.

Question 5.10. Let $(N, \rho, \mathcal{M})$ and $(R, \chi, \mathcal{L})$ be soft lc-regular spaces. Is $(N \times R, pr(\rho \times \chi), \mathcal{M} \times \mathcal{L})$ soft lc-regular?

Theorem 5.11. Let $(N, \rho, \mathcal{M})$ be a soft lc-regular space and let $\varnothing \neq Y \subseteq N$ such that $C_Y \in LC(\rho)$, then $(Y, \rho_Y, \mathcal{M})$ is soft lc-regular.

Proof. Let $T \in LC((\rho)_Y)$ and $a_y \widetilde{\in} C_Y - T$. By Theorem 2.21, $T \in LC(\rho)$. Since $a_y \widetilde{\in} C_Y - T$, then $a_y \widetilde{\in} 1_{\mathcal{M}} - T$. Since $(N, \rho, \mathcal{M})$ is soft lc-regular, then there are $\{G, K\} \subseteq \rho$ such that $a_y \widetilde{\in} G$, $T \widetilde{\subseteq} K$, and $G \widetilde{\cap} K = 0_{\mathcal{M}}$. Thus, we have $\{G \widetilde{\cap} C_Y, K \widetilde{\cap} C_Y\} \subseteq \rho_Y$, $a_y \widetilde{\in} G \widetilde{\cap} C_Y$, and $T = T \widetilde{\cap} C_Y \widetilde{\subseteq} G \widetilde{\cap} C_Y$. Therefore, $(Y, \rho_Y, \mathcal{M})$ is soft lc-regular.

6 Conclusion

In this paper, we continue the study of soft locally closed sets. We present characterizations of these sets and investigate their behaviors using specialized soft topologies. Also, we define and investigate soft dense-in-itself spaces; in particular, we characterize soft dense-in-itself subspaces in terms of locally closed sets. Given a soft TS $(N, \rho, \mathcal{M})$, $LC(\rho)$ forms a soft base for a soft topology on N relative to $\mathcal{M}$, which is denoted by ρ_l. We study some relations between $(N, \rho, \mathcal{M})$ and $(N, \rho_l, \mathcal{M})$. In particular, we show that $(N, \rho, \mathcal{M})$ is soft locally indiscrete if and only if $\rho = \rho_l$. Moreover, several characterizations and relationships of both soft locally indiscrete spaces and soft submaximal spaces are given. In addition to these, we define and explore soft lc-regularity as a stronger form of soft regularity. Finally, we discuss the relationship between some soft topological notions and their classical counterparts.

We intend to do the following in the next papers:

(i) Define new continuity concepts between soft topological spaces using soft locally closed sets.

(ii) Define some generalizations of soft locally closed sets.

(iii) Define soft lc-normality.

(iv) Investigate how our new concepts and findings can be applied to digital and approximation environments, as well as decision-making difficulties.

(v) Trying to solve the Questions 3.19 and 5.10.

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All of the contributors wrote this article in collaboration. The final manuscript was read and approved by all writers.

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There are no competing interests declared by the authors.

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