Application of Neutrosophic Pentagonal Controlled Metric Space via Orthogonality in Traffic Flow Network Using Integral Equation

M. Rathivel[1,*], M. Jeyaraman[2], Rahul Shukla[3]
[1]Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India
[2]P.G. and Associate Professor, Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India
[3]Faculty of Natural Sciences, Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa

Emails: rathiravi52379@gmail.com; jeya.math@gmail.com; rshukla@wsu.ac.za

Abstract

In this paper, we researched and confirmed some of the axioms of $\mathfrak{NOPCMS}$ (Neutrosophic orthogonal pentagonal controlled metric space). We used $\mathfrak{NOPCMS}$ to translate the Banach contraction principle in the formerly defined spaces. Several cases were numerically evaluated, and certain findings were supported, in order to review what we found. Furthermore, by demonstrating their existence with a unique and comprehensive solution, we deliver proof of usage and implementation.

Keywords: Fixed point; Neutrosophic orthogonal Pentagonal Controlled Metric Space; Integral equation

1 Introduction

Numerous writers have introduced and popularised numerous spaces, explored fixed point theory directly, and added interest to the field. Bakhtin[2] and Czerwik[4] suggested a generalisation of metric spaces in this connectivity, which they called a b-metric space. Fuzzy sets were created by Zadeh.[19] Several authors have benefited in different ways from the development of fuzzy metric spaces, which are metric spaces and fuzzy sets.

Banach[3] made various significant findings that established the foundation for the FP hypothesis' abstraction. Using a mathematical method, the fuzzy set term let us comprehend the level of ambiguity in the elements. Kramosil and Michalek coined the term fuzzy metric space ($\mathfrak{FMS}$).[11] In,[15] Schweizer and Sklar coined the phrase "continuous ($\mathfrak{CTS}$) t- norms".It is well accepted that the Banach contraction principle was crucial to the onset of $\mathfrak{FMS}$. A vague version of the Banach contraction principle was discussed by Grabiec.[8] With great work, Park[13] developed an intuitionistic fuzzy metric space.

Smarandache coined the phrase "Neutrosophic set" in 1998 and presented it alongside Sowndrarajan,[17] highlighting some significant $\mathfrak{NMS}$ discoveries. Sowdrarajan and Jeyaraman et al.[10] verified certain FP findings in $\mathfrak{NMS}$ in 2020. While the intuitionistic and fuzzy do not address naturalness, the Neutrosophic does. Sezen[16] proposed the premise of controlled fuzzy metric space ($\mathfrak{CFMS}$), while Milaiki[12] suggested the notion of controlled metric space ($\mathfrak{CMS}$). Both authors illustrated different contraction mapping results.

As a further extension of fuzzy b-metric spaces, the authors of[9] established the idea of an extended fuzzy b-metric space. The idea of controlled metric type spaces was initially put forward in the work.[11] A formulation of controlled fuzzy metric spaces, which are also generalisations of extended fuzzy b-metric spaces, was recently proposed in[13] as a generalisation of the concept of controlled - type metric spaces.

Eshaghiet al.[6] derived the Banach fixed point result while developing the idea of an orthogonal set. Numerous writers kept working on orthogonal spaces. The idea of control fuzzy metric spaces is being further generalised in this work.[13] In particular, we explore the notion of fuzzy metric spaces for orthogonal control.

First, let's review some fundamental terms associated with the given notions.

2 Preliminaries

Definition 2.1. [10] A 6-tuple $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond)$ is called $\mathfrak{NMS}$ if $\mathfrak{K}$ is an arbitrary non empty set, $\star$ neutrosophic $\mathfrak{CTN}$, $\diamond$ neutrosophic $\mathfrak{CTC}$ and $\mathfrak{A}, \mathfrak{I}, \mathfrak{D}$ are neutrosophic sets on $\mathfrak{K}\times\mathfrak{K}\times(0,\infty)$ satisfying the following condition: For all $\varsigma, \Theta, \eta \in \mathfrak{K}, \mathfrak{z} \in (0,\infty)$ a) $0 \leq \mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) \leq 1; 0 \leq \mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) \leq 1; 0 \leq \mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) \leq 1$;
b) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) + \mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) + \mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) \leq 3$;
c) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) = 1, \forall \mathfrak{z} > 0, \Leftrightarrow \varsigma = \Theta$;
d) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{A}(\Theta, \varsigma, \mathfrak{z})$; for $\mathfrak{z} > 0$;
e) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) \star \mathfrak{A}(\Theta, \eta, \mathfrak{x}) \geq \mathfrak{A}(\varsigma, \eta, \mathfrak{z}+\mathfrak{x}) \forall \mathfrak{z}, \mathfrak{x} > 0$;
f) $\mathfrak{A}(\varsigma, \Theta, .) : (0, +\infty) \to [0, 1]$ is neutrosophic $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) = 1$;
g) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = 0, \forall \mathfrak{z} > 0, \Leftrightarrow \varsigma = \Theta$;
h) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{I}(\Theta, \varsigma, \mathfrak{z})$; for $\mathfrak{z} > 0$;
i) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) \star \mathfrak{I}(\Theta, \eta, \mathfrak{x}) \leq \mathfrak{I}(\varsigma, \eta, \mathfrak{z}+\mathfrak{x}) \forall \mathfrak{z}, \mathfrak{x} > 0$;
j) $\mathfrak{I}(\varsigma, \Theta, .) : (0, +\infty) \to [0, 1]$ is neutrosophic $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = 0$;
k) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = 0, \forall \mathfrak{z} > 0, \Leftrightarrow \varsigma = \Theta$;
l) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{D}(\Theta, \varsigma, \mathfrak{z})$; for $\mathfrak{z} > 0$;
m) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) \star \mathfrak{D}(\Theta, \eta, \mathfrak{x}) \leq \mathfrak{D}(\varsigma, \eta, \mathfrak{z}+\mathfrak{x}) \forall \mathfrak{z}, \mathfrak{x} > 0$;
n) $\mathfrak{D}(\varsigma, \Theta, .) : (0, +\infty) \to [0, 1]$ is neutrosophic $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = 0$;
Then, $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond)$ is called a $\mathfrak{NMS}$.

Definition 2.2. [12] Given $\urcorner$, let $\mathfrak{K}$ be a non empty set and $\urcorner : \mathfrak{K} \times \mathfrak{K} \to [1, +\infty)$ are incompetent mapping, if $\delta : \mathfrak{K} \times \mathfrak{K} \to (0, +\infty)$ is called as a Controlled metric type ($\mathfrak{CMS}$) if
a) $\delta(\varsigma, \Theta,) = 0$ iff $\varsigma = \Theta$;
b) $\delta(\varsigma, \Theta) = \delta(\Theta, \varsigma)$;
c) $\delta(\varsigma, \Theta) \leq \urcorner(\varsigma, \eta)\delta(\varsigma, \eta) + \urcorner(\eta, \Theta)\delta(\eta, \Theta)$; for every $\varsigma, \Theta, \eta \in \mathfrak{K}$.

Definition 2.3. :[14] Let $\mathfrak{K}$ be a non empty set and $\urcorner : \mathfrak{K} \times \mathfrak{K} \to [1, +\infty), \star$ neutrosophic $\mathfrak{CTN}, \diamond$ neutrosophic $\mathfrak{CTC}$ and $\mathfrak{A}, \mathfrak{I}, \mathfrak{D}$ are neutrosophic sets on $\mathfrak{K}\times\mathfrak{K}\times(0,\infty)$ satisfying the following condition: For all $\varsigma, \Theta, \eta \in \mathfrak{K}$,
a) $0 \leq \mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) \leq 1; 0 \leq \mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) \leq 1; 0 \leq \mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) \leq 1$;
b) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) + \mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) + \mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) \leq 3$;
c) $\mathfrak{A}(\varsigma, \Theta, 0) = 0$
d) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) = 1, \forall \mathfrak{z} > 0, \Leftrightarrow \varsigma = \Theta$;
e) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{A}(\Theta, \varsigma, \mathfrak{z})$;
f) $\mathfrak{A}(\varsigma, \eta, \mathfrak{z}+\mathfrak{x}) \geq \mathfrak{A}\left(\varsigma, \Theta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\Theta)}\right) \star \mathfrak{A}\left(\Theta, \eta, \frac{\mathfrak{x}}{\urcorner(\Theta,\eta)}\right)$;
g) $\mathfrak{A}(\varsigma, \Theta, .) : (0, +\infty) \to [0, 1]$ is $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{A}(\varsigma, \Theta, \mathfrak{z}) = 1$;
h) $\mathfrak{I}(\varsigma, \Theta, 0) = 1$
i) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = 0, \forall \mathfrak{z} > 0 \Leftrightarrow \varsigma = \Theta$;
j) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{I}(\Theta, \varsigma, \mathfrak{z})$;
k) $\mathfrak{I}(\varsigma, \eta, \mathfrak{z}+\mathfrak{x}) \leq \mathfrak{I}\left(\varsigma, \Theta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\Theta)}\right) \diamond \mathfrak{I}\left(\Theta, \eta, \frac{\mathfrak{x}}{\urcorner(\Theta,\eta)}\right)$;
l) $\mathfrak{I}(\varsigma, \Theta, .) : (0, +\infty) \to [0, 1]$ is $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = 0$;
m) $\mathfrak{D}(\varsigma, \Theta, 0) = 1$
n) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = 0, \forall \mathfrak{z} > 0 \Leftrightarrow \varsigma = \Theta$;
o) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{I}(\Theta, \varsigma, \mathfrak{z})$;
p) $\mathfrak{D}(\varsigma, \eta, \mathfrak{z}+\mathfrak{x}) \leq \mathfrak{I}\left(\varsigma, \Theta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\Theta)}\right) \diamond \mathfrak{I}\left(\Theta, \eta, \frac{\mathfrak{x}}{\urcorner(\Theta,\eta)}\right)$;

q) $(\varsigma,\Theta,.) : (0,+\infty) \to [0,1]$ is $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) = 0$
Then, $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},\star,\diamond)$ is called a $\mathfrak{NCMS}$.

Definition 2.4. [6] Assume $\mathfrak{K}$ be a set and $\mu : \mathfrak{K} \to \mathfrak{K}$ and $\mathfrak{O}(\tau) = \{\tau_0, \mu\tau_0, \mu^2\tau_0 \dots\}$ for some $\tau_0 \in \mathfrak{K}$, be the orbit of μ. A function $\mathfrak{Y} : \mathfrak{K} \to \mathfrak{K}$ is said to be μ - orbitally lower semi continuous at $m \in \mathfrak{K}$ if $\{\tau_n\} \in \mathfrak{O}(\tau_0)$ such that $\tau_n \to m$ then we get $\mathfrak{Y}(m) \geq \lim_{n\to\infty} \inf \mathfrak{Y}(\tau_n)$.

Definition 2.5. [6] Suppose $\mathfrak{K} \neq \emptyset$ and let $\perp \in \mathfrak{K}\times\mathfrak{K}$ be a binary relation. Suppose there exist $\tau_0 \in \mathfrak{K}$ such that $\tau_0 \perp \tau$ or $\tau \perp \tau_0$ for all $\tau \in \mathfrak{K}$. Thus, $\mathfrak{K}$ is known as Orthogonal Set (OS) and is denoted by $(\mathfrak{K},\perp)$.

Definition 2.6. [6] Assume that $(\mathfrak{K},\perp)$ is an OS. A sequence $\{\tau_n\}$ for $n \in \mathbb{N}$ is known to be an $\mathfrak{O}$ - sequence if $(\forall n, \tau_n \perp \tau_{n+1})$ or $(\forall n, \tau_{n+1} \perp \tau_n)$

Definition 2.7. [18] Suppose $\mathfrak{K} \neq \emptyset$ and $\urcorner, \varphi, \xi, \psi, \delta : \mathfrak{K}\times\mathfrak{K} \to [1,+\infty)$ are considered as a incompetant mappings, $\star$ as $\mathfrak{CTN}$, $\diamond$ as $\mathfrak{CTC}$, and $\mathfrak{A},\mathfrak{I},\mathfrak{D}$ are neutrosophic sets on $\mathfrak{K}\times\mathfrak{K}\times(0,+\infty)$ is characterized $\mathfrak{NPCMS}$ on $\mathfrak{K}$, if for each one $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},\star,\diamond)$ fulfills all $\varsigma,\Theta,\eta,\omega,\epsilon,\sigma \in \mathfrak{K}, \varsigma \neq \eta, \eta \neq \omega, \omega \neq \epsilon, \epsilon \neq \sigma, \sigma \neq \Theta$, and $\mathfrak{z},\mathfrak{x},d,h,v > 0$ holds the following:
a) $0 \leq \mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) \leq 1; 0 \leq \mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) \leq 1; 0 \leq \mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) \leq 1;$
b) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) + \mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) + \mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) \leq 3;$
c) $\mathfrak{A}(\varsigma,\Theta,0) = 0;$
d) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) = 1, \forall \mathfrak{z} > 0, \Leftrightarrow \varsigma = \Theta;$
e) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) = \mathfrak{A}(\Theta,\varsigma,\mathfrak{z});$
f) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\geq \mathfrak{A}\left(\varsigma,\eta,\frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \star \mathfrak{A}\left(\eta,\omega,\frac{\mathfrak{x}}{\varphi(\eta,\omega)}\right) \star \mathfrak{A}\left(\omega,\epsilon,\frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \star \mathfrak{A}\left(\epsilon,\sigma,\frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \star \mathfrak{A}\left(\sigma,\Theta,\frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right);$
g) $\mathfrak{A}(\varsigma,\Theta,.) : (0,+\infty) \to [0,1]$ is $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) = 1;$
h) $\mathfrak{I}(\varsigma,\Theta,0) = 0;$
i) $\mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) = 0, \forall \mathfrak{z} > 0 \Leftrightarrow \varsigma = \Theta;$
j) $\mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) = \mathfrak{I}(\Theta,\varsigma,\mathfrak{z});$
k) $\mathfrak{I}(\varsigma,\Theta,\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\leq \mathfrak{I}\left(\varsigma,\eta,\frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \diamond \mathfrak{I}\left(\eta,\omega,\frac{\mathfrak{x}}{\varphi(\eta,\omega)}\right) \diamond \mathfrak{I}\left(\omega,\epsilon,\frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \diamond \mathfrak{I}\left(\epsilon,\sigma,\frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \diamond \mathfrak{I}\left(\sigma,\Theta,\frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right);$
l) $\mathfrak{I}(\varsigma,\Theta,.) : (0,+\infty) \to [0,1]$ is $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) = 0;$
m) $\mathfrak{D}(\varsigma,\Theta,0) = 0;$
n) $\mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) = 0, \forall \mathfrak{z} > 0 \Leftrightarrow \varsigma = \Theta;$
o) $\mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) = \mathfrak{D}(\Theta,\varsigma,\mathfrak{z});$
p) $\mathfrak{D}(\varsigma,\Theta,\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\leq \mathfrak{D}\left(\varsigma,\eta,\frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \diamond \mathfrak{D}\left(\eta,\omega,\frac{\mathfrak{x}}{\varphi(\eta,\omega)}\right) \diamond \mathfrak{D}\left(\omega,\epsilon,\frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \diamond \mathfrak{D}\left(\epsilon,\sigma,\frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \diamond \mathfrak{D}\left(\sigma,\Theta,\frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right);$
q) $\mathfrak{D}(\varsigma,\Theta,.) : (0,+\infty) \to [0,1]$ is $\mathfrak{CTS}$ and $\lim_{z\to+\infty} \mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) = 0;$
Then, $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},\star,\diamond)$ is called a $\mathfrak{NPCMS}$.

3 Main results

We have now clarified the meaning of the $\mathfrak{NOPCMS}$ and included illustrations to support certain of the arguments.

Definition 3.1. A 7- tuple $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},\star,\diamond,\perp)$ is called an neutrosophic orthogonal pentagonal controlled metric space $\mathfrak{NOPCMS}$ if $\mathfrak{K}$ is an (non empty) orthogonal set, $\urcorner, \varphi, \xi, \psi, \delta : \mathfrak{K}\times\mathfrak{K} \to [1,+\infty)$, where $\star$ is a $\mathfrak{CTN}$ and $\diamond$ as $\mathfrak{CTC}$ and $\mathfrak{A},\mathfrak{I},\mathfrak{D}$ are neutrosophic sets on $\mathfrak{K}\times\mathfrak{K}\times(0,+\infty)$, satisfying the following conditions:
a) $0 \leq \mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) \leq 1; 0 \leq \mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) \leq 1; 0 \leq \mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) \leq 1;$
b) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) + \mathfrak{I}(\varsigma,\Theta,\mathfrak{z}) + \mathfrak{D}(\varsigma,\Theta,\mathfrak{z}) \leq 3;$
c) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) > 0, \forall \varsigma,\Theta \in \mathfrak{K}, \mathfrak{z} > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
d) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) = 1 \Leftrightarrow \varsigma = \Theta \forall \mathfrak{z} > 0, \varsigma,\Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
e) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}) = \mathfrak{A}(\Theta,\varsigma,\mathfrak{z}) \forall \mathfrak{z} > 0, \varsigma,\Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
f) $\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$

$\geq \mathfrak{A}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \star \mathfrak{A}\left(\eta, \omega, \frac{\mathfrak{r}}{\varphi(\eta,\omega)}\right) \star \mathfrak{A}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \star \mathfrak{A}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \star \mathfrak{A}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right) \star \mathfrak{A}\left(\epsilon, \sigma, \mathfrak{h}\right) \star \mathfrak{A}\left(\sigma, \Theta, \mathfrak{v}\right)$; Or
$\mathfrak{A}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\geq \mathfrak{A}(\varsigma,\eta,\mathfrak{z}) \star \mathfrak{A}(\eta,\omega,\mathfrak{r}) \star \mathfrak{A}(\omega,\epsilon,\mathfrak{d}) \star \mathfrak{A}(\epsilon,\sigma,\mathfrak{h}) \star \mathfrak{A}(\sigma,\Theta,\mathfrak{v})$
$\forall \varsigma, \Theta, \eta, \omega, \epsilon, \sigma \in \mathfrak{K}, \mathfrak{z}, \mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \mathfrak{v} > 0$ such that $\varsigma \perp \eta, \eta \perp \omega, \omega \perp \epsilon, \epsilon \perp \sigma, \sigma \perp \Theta$ and $\varsigma \perp \Theta$;
g) $\mathfrak{A}(\varsigma, \Theta, .) : (0, +\infty) \to [0,1]$ is $\mathfrak{CTS} \forall \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
h) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) < 0, \forall \varsigma, \Theta \in \mathfrak{K}, \mathfrak{z} > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
i) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = 0 \Leftrightarrow \varsigma = \Theta \forall \mathfrak{z} > 0, \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
j) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{I}(\Theta, \varsigma, \mathfrak{z}) \forall \mathfrak{z} > 0, \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
k) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\leq \mathfrak{I}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \diamond \mathfrak{I}\left(\eta, \omega, \frac{\mathfrak{r}}{\varphi(\eta,\omega)}\right) \diamond \mathfrak{I}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \diamond \mathfrak{I}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \diamond \mathfrak{I}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$; Or;
$\mathfrak{I}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\leq \mathfrak{I}(\varsigma,\eta,\mathfrak{z}) \diamond \mathfrak{I}(\eta,\omega,\mathfrak{r}) \diamond \mathfrak{I}(\omega,\epsilon,\mathfrak{d}) \diamond \mathfrak{I}(\epsilon,\sigma,\mathfrak{h}) \diamond \mathfrak{I}(\sigma,\Theta,\delta(\sigma,\Theta)\mathfrak{v})$
$\forall \varsigma, \Theta, \eta, \omega, \epsilon, \sigma \in \mathfrak{K}, \mathfrak{z}, \mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \mathfrak{v} > 0$ such that $\varsigma \perp \eta, \eta \perp \omega, \omega \perp \epsilon, \epsilon \perp \sigma, \sigma \perp \Theta$ and $\varsigma \perp \Theta$;
l) $\mathfrak{I}(\varsigma, \Theta, .) : (0, +\infty) \to [0,1]$ is $\mathfrak{CTS} \forall \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
m) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) < 0, \forall \varsigma, \Theta \in \mathfrak{K}, \mathfrak{z} > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
n) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = 0 \Leftrightarrow \varsigma = \Theta \forall \mathfrak{z} > 0, \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
o) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}) = \mathfrak{D}(\Theta, \varsigma, \mathfrak{z}) \forall \mathfrak{z} > 0, \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$;
p) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\leq \mathfrak{D}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \diamond \mathfrak{D}\left(\eta, \omega, \frac{\mathfrak{r}}{\varphi(\eta,\omega)}\right) \diamond \mathfrak{D}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \diamond \mathfrak{D}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \diamond \mathfrak{D}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$; Or
$\mathfrak{D}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\leq \mathfrak{D}(\varsigma,\eta,\mathfrak{z}) \diamond \mathfrak{D}(\eta,\omega,\mathfrak{r}) \diamond \mathfrak{D}(\omega,\epsilon,\mathfrak{d}) \diamond \mathfrak{D}(\epsilon,\sigma,\mathfrak{h}) \diamond \mathfrak{D}(\sigma,\Theta,\mathfrak{v})$
$\forall \varsigma, \Theta, \eta, \omega, \epsilon, \sigma \in \mathfrak{K}, \mathfrak{z}, \mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \mathfrak{v} > 0$ such that $\varsigma \perp \eta, \eta \perp \omega, \omega \perp \epsilon, \epsilon \perp \sigma, \sigma \perp \Theta$ and $\varsigma \perp \Theta$.
q) $\mathfrak{D}(\varsigma, \Theta, .) : (0, +\infty) \to [0,1]$ is $\mathfrak{CTS} \forall \varsigma, \Theta \in \mathfrak{K}$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$.

Note:
From the above definition, we show that the following are equivalent:
(i) $\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\geq \mathfrak{A}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \star \mathfrak{A}\left(\eta, \omega, \frac{\mathfrak{r}}{\varphi(\eta,\omega)}\right) \star \mathfrak{A}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \star \mathfrak{A}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \star \mathfrak{A}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$
(ii) $\mathfrak{A}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\geq \mathfrak{A}(\varsigma,\eta,\mathfrak{z}) \star \mathfrak{A}(\eta,\omega,\mathfrak{r}) \star \mathfrak{A}(\omega,\epsilon,\mathfrak{d}) \star \mathfrak{A}(\epsilon,\sigma,\mathfrak{h}) \star \mathfrak{A}(\sigma,\Theta,\mathfrak{v})$;
(iii) $\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\leq \mathfrak{I}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \diamond \mathfrak{I}\left(\eta, \omega, \frac{\mathfrak{r}}{\varphi(\eta,\omega)}\right) \diamond \mathfrak{I}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \diamond \mathfrak{I}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \diamond \mathfrak{I}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$
iv) $\mathfrak{I}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\leq \mathfrak{I}(\varsigma,\eta,\mathfrak{z}) \diamond \mathfrak{I}(\eta,\omega,\mathfrak{r}) \diamond \mathfrak{I}(\omega,\epsilon,\mathfrak{d}) \diamond \mathfrak{I}(\epsilon,\sigma,\mathfrak{h}) \diamond \mathfrak{I}(\sigma,\Theta,\mathfrak{v})$
v) $\mathfrak{D}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\leq \mathfrak{D}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right) \diamond \mathfrak{D}\left(\eta, \omega, \frac{\mathfrak{r}}{\varphi(\eta,\omega)}\right) \diamond \mathfrak{D}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right) \diamond \mathfrak{D}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right) \diamond \mathfrak{D}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$
vi) $\mathfrak{D}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\leq \mathfrak{D}(\varsigma,\eta,\mathfrak{z}) \diamond \mathfrak{D}(\eta,\omega,\mathfrak{r}) \diamond \mathfrak{D}(\omega,\epsilon,\mathfrak{d}) \diamond \mathfrak{D}(\epsilon,\sigma,\mathfrak{h}) \diamond \mathfrak{D}(\sigma,\Theta,\mathfrak{v})$

Proof. (i) $\Rightarrow$ (ii)
$\mathfrak{A}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$= \mathfrak{A}(\varsigma,\eta,\mathfrak{z}) \star \mathfrak{A}(\eta,\omega,\mathfrak{r}) \star \mathfrak{A}(\omega,\epsilon,\mathfrak{d}) \star \mathfrak{A}(\epsilon,\sigma,\mathfrak{h}) \star \mathfrak{A}(\sigma,\Theta,\mathfrak{v})$
$\geq \mathfrak{A}\left(\varsigma, \eta, \frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{z}}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right) \star \mathfrak{A}\left(\eta, \omega, \frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{r}}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right) \star$
$\mathfrak{A}\left(\omega, \epsilon, \frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{d}}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right) \star \mathfrak{A}\left(\epsilon, \sigma, \frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{h}}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right) \star$
$\mathfrak{A}\left(\sigma, \Theta, \frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{v}}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right)$;
$\geq \mathfrak{A}\left((\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{z}\right.$
$+\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{r} + \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{d}$
$\left.+\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{h} + \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{v})\right)$
$\geq \mathfrak{A}(\varsigma, \Theta, \urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
Similarly, we can easily prove (ii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii)
$\mathfrak{I}(\varsigma, \Theta, \mathfrak{z}+\mathfrak{r}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$= \mathfrak{I}(\varsigma,\eta,\mathfrak{z}) \diamond \mathfrak{I}(\eta,\omega,\mathfrak{r}) \diamond \mathfrak{I}(\omega,\epsilon,\mathfrak{d}) \diamond \mathfrak{I}(\epsilon,\sigma,\mathfrak{h}) \diamond \mathfrak{I}(\sigma,\Theta,\mathfrak{v})$

$\leq \mathfrak{I}\left(\varsigma,\eta,\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{z}}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right)\Diamond\mathfrak{I}\left(\eta,\omega,\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{x}}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right)\Diamond$
$\mathfrak{I}\left(\omega,\epsilon,\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{d}}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right)\Diamond\mathfrak{I}\left(\epsilon,\sigma,\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{h}}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right)\Diamond$
$\mathfrak{I}\left(\sigma,\Theta,\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{v}}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}\right);$
$\geq \mathfrak{I}\left((\varsigma,\Theta,\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{z}\right.$
$+\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{x}+\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{d}$
$+\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{h}+\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)\mathfrak{v}))$
$\geq \mathfrak{I}(\varsigma,\Theta,\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
Similarly, we can easily prove (ii) $\Rightarrow$ (i). In the same manner, following will be verified
$\mathfrak{D}(\varsigma,\Theta,\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\leq\mathfrak{D}\left(\varsigma,\eta,\frac{\mathfrak{z}}{\daleth(\varsigma,\eta)}\right)\Diamond\mathfrak{D}\left(\eta,\omega,\frac{\mathfrak{x}}{\varphi(\eta,\omega)}\right)\Diamond\mathfrak{D}\left(\omega,\epsilon,\frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right)$
$\Diamond\mathfrak{D}\left(\epsilon,\sigma,\frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right)\Diamond\mathfrak{D}\left(\sigma,\Theta,\frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$ or
$\mathfrak{D}(\varsigma,\Theta,\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\leq\mathfrak{D}(\varsigma,\eta,\mathfrak{z})\Diamond\mathfrak{D}(\eta,\omega,\mathfrak{x})\Diamond\mathfrak{D}(\omega,\epsilon,\mathfrak{d})\Diamond\mathfrak{D}(\epsilon,\sigma,\psi(\epsilon,\sigma)\mathfrak{h})\Diamond\mathfrak{D}(\sigma,\Theta,\delta(\sigma,\Theta)\mathfrak{v})$ □

Example 3.2. Let $\mathfrak{K}=\mathfrak{Z}=\mathfrak{M}\cup\mathfrak{B}$, where $\mathfrak{M}=\{-2,-1\}$ and $\mathfrak{B}=\{1,2\}$. Define a binary relation $\perp$ by $\varsigma\perp\Theta\Leftrightarrow\varsigma+\Theta\geq 0$. Given $\mathfrak{A},\mathfrak{I},\mathfrak{D}:\mathfrak{K}\times\mathfrak{K}\times[0,\infty)\to[0,1]$ as

$$\mathfrak{A}(\varsigma,\Theta,\iota)=\begin{cases}\frac{\iota}{\iota+|\varsigma-\Theta|^6} & \text{both}\,\varsigma,\Theta\in\mathfrak{B}\\ e^{\frac{1}{\varsigma\Theta\iota}} & \text{otherwise}\end{cases}$$
$$\mathfrak{I}(\varsigma,\Theta,\iota)=\begin{cases}\frac{|\varsigma-\Theta|^6}{\iota+|\varsigma-\Theta|^6} & \text{both}\,\varsigma,\Theta\in\mathfrak{B}\\ 1-e^{\frac{1}{\varsigma\Theta\iota}} & \text{otherwise}\end{cases}$$
$$\mathfrak{D}(\varsigma,\Theta,\iota)=\begin{cases}\frac{|\varsigma-\Theta|^6}{\iota} & \text{both}\,\varsigma,\Theta\in\mathfrak{B}\\ e^{-\frac{1}{\varsigma\Theta\iota}}-1 & \text{otherwise}\end{cases}$$

with a $\mathfrak{CTN}$ $*$ defined by $\iota_1*\iota_2=\iota_1\cdot\iota_2$ and $\mathfrak{CTC}$ by $\iota_1\Diamond\iota_2=\iota_1\cdot\iota_2$ and $\mathfrak{z},\mathfrak{x},\mathfrak{d},\mathfrak{h},\mathfrak{v}>0$ where $\iota=\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}$. Define $\daleth,\varphi,\xi,\psi,\delta:\mathfrak{K}\times\mathfrak{K}\to[1,+\infty)$ by

$$\daleth(\varsigma,\Theta)=\begin{cases}1, & \text{if}\,\varsigma,\Theta\in\mathfrak{M},\\ \max\{\varsigma,\Theta\}, & \text{otherwise}\end{cases}$$
$$\varphi(\varsigma,\Theta)=\begin{cases}1, & \text{if}\,\varsigma,\Theta\in\mathfrak{M},\\ \frac{1}{\min\{|\varsigma|,|\Theta|\}}, & \text{otherwise}\end{cases}$$
$$\xi(\varsigma,\Theta)=\begin{cases}1, & \text{if}\,\varsigma,\Theta\in\mathfrak{M},\\ \frac{1}{\max\{\varsigma,\Theta\}}, & \text{otherwise}\end{cases}$$
$$\psi(\varsigma,\Theta)=\begin{cases}1, & \text{if}\,\varsigma,\Theta\in\mathfrak{M},\\ \min\{|\varsigma|,|\Theta|\}, & \text{otherwise}\end{cases}$$
$$\delta(\varsigma,\Theta)=\begin{cases}1, & \text{if}\,\varsigma,\Theta\in\mathfrak{M},\\ \max\{\varsigma+\Theta,|\varsigma-\Theta|\}, & \text{otherwise}\end{cases}$$

Then, $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},\star,\Diamond,\perp)$ is a neutrosophic orthogonal pentagonal controlled metric space ($\mathfrak{NOPCMS}$) under given suitable domain but it is not a control fuzzy metric space at
$\varsigma=\Theta=\eta=\omega=\epsilon=\sigma=-1$ and $\mathfrak{x},\mathfrak{d},\mathfrak{h},\mathfrak{v}=1$ then, $\daleth=\varphi=\xi=\psi=\delta=1$
$\mathfrak{A}(\varsigma,\Theta,\iota)$
$=e^{\frac{1}{\varsigma\Theta\iota}}\ngeq\mathfrak{A}\left(\varsigma,\eta,\frac{\mathfrak{z}}{\daleth(\varsigma,\eta)}\right)\star\mathfrak{A}\left(\eta,\omega,\frac{\mathfrak{x}}{\varphi(\eta,\omega)}\right)\star\mathfrak{A}\left(\omega,\epsilon,\frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right)\star\mathfrak{A}\left(\epsilon,\sigma,\frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right)\star\mathfrak{A}\left(\sigma,\Theta,\frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right).$

Example 3.3. Let $\mathfrak{K}=\mathbb{Z}=\mathfrak{M}\cup\mathfrak{B}\cup 0$, where $\mathfrak{M}=\{-\infty,\ldots,-3,-2,-1\}$ and
$\mathfrak{B}=\{1,2,3,4\ldots\infty\}$. Define a binary relation $\perp$ by $\varsigma\perp\Theta\Leftrightarrow\varsigma+\Theta\geq 0$.
Given $\mathfrak{A},\mathfrak{I},\mathfrak{D}:\mathfrak{K}\times\mathfrak{K}\times[0,\infty)\to[0,1]$ by

$$\mathfrak{M}(\varsigma,\Theta,\iota)=\frac{\iota}{\iota+\max\{\varsigma,\Theta\}},\quad (1)$$
$$\mathfrak{I}(\varsigma,\Theta,\iota)=\frac{\max\{\varsigma,\Theta\}}{\iota+\max\{\varsigma,\Theta\}},\quad (2)$$
$$\mathfrak{D}(\varsigma,\Theta,\iota)=\frac{\max\{\varsigma,\Theta\}}{\iota}\quad (3)$$

with a $\mathfrak{CTN}$ $*$ defined by $\iota_1 * \iota_2 = \iota_1 \cdot \iota_2$ and $\mathfrak{CTC}$ by $\iota_1 \Delta \iota_2 = \iota_1 \cdot \iota_2, \forall \mathfrak{z}, \mathfrak{x}, \mathfrak{d}, \mathfrak{h}, \mathfrak{v} > 0$ where $\iota = \mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}$. Define $\daleth, \varphi, , \psi, \delta : \mathfrak{K} \times \mathfrak{K} \to [1, +\infty)$ by

$$\daleth(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \max\{\varsigma, \Theta\}, & \text{otherwise.} \end{cases} \tag{4}$$

$$\varphi(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \frac{1}{\min\{|\varsigma|, |\Theta|\}}, & \text{otherwise.} \end{cases} \tag{5}$$

$$\xi(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \frac{1}{\max\{\varsigma, \Theta\}}, & \text{otherwise.} \end{cases} \tag{6}$$

$$\psi(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \min\{|\varsigma|, |\Theta|\}, & \text{otherwise.} \end{cases} \tag{7}$$

$$\delta(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \max\{\varsigma + \Theta, |\varsigma - \Theta|\}, & \text{otherwise.} \end{cases} \tag{8}$$

Then, $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ is called an neutrosophic orthogonal pentagonal controlled metric space $\mathfrak{NOPCMS}$ but it is not a control fuzzy metric space.

Proof. First, we show that $\mathfrak{D}$ is an neutrosophic orthogonal pentagonal controlled metric space. The conditions except $\{(d), (i), (n)\}, \{(e), (j), (o)\}, \{(f), (k), (p)\}$ are obvious. Here, we prove $\{(d), (i), (n)\}, \{(e), (j), (o)\}$ and $\{(f), (k), (p)\}$:

(d) $\mathfrak{A}(\varsigma, \Theta, \iota) = 1 \Leftrightarrow \varsigma = \Theta \, \forall \varsigma, \Theta \in \mathfrak{K}, \iota > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$:
$\mathfrak{A}(\varsigma, \Theta, \iota) = 1, \Leftrightarrow \frac{\iota}{\iota + \max\{\varsigma, \Theta\}} = 1, \Leftrightarrow \iota = \iota + \max\{\varsigma, \Theta\} \Leftrightarrow \max\{\varsigma, \Theta\} = 0, \Leftrightarrow \varsigma = \Theta$
(i) $\mathfrak{I}(\varsigma, \Theta, \iota) = 0 \Leftrightarrow \varsigma = \Theta \, \forall \varsigma, \Theta \in \mathfrak{K}, \iota > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$:
$\mathfrak{I}(\varsigma, \Theta, \iota) = 0, \Leftrightarrow \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}} = 0, \Leftrightarrow \max\{\varsigma, \Theta\} = 0, \Leftrightarrow \varsigma = \Theta$
(n) $\mathfrak{D}(\varsigma, \Theta, \iota) = 0 \Leftrightarrow \varsigma = \Theta \, \forall \varsigma, \Theta \in \mathfrak{K}, \iota > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$:
$\mathfrak{D}(\varsigma, \Theta, \iota) = 0, \Leftrightarrow \frac{\max\{\varsigma, \Theta\}}{\iota} = 0, \Leftrightarrow \max\{\varsigma, \Theta\} = 0, \Leftrightarrow \varsigma = \Theta$
(e) $\mathfrak{A}(\varsigma, \Theta, \iota) = \mathfrak{A}(\Theta, \varsigma, \iota), \forall \varsigma, \Theta \in \mathfrak{K}, \iota > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$
$\mathfrak{A}(\varsigma, \Theta, \iota) = \frac{\iota}{\iota + \max\{\varsigma, \Theta\}} = \frac{\iota}{\iota + \max\{\Theta, \varsigma\}} = \mathfrak{A}(\Theta, \varsigma, \iota)$
(j) $\mathfrak{I}(\varsigma, \Theta, \iota) = \mathfrak{I}(\Theta, \varsigma, \iota), \forall \varsigma, \Theta \in \mathfrak{K}, \iota > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$
$\mathfrak{I}(\varsigma, \Theta, \iota) = \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}} = \frac{\max\{\Theta, \varsigma\}}{\iota + \max\{\Theta, \varsigma\}} = \mathfrak{I}(\Theta, \varsigma, \iota)$
(o) $\mathfrak{D}(\varsigma, \Theta, \iota) = \mathfrak{D}(\Theta, \varsigma, \iota), \forall \varsigma, \Theta \in \mathfrak{K}, \iota > 0$ such that $\varsigma \perp \Theta$ and $\Theta \perp \varsigma$
$\mathfrak{D}(\varsigma, \Theta, \iota) = \frac{\max\{\varsigma, \Theta\}}{\iota} = \frac{\max\{\Theta, \varsigma\}}{\iota} = \mathfrak{D}(\Theta, \varsigma, \iota)$
(f) $\mathfrak{A}(\varsigma, \Theta, \daleth(\varsigma, \eta)\varphi(\eta, \omega)\xi(\omega, \epsilon)\psi(\epsilon, \sigma)\delta(\sigma, \Theta)(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v}))$
$\geq \mathfrak{A}(\varsigma, \eta, \mathfrak{z}) \star \mathfrak{A}(\eta, \omega, \mathfrak{x}) \star \mathfrak{A}(\omega, \epsilon, \mathfrak{d}) \star \mathfrak{A}(\epsilon, \sigma, \mathfrak{h}) \star \mathfrak{A}(\sigma, \Theta, \mathfrak{v})$
$\forall \varsigma, \Theta, \eta, \omega, \epsilon, \sigma \in \mathfrak{K}, \mathfrak{z}, \mathfrak{x}, \mathfrak{d}, \mathfrak{h}, \mathfrak{v} > 0$ such that $\varsigma \perp \eta, \eta \perp \omega, \omega \perp \epsilon, \epsilon \perp \sigma, \sigma \perp \Theta$ and $\varsigma \perp \Theta$.

$$\begin{aligned} \mathfrak{A}(\Theta, \varsigma, \iota) = \mathfrak{A}\left(\varsigma, \eta, \frac{\mathfrak{z}}{\daleth(\varsigma, \eta)}\right) \star \mathfrak{A}\left(\eta, \omega, \frac{\mathfrak{x}}{\varphi(\eta, \omega)}\right) \star \mathfrak{A}\left(\omega, \epsilon, \frac{\mathfrak{d}}{\xi(\omega, \epsilon)}\right) \star \\ \mathfrak{A}\left(\epsilon, \sigma, \frac{\mathfrak{h}}{\psi(\epsilon, \sigma)}\right) \star \mathfrak{A}\left(\sigma, \Theta, \frac{\mathfrak{v}}{\delta(\sigma, \Theta)}\right) \end{aligned} \tag{9}$$

$\Rightarrow \max\{\varsigma, \Theta\} \leq \daleth(\varsigma, \eta)[\max\{\varsigma, \eta\}] + \varphi(\eta, \omega)[\max\{\eta, \omega\}] + \xi(\omega, \epsilon)[\max\{\omega, \epsilon\}]$
$+\psi(\epsilon, \sigma)[\max\{\epsilon, \sigma\}] + \delta(\sigma, \Theta)[\max\{\sigma, \Theta\}]$
$\Rightarrow \mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} \max\{\varsigma, \Theta\} \leq \daleth(\varsigma, \eta)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{x}^2\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{x}\Theta^2\mathfrak{h}\mathfrak{v} + \mathfrak{x}\mathfrak{d}\mathfrak{h}^2\mathfrak{v} + \mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}^2)[\max\{\varsigma, \eta\}]$
$+\varphi(\eta, \omega)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{z}^2\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{z}\Theta^2\mathfrak{h}\mathfrak{v} + \mathfrak{z}\mathfrak{d}\mathfrak{h}^2\mathfrak{v} + \mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}^2)[\max\{\eta, \omega\}]$
$+\xi(\omega, \epsilon)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{z}^2\mathfrak{x}\mathfrak{h}\mathfrak{v} + \mathfrak{z}\mathfrak{x}^2\mathfrak{h}\mathfrak{v} + \mathfrak{z}\mathfrak{x}\mathfrak{h}^2\mathfrak{v} + \mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}^2)[\max\{\omega, \epsilon\}]$
$+\psi(\epsilon, \sigma)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{z}^2\mathfrak{x}\mathfrak{d}\mathfrak{v} + \mathfrak{z}\mathfrak{x}^2\mathfrak{d}\mathfrak{v} + \mathfrak{z}\mathfrak{x}\Theta^2\mathfrak{v} + \mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}^2)[\max\{\epsilon, \sigma\}]$
$+\delta(\sigma, \Theta)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} + \mathfrak{z}^2\mathfrak{x}\mathfrak{d}\mathfrak{h} + \mathfrak{z}\mathfrak{x}^2\mathfrak{d}\mathfrak{h} + \mathfrak{z}\mathfrak{x}\Theta^2\mathfrak{h} + \mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}^2)[\max\{\sigma, \Theta\}]$
$\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} \max\{\varsigma, \Theta\} \leq \daleth(\varsigma, \eta)\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v})[\max\{\varsigma, \eta\}]$
$+\varphi(\eta, \omega)\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v})[\max\{\eta, \omega\}]$
$+\xi(\omega, \epsilon)\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v})[\max\{\omega, \epsilon\}]$
$+\psi(\epsilon, \sigma)\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v})[\max\{\epsilon, \sigma\}]$
$+\delta(\sigma, \Theta)\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v})[\max\{\sigma, \Theta\}]$
$\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v} \max\{\varsigma, \Theta\} \leq \daleth(\varsigma, \eta)\varphi(\eta, \omega)\xi(\omega, \epsilon)\psi(\epsilon, \sigma)\delta(\sigma, \Theta)(\mathfrak{z} + \mathfrak{x} + \mathfrak{d} + \mathfrak{h} + \mathfrak{v})$
$\Big[\frac{\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma, \eta\})}{\varphi(\eta, \omega)\xi(\omega, \epsilon)\psi(\epsilon, \sigma)\delta(\sigma, \Theta)} + \frac{\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta, \omega\})}{\daleth(\varsigma, \eta)\xi(\omega, \epsilon)\psi(\epsilon, \sigma)\delta(\sigma, \Theta)} + \frac{\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega, \epsilon\})}{\daleth(\varsigma, \eta)\varphi(\eta, \omega)\psi(\epsilon, \sigma)\delta(\sigma, \Theta)}$

$+\frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\delta(\sigma,\Theta)}+\frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)}\Big]$
$\leq \daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})$
$+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})]$
$\leq \daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})$
$+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})]$
$+\max\{\varsigma,\eta\}\max\{\eta,\omega\}\max\{\omega,\epsilon\}\max\{\epsilon,\sigma\}\max\{\sigma,\Theta\}$
$\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}$
$\leq \daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}$
$+\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})]$
$+\max\{\varsigma,\eta\}\max\{\eta,\omega\}\max\{\omega,\epsilon\}\max\{\epsilon,\sigma\}\max\{\sigma,\Theta\}$
$\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}$
$\leq \daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+[[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})$
$+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})]$
$+\max\{\varsigma,\eta\}\max\{\eta,\omega\}\max\{\omega,\epsilon\}\max\{\epsilon,\sigma\}\max\{\sigma,\Theta\}]]$
$\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}[\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}]$
$\leq \daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[[\mathfrak{z}+(\max\{\varsigma,\eta\}]\,[\mathfrak{x}+(\max\{\eta,\omega\})]\,[\Theta+[\max\{\omega,\epsilon\}]]\,[\mathfrak{h}+[\max\{\epsilon,\sigma\}]]\,[\mathfrak{v}+[\max\{\sigma,\Theta\}]]]$
$\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}}$
$\geq \frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}}{[\mathfrak{z}+(\max\{\varsigma,\eta\}][\mathfrak{x}+(\max\{\eta,\omega\})][\Theta+[\max\{\omega,\epsilon\}]][\mathfrak{h}+[\max\{\epsilon,\sigma\}]][\mathfrak{v}+[\max\{\sigma,\Theta\}]]}$
$\frac{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}}$
$\geq \frac{\mathfrak{z}}{[\mathfrak{z}+\max\{\varsigma,\eta\}]}\cdot\frac{\mathfrak{x}}{[\mathfrak{x}+(\max\{\eta,\omega\})]}\cdot\frac{\mathfrak{d}}{[\Theta+[\max\{\omega,\epsilon\}]]}\cdot\frac{\mathfrak{h}}{[\mathfrak{h}+[\max\{\epsilon,\sigma\}]]}\cdot\frac{\mathfrak{v}}{[\mathfrak{v}+[\max\{\sigma,\Theta\}]]}$
$\Rightarrow \mathfrak{A}(\varsigma,\Theta,\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\geq \mathfrak{A}(\varsigma,\eta,\mathfrak{z})\star\mathfrak{A}(\eta,\omega,\mathfrak{x})\star\mathfrak{A}(\omega,\epsilon,\Theta)\star\mathfrak{A}(\epsilon,\sigma,\mathfrak{h})\star\mathfrak{A}(\sigma,\Theta,\mathfrak{v})$.
In a similar way we proceed as following,

$$\mathfrak{I}(\varsigma,\Theta,\iota)=\frac{\max\{\varsigma,\Theta\}}{\iota+\max\{\varsigma,\Theta\}}=\frac{\max\{\varsigma,\Theta\}}{\iota+\max\{\Theta,\varsigma\}}=\mathfrak{I}(\Theta,\varsigma,\iota) \tag{10}$$

(k) $\mathfrak{I}(\varsigma,\Theta,\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\leq \mathfrak{I}(\varsigma,\eta,\daleth(\varsigma,\eta)\mathfrak{z})\diamond\mathfrak{I}(\eta,\omega,\varphi(\eta,\omega)\mathfrak{x})\diamond\mathfrak{I}(\omega,\epsilon,\xi(\omega,\epsilon)\mathfrak{d})\diamond\mathfrak{I}(\epsilon,\sigma,\psi(\epsilon,\sigma)\mathfrak{h})\diamond\mathfrak{I}(\sigma,\Theta,\delta(\sigma,\Theta)\mathfrak{v})$
$\forall\varsigma,\Theta,\eta,\omega,\epsilon,\sigma\in\mathfrak{K},\mathfrak{z},\mathfrak{x},\mathfrak{d},\mathfrak{h},\mathfrak{v}>0$ such that $\varsigma\perp\eta,\eta\perp\omega,\omega\perp\epsilon,\epsilon\perp\sigma,\sigma\perp\Theta$ and $\varsigma\perp\Theta$.
$\Rightarrow \max\{\varsigma,\Theta\}\leq\daleth(\varsigma,\eta)[\max\{\varsigma,\eta\}]+\varphi(\eta,\omega)[\max\{\eta,\omega\}]+\xi(\omega,\epsilon)[\max\{\omega,\epsilon\}]$
$+\psi(\epsilon,\sigma)[\max\{\epsilon,\sigma\}]+\delta(\sigma,\Theta)[\max\{\sigma,\Theta\}]$
$\Rightarrow 1-\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}\geq 1-(\daleth(\varsigma,\eta)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{x}^2\mathfrak{d}\mathfrak{h}\mathfrak{v}\mathfrak{x}\Theta^2\mathfrak{h}\mathfrak{v}+\mathfrak{x}\mathfrak{d}\mathfrak{h}^2\mathfrak{v}+\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}^2)[\max\{\varsigma,\eta\}]$
$+\varphi(\eta,\omega)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}^2\mathfrak{d}\mathfrak{h}\mathfrak{v}\mathfrak{z}\Theta^2\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{d}\mathfrak{h}^2\mathfrak{v}+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}^2)[\max\{\eta,\omega\}]$
$+\xi(\omega,\epsilon)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}^2\mathfrak{x}\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{x}^2\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{h}^2\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}^2)[\max\{\omega,\epsilon\}]$
$+\psi(\epsilon,\sigma)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}^2\mathfrak{x}\mathfrak{d}\mathfrak{v}+\mathfrak{z}\mathfrak{x}^2\mathfrak{d}\mathfrak{v}+\mathfrak{z}\mathfrak{x}\Theta^2\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}^2)[\max\{\epsilon,\sigma\}]$
$+\delta(\sigma,\Theta)(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}^2\mathfrak{x}\mathfrak{d}\mathfrak{h}+\mathfrak{z}\mathfrak{x}^2\mathfrak{d}\mathfrak{h}+\mathfrak{z}\mathfrak{x}\Theta^2\mathfrak{h}+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}^2)[\max\{\sigma,\Theta\}])$
$1-\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}\geq 1-(\daleth(\varsigma,\eta)\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})[\max\{\varsigma,\eta\}]$
$+\varphi(\eta,\omega)\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})[\max\{\eta,\omega\}]$
$+\xi(\omega,\epsilon)zxhv(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})[\max\{\omega,\epsilon\}]$
$+\psi(\epsilon,\sigma)\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})[\max\{\epsilon,\sigma\}]$
$+\delta(\sigma,\Theta)\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})[\max\{\sigma,\Theta\}])$
$1-\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}\geq 1-(\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\Big[\frac{\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})}{\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}+\frac{\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})}{\daleth(\varsigma,\eta)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}$
$+\frac{\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)}+\frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\delta(\sigma,\Theta)}+\frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})}{\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)}\Big]\Big)$
$\geq 1-(\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})])$
$\geq 1-(\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})$
$+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})]+\max\{\varsigma,\eta\}\max\{\eta,\omega\}\max\{\omega,\epsilon\}\max\{\epsilon,\sigma\}\max\{\sigma,\Theta\})$
$1-[\daleth(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}]$

$\geq 1-(\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}$
$+\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})$
$+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})+\max\{\varsigma,\eta\}\max\{\eta,\omega\}\max\{\omega,\epsilon\}\max\{\epsilon,\sigma\}\max\{\sigma,\Theta\}])$
$1-[\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}\max\{\varsigma,\Theta\}]$
$\geq 1-(\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})\,[\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}+\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\varsigma,\eta\})$
$+\mathfrak{z}\mathfrak{d}\mathfrak{h}\mathfrak{v}(\max\{\eta,\omega\})+\mathfrak{z}\mathfrak{x}\mathfrak{h}\mathfrak{v}(\max\{\omega,\epsilon\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{v}(\max\{\epsilon,\sigma\})+\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}(\max\{\sigma,\Theta\})$
$+\max\{\varsigma,\eta\}\max\{\eta,\omega\}\max\{\omega,\epsilon\}\max\{\epsilon,\sigma\}\max\{\sigma,\Theta\}])$
$1-(\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}[\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}])$
$\geq 1-(\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$[[\mathfrak{z}+(\max\{\varsigma,\eta\}][\mathfrak{x}+(\max\{\eta,\omega\})][d+[\max\{\omega,\epsilon\}]][\mathfrak{h}+[\max\{\epsilon,\sigma\}]][\mathfrak{v}+[\max\{\sigma,\Theta\}]]]))$
$1-\left(\frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}}\right)$
$\leq 1-\left(\frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}}{[[\mathfrak{z}+\max\{\varsigma,\eta\}][\mathfrak{x}+(\max\{\eta,\omega\})][\Theta+[\max\{\omega,\epsilon\}]][\mathfrak{h}+[\max\{\epsilon,\sigma\}]][\mathfrak{v}+[\max\{\sigma,\Theta\}]]]}\right)$
$\frac{\max\{\varsigma,\Theta\}}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}}$
$\geq \frac{\max\{\varsigma,\eta\}}{[\mathfrak{z}+\max\{\varsigma,\eta\}]}\cdot\frac{\max\{\eta,\omega\}}{[\mathfrak{x}+(\max\{\eta,\omega\})]}\cdot\frac{\max\{\omega,\epsilon\}}{[\Theta+[\max\{\omega,\epsilon\}]]}\cdot\frac{\max\{\epsilon,\sigma\}}{[\mathfrak{h}+[\max\{\epsilon,\sigma\}]]}\cdot\frac{\max\{\sigma,\Theta\}}{[\mathfrak{v}+[\max\{\sigma,\Theta\}]]}$
$\Rightarrow \mathfrak{I}(\varsigma,\Theta,\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\geq \mathfrak{I}(\varsigma,\eta,\mathfrak{z})\diamond\mathfrak{I}(\eta,\omega,\mathfrak{x})\diamond\mathfrak{I}(\omega,\epsilon,\Theta)\diamond\mathfrak{I}(\epsilon,\sigma,\mathfrak{h})\diamond\mathfrak{I}(\sigma,\Theta,\mathfrak{v})$.
Like wise, In a similar way we proceed prove the following,
p$\Rightarrow \mathfrak{D}(\varsigma,\Theta,\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\geq \mathfrak{D}(\varsigma,\eta,\mathfrak{z})\diamond\mathfrak{D}(\eta,\omega,\mathfrak{x})\diamond\mathfrak{D}(\omega,\epsilon,\Theta)\diamond\mathfrak{D}(\epsilon,\sigma,\mathfrak{h})\diamond\mathfrak{D}(\sigma,\Theta,\mathfrak{v})$.
Now, we show that $\mathfrak{A}$ is not a control fuzzy metric space. Indeed
$\mathfrak{A}(\varsigma,\Theta,\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}))$
$\geq \mathfrak{A}(\varsigma,\eta,\mathfrak{z})\star\mathfrak{A}(\eta,\omega,\mathfrak{x})\star\mathfrak{A}(\omega,\epsilon,\Theta)\star\mathfrak{A}(\epsilon,\sigma,\mathfrak{h})\star\mathfrak{A}(\sigma,\Theta,\mathfrak{v})$.
$=\frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}}$
$\mathfrak{A}(\varsigma,\eta,\mathfrak{z})=\frac{\mathfrak{z}}{\mathfrak{z}+\max\{\varsigma,\eta\}},\mathfrak{A}(\eta,\omega,\mathfrak{x})=\frac{\mathfrak{x}}{\mathfrak{x}+\max\{\eta,\omega\}},\mathfrak{A}(\omega,\epsilon,\Theta)=\frac{\mathfrak{d}}{\Theta+\max\{\omega,\epsilon\}}$
$\mathfrak{A}(\epsilon,\sigma,\mathfrak{h})=\frac{\mathfrak{h}}{\mathfrak{h}+\max\{\epsilon,\sigma\}},\mathfrak{A}(\sigma,\Theta,\mathfrak{v})=\frac{\mathfrak{v}}{\mathfrak{v}+\max\{\sigma,\Theta\}}$
This implies
$(f)\mathfrak{A}(\varsigma,\Theta,\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})$
$\geq \mathfrak{A}\left(\varsigma,\eta,\frac{\mathfrak{z}}{\urcorner(\varsigma,\eta)}\right)\star\mathfrak{A}\left(\eta,\omega,\frac{\mathfrak{x}}{\varphi(\eta,\omega)}\right)\star\mathfrak{A}\left(\omega,\epsilon,\frac{\mathfrak{d}}{\xi(\omega,\epsilon)}\right)\star\mathfrak{A}\left(\epsilon,\sigma,\frac{\mathfrak{h}}{\psi(\epsilon,\sigma)}\right)\star\mathfrak{A}\left(\sigma,\Theta,\frac{\mathfrak{v}}{\delta(\sigma,\Theta)}\right)$
$\forall\varsigma,\Theta,\eta,\omega,\epsilon,\sigma\in\mathfrak{K},\mathfrak{z},\mathfrak{x},\mathfrak{d},\mathfrak{h},\mathfrak{v}>0$ such that $\varsigma\perp\eta,\eta\perp\omega,\omega\perp\epsilon,\epsilon\perp\sigma,\sigma\perp\Theta$ and $\varsigma\perp\Theta$.
$\frac{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})}{\urcorner(\varsigma,\eta)\varphi(\eta,\omega)\xi(\omega,\epsilon)\psi(\epsilon,\sigma)\delta(\sigma,\Theta)(\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v})+\max\{\varsigma,\Theta\}}$
$\geq \frac{\mathfrak{z}}{\mathfrak{z}+\max\{\varsigma,\eta\}}\cdot\frac{\mathfrak{x}}{\mathfrak{x}+\max\{\eta,\omega\}}\cdot\frac{\mathfrak{d}}{\Theta+\max\{\omega,\epsilon\}}\cdot\frac{\mathfrak{h}}{\mathfrak{h}+\max\{\epsilon,\sigma\}}\cdot\frac{\mathfrak{v}}{\mathfrak{v}+\max\{\sigma,\Theta\}}$
Now, let $\varsigma=\mathfrak{d}=\eta=\omega=\epsilon=\sigma=-1$; then
$\urcorner(\varsigma,\eta)=\varphi(\eta,\omega)=\xi(\omega,\epsilon)=\psi(\epsilon,\sigma)=\delta(\sigma,\Theta)=1$ and
$\max\{\varsigma,\eta\}=\max\{\eta,\omega\}=\max\{\omega,\epsilon\}=\max\{\epsilon,\sigma\}=\max\{\sigma,\Theta\}=-1$. This implies that

$$\frac{\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}}{\mathfrak{z}+\mathfrak{x}+\mathfrak{d}+\mathfrak{h}+\mathfrak{v}-1}\geq\frac{\mathfrak{z}}{\mathfrak{z}-1}\cdot\frac{\mathfrak{x}}{\mathfrak{x}-1}\cdot\frac{\mathfrak{d}}{\mathfrak{d}-1}\cdot\frac{\mathfrak{h}}{\mathfrak{h}-1}\cdot\frac{\mathfrak{v}}{\mathfrak{v}-1}$$
$$=\frac{\mathfrak{z}\mathfrak{x}\mathfrak{d}\mathfrak{h}\mathfrak{v}}{(\mathfrak{z}-1)(\mathfrak{x}-1)(\mathfrak{d}-1)(\mathfrak{h}-1)(\mathfrak{v}-1)},\mathfrak{z},\mathfrak{x},\mathfrak{d},\mathfrak{h},\mathfrak{v}\neq 1. \tag{11}$$

Taking $\mathfrak{z}=\mathfrak{x}=\mathfrak{d}=\mathfrak{h}=\mathfrak{v}=2$, we get a contradiction. □

Theorem 3.4. *Let $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},\star,\diamond,\perp)$ be an orthogonal complete $\mathfrak{NOPCMS}$ and define $\mathfrak{A},\mathfrak{I},\mathfrak{D}:\mathfrak{K}\times\mathfrak{K}\to[1,+\infty)$ by such that*

$$\lim_{\alpha\to\infty}\mathfrak{A}(\varsigma,\Theta,\iota)=1,\lim_{n\to\infty}\mathfrak{I}(\varsigma,\Theta,\iota)=0,\lim_{n\to\infty}\mathfrak{D}(\varsigma,\Theta,\iota)=0 \tag{12}$$

for all $\iota>0$ and $\varsigma,\Theta\in\mathfrak{K}$.
Let $\mathfrak{Y}:\mathfrak{K}\to\mathfrak{K}$ is an $\perp$-continuous, $\perp$-contraction and $\perp$-preserving mapping satisfying

$$\mathfrak{A}(\mathfrak{Y}\varsigma,\mathfrak{Y}\Theta,\varpi\iota)\geq\mathfrak{A}(\varsigma,\Theta,\iota), \tag{13}$$

and

$$\mathfrak{I}(\mathfrak{Y}\varsigma,\mathfrak{Y}\Theta,\varpi\iota)\leq\mathfrak{I}(\varsigma,\Theta,\iota), \tag{14}$$

$$\mathfrak{D}(\mathfrak{Y}\varsigma,\mathfrak{Y}\Theta,\varpi\iota)\leq\mathfrak{D}(\varsigma,\Theta,\iota) \tag{15}$$

for all $\iota > 0$ and $\varsigma, \Theta \in \mathfrak{K}$, where $0 < p < 1$. Furthermore, if for $\varsigma_0 \in \mathfrak{K}, \varpi \in [0,1]$ and $n \in 1, 2, 3, \ldots$.

$$\begin{aligned}&\lim_{n\to\infty} \urcorner(\varsigma_n, \Theta), \lim_{n\to\infty} \urcorner(\Theta, \varsigma_n), \lim_{n\to\infty} \varphi(\varsigma_n, \Theta), \lim_{n\to\infty} \varphi(\Theta, \varsigma_n),\\ &\lim_{n\to\infty} \xi(\varsigma_n, \Theta), \lim_{n\to\infty} \xi(\Theta, \varsigma_n), \lim_{n\to\infty} \psi(\varsigma_n, \Theta), \lim_{n\to\infty} \psi(\Theta, \varsigma_n),\\ &\lim_{n\to\infty} \delta(\varsigma_n, \Theta), \lim_{n\to\infty} \delta(\Theta, \varsigma_n)\end{aligned} \tag{16}$$

are all exist and are finite. Also,that,

$$\lim_{\alpha\to\infty} \mathfrak{A}(\mathfrak{Y}^n u, u, \iota) = \mathfrak{A}(u, u, \iota), \lim_{\alpha\to\infty} \mathfrak{I}(\mathfrak{F}^n u, u, \iota) = \mathfrak{I}(u, u, \iota), \text{ and } \lim_{\alpha\to\infty} \mathfrak{D}(\mathfrak{Y}^n u, u, \iota) = \mathfrak{D}(u, u, \iota), \tag{17}$$

for all $\iota > 0$ and $\varsigma, \Theta \in \mathfrak{K}$, where $\varsigma_n = \mathfrak{Y}^n \varsigma_0$, then $\mathfrak{Y}$ has a unique fixed point in $\mathfrak{K}$.

Proof. Since $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ is an Neutrosophic orthogonal G-complete pentagonal controlled metric space, there exists $\varsigma_0 \in \mathfrak{K}$ and define a sequence $\{\varsigma_n\}$ by $\varsigma_n = \mathfrak{Y}\varsigma_{n-1}$ for all $n \in \{1, 2, 3, \ldots\}$ such that $\varsigma_0 \perp \mathfrak{d}$, for all $\mathfrak{d} \in \mathfrak{K}$.
This yields that $\varsigma_0 \perp \mathfrak{Y}\varsigma_0$. Consider
$\varsigma_1 = \mathfrak{Y}\varsigma_0, \varsigma_2 = \mathfrak{Y}^2\varsigma_0 = \mathfrak{Y}\varsigma_1, \ldots, \varsigma_n = \mathfrak{Y}^n\varsigma_0 = \zeta\varsigma_{n-1}$.
If $\varsigma_n = \varsigma_{n-1}$, then ς_n is a fixed point of ζ. Suppose that $\varsigma_n \neq \varsigma_{n-1}$ for all $n \in \mathbb{N}$. Since $\mathfrak{Y}$ is $\perp$-preserving, $\{\varsigma_n\}$ is an orthogonal sequence. Since $\mathfrak{Y}$ is an $\perp$-contraction, we have and define a sequence $\{\varsigma_n\}$ by $\varsigma_n = \mathfrak{Y}\varsigma_{n-1}$ for all $n \in \{1, 2, 3, \ldots\}$.
Without loss of generality, assume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \in \{0, 1, 2, 3, \ldots\}$. With the help of (13) and (14), we deduce

$$\mathfrak{A}(\varsigma_n, \varsigma_{n+1}, \iota) = \mathfrak{A}(\mathfrak{Y}\varsigma_{n-1}, \mathfrak{Y}\varsigma_n, \iota) \geq \mathfrak{A}\left(\varsigma_{n-1}, \varsigma_n, \frac{\iota}{p}\right) \cdots \geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{p^n}\right). \tag{18}$$

Keeping on the same lines, we obtain

$$\mathfrak{A}(\varsigma_n, \varsigma_{n+2}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_2, \frac{\iota}{p^n}\right), \mathfrak{A}(\varsigma_n, \varsigma_{n+3}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_3, \frac{\iota}{p^n}\right), \mathfrak{A}(\varsigma_n, \varsigma_{n+4}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_4, \frac{\iota}{p^n}\right), \tag{19}$$

It symbolizes, if $m = 1, 2, 3, \ldots$,

$$\mathfrak{A}(\varsigma_n, \varsigma_{n+4m+1}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_{4m+1}, \frac{\iota}{p^n}\right), \tag{20}$$

$$\mathfrak{A}(\varsigma_n, \varsigma_{n+4m+2}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_{4m+2}, \frac{\iota}{p^n}\right), \tag{21}$$

$$\mathfrak{A}(\varsigma_n, \varsigma_{n+4m+3}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_{4m+3}, \frac{\iota}{p^n}\right), \tag{22}$$

$$\mathfrak{A}(\varsigma_n, \varsigma_{n+4m+4}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_{4m+4}, \frac{\iota}{p^n}\right), \tag{23}$$

and

$$\mathfrak{I}(\varsigma_n, \varsigma_{n+1}, \iota) = \mathfrak{I}(\mathfrak{Y}\varsigma_{n-1}, \mathfrak{Y}\varsigma_n, \iota) \leq \mathfrak{I}\left(\varsigma_{n-1}, \varsigma_n, \frac{\iota}{p}\right) \cdots \leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{p^n}\right). \tag{24}$$

Keeping on the same lines, we obtain

$$\mathfrak{I}(\varsigma_n, \varsigma_{n+2}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_2, \frac{\iota}{p^n}\right), \mathfrak{I}(\varsigma_n, \varsigma_{n+3}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_3, \frac{\iota}{p^n}\right), \mathfrak{I}(\varsigma_n, \varsigma_{n+4}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_4, \frac{\iota}{p^n}\right), \tag{25}$$

It implies, if $m = 1, 2, 3, \ldots$,

$$\mathfrak{I}(\varsigma_n, \varsigma_{n+4m+1}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_{4m+1}, \frac{\iota}{p^n}\right), \tag{26}$$

$$\mathfrak{I}(\varsigma_n, \varsigma_{n+4m+2}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_{4m+2}, \frac{\iota}{p^n}\right), \tag{27}$$

$$\mathfrak{I}(\varsigma_n, \varsigma_{n+4m+3}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_{4m+3}, \frac{\iota}{p^n}\right), \tag{28}$$

$$\mathfrak{I}(\varsigma_n, \varsigma_{n+4m+4}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_{4m+4}, \frac{\iota}{p^n}\right), \tag{29}$$

Similarly,

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+1}, \iota) = \mathfrak{D}(\mathfrak{Y}\varsigma_{n-1}, \mathfrak{Y}\varsigma_n, \iota) \leq \mathfrak{D}\left(\varsigma_{n-1}, \varsigma_n, \frac{\iota}{p}\right) \cdots \leq \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{p^n}\right). \tag{30}$$

Keeping on the same lines, we obtain

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+2}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_2, \frac{\iota}{p^n}\right), \tag{31}$$

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+3}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_3, \frac{\iota}{p^n}\right), \tag{32}$$

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+4}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_4, \frac{\iota}{p^n}\right), \tag{33}$$

It implies, if $m = 1, 2, 3, \ldots,$

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+4m+1}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_{4m+1}, \frac{\iota}{p^n}\right), \tag{34}$$

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+4m+2}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_{4m+2}, \frac{\iota}{p^n}\right), \tag{35}$$

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+4m+3}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_{4m+3}, \frac{\iota}{p^n}\right), \tag{36}$$

$$\mathfrak{D}(\varsigma_n, \varsigma_{n+4m+4}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_{4m+4}, \frac{\iota}{p^n}\right), \tag{37}$$

Expressing $\iota = \frac{\iota}{5} + \frac{\iota}{5} + \frac{\iota}{5} + \frac{\iota}{5} + \frac{\iota}{5}$ and by using (18), (23),(24), (29), (30) and (37), we obtain

$$\begin{aligned}
\mathfrak{A}(\varsigma_0, \varsigma_5, \iota) &\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^4\delta(\varsigma_4, \varsigma_5)}\right), \\
\mathfrak{I}(\varsigma_0, \varsigma_5, \iota) &\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^4\delta(\varsigma_4, \varsigma_5)}\right) \text{ and} \\
\mathfrak{D}(\varsigma_0, \varsigma_5, \iota) &\leq \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^4\delta(\varsigma_4, \varsigma_5)}\right),
\end{aligned}$$

In similar manner, we can deduce

$$\begin{aligned}
\mathfrak{A}(\varsigma_0, \varsigma_9, \iota) &\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^4\delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_4, \varsigma_5)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^5\delta(\varsigma_4, \varsigma_9)\varphi(\varsigma_5, \varsigma_6)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^6\delta(\varsigma_4, \varsigma_9)\xi(\varsigma_6, \varsigma_7)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^7\delta(\varsigma_4, \varsigma_9)\psi(\varsigma_7, \varsigma_8)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^8\delta(\varsigma_4, \varsigma_9)\delta(\varsigma_8, \varsigma_9)}\right), \\
\mathfrak{I}(\varsigma_0, \varsigma_9, \iota) &\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right)
\end{aligned}$$

$$\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^4 \delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_4, \varsigma_5)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^5 \delta(\varsigma_4, \varsigma_9)\varphi(\varsigma_5, \varsigma_6)}\right)$$
$$\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^6 \delta(\varsigma_4, \varsigma_9)\xi(\varsigma_6, \varsigma_7)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^7 \delta(\varsigma_4, \varsigma_9)\psi(\varsigma_7, \varsigma_8)}\right)$$
$$\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^8 \delta(\varsigma_4, \varsigma_9)\delta(\varsigma_8, \varsigma_9)}\right) \text{ and}$$
$$\mathfrak{D}(\varsigma_0, \varsigma_9, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^4 \delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_4, \varsigma_5)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^5 \delta(\varsigma_4, \varsigma_9)\varphi(\varsigma_5, \varsigma_6)}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^6 \delta(\varsigma_4, \varsigma_9)\xi(\varsigma_6, \varsigma_7)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^7 \delta(\varsigma_4, \varsigma_9)\psi(\varsigma_7, \varsigma_8)}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^8 \delta(\varsigma_4, \varsigma_9)\delta(\varsigma_8, \varsigma_9)}\right),$$

We obtain for each m=1,2,3,. . . ,

$$\mathfrak{A}(\varsigma_0, \varsigma_{4m+1}, \iota) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_1, \varsigma_2, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right)$$
$$* \mathfrak{A}\left(\varsigma_2, \varsigma_3, \frac{\iota}{5\xi(\varsigma_2, \varsigma_3)}\right) * \mathfrak{A}\left(\varsigma_3, \varsigma_4, \frac{\iota}{5\psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_4, \varsigma_{4m+1}, \frac{\iota}{5\delta(\varsigma_4, \varsigma_{4m+1})}\right)$$
$$\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^4 \delta(\varsigma_4, \varsigma_{4m+1})\daleth(\varsigma_4, \varsigma_5)}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^5 \delta(\varsigma_4, \varsigma_{4m+1})\varphi(\varsigma_5, \varsigma_6)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^6 \delta(\varsigma_4, \varsigma_{4m+1})\xi(\varsigma_6, \varsigma_7)}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^7 \delta(\varsigma_4, \varsigma_{4m+1})\psi(\varsigma_7, \varsigma_8)}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^8 \delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\daleth(\varsigma_8, \varsigma_9)}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^9 \delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\varphi(\varsigma_9, \varsigma_{10})}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^1 0\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\xi(\varsigma_{10}, \varsigma_{11})}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^1 1\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\psi(\varsigma_{11}, \varsigma_{12})}\right)$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^4 p^1 2\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\daleth(\varsigma_{12}, \varsigma_{13})}\right) * \ldots$$
$$* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^m p^4 m\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\ldots\delta(\varsigma_{4m-4}, \varsigma_{4m+1})\delta(\varsigma_4 m, \varsigma_{4m+1})}\right),$$

and

$$\mathfrak{I}(\varsigma_0, \varsigma_{4m+1}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_1, \varsigma_2, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right)$$
$$\diamond \mathfrak{I}\left(\varsigma_2, \varsigma_3, \frac{\iota}{5\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{I}\left(\varsigma_3, \varsigma_4, \frac{\iota}{5\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_4, \varsigma_{4m+1}, \frac{\iota}{5\delta(\varsigma_4, \varsigma_{4m+1})}\right)$$
$$\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right)$$
$$\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^4 \delta(\varsigma_4, \varsigma_{4m+1})\daleth(\varsigma_4, \varsigma_5)}\right)$$

$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^5\delta(\varsigma_4,\varsigma_{4m+1})\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^6\delta(\varsigma_4,\varsigma_{4m+1})\xi(\varsigma_6,\varsigma_7)}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^7\delta(\varsigma_4,\varsigma_{4m+1})\psi(\varsigma_7,\varsigma_8)}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^8\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\daleth(\varsigma_8,\varsigma_9)}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^9\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\varphi(\varsigma_9,\varsigma_{10})}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^10\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\xi(\varsigma_{10},\varsigma_{11})}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^11\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\psi(\varsigma_{11},\varsigma_{12})}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^12\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\delta(\varsigma_{12},\varsigma_{4m+1})\daleth(\varsigma_{12},\varsigma_{13})}\right)\diamond\dots$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^mp^4m\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\delta(\varsigma_{12},\varsigma_{4m+1})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+1})\delta(\varsigma_4m,\varsigma_{4m+1})}\right),$$

Likewise

$$\mathfrak{D}(\varsigma_0,\varsigma_{4m+1},\iota)\le\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\daleth(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5\varphi(\varsigma_1,\varsigma_2)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5\xi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{D}\left(\varsigma_3,\varsigma_4,\frac{\iota}{5\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{D}\left(\varsigma_4,\varsigma_{4m+1},\frac{\iota}{5\delta(\varsigma_4,\varsigma_{4m+1})}\right)$$
$$\le\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\daleth(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p\varphi(\varsigma_1,\varsigma_2)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^2\xi(\varsigma_2,\varsigma_3)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^3\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^4\delta(\varsigma_4,\varsigma_{4m+1})\daleth(\varsigma_4,\varsigma_5)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^5\delta(\varsigma_4,\varsigma_{4m+1})\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^6\delta(\varsigma_4,\varsigma_{4m+1})\xi(\varsigma_6,\varsigma_7)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^7\delta(\varsigma_4,\varsigma_{4m+1})\psi(\varsigma_7,\varsigma_8)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^8\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\daleth(\varsigma_8,\varsigma_9)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^9\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\varphi(\varsigma_9,\varsigma_{10})}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^10\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\xi(\varsigma_{10},\varsigma_{11})}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^11\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\psi(\varsigma_{11},\varsigma_{12})}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^12\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\delta(\varsigma_{12},\varsigma_{4m+1})\daleth(\varsigma_{12},\varsigma_{13})}\right)\diamond\dots$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^mp^4m\delta(\varsigma_4,\varsigma_{4m+1})\delta(\varsigma_8,\varsigma_{4m+1})\delta(\varsigma_{12},\varsigma_{4m+1})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+1})\delta(\varsigma_4m,\varsigma_{4m+1})}\right),$$

Now, using (20), (26) and (34), we deduce that

$$\mathfrak{A}(\varsigma_n,\varsigma_{n+4m+1},\iota)\ge\mathfrak{A}\left(\varsigma_0,\varsigma_{4m+1},\frac{\iota}{p^n}\right)$$
$$\ge\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^n\daleth(\varsigma_0,\varsigma_1)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+1}\varphi(\varsigma_1,\varsigma_2)}\right)$$
$$*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+2}\xi(\varsigma_2,\varsigma_3)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+3}\psi(\varsigma_3,\varsigma_4)}\right)$$

$$
\begin{aligned}
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+4}\delta(\varsigma_4, \varsigma_{4m+1})\daleth(\varsigma_4, \varsigma_5)}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+5}\delta(\varsigma_4, \varsigma_{4m+1})\varphi(\varsigma_5, \varsigma_6)}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+6}\delta(\varsigma_4, \varsigma_{4m+1})\xi(\varsigma_6, \varsigma_7)}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+7}\delta(\varsigma_4, \varsigma_{4m+1})\psi(\varsigma_7, \varsigma_8)}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+8}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\daleth(\varsigma_8, \varsigma_9)}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+9}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\varphi(\varsigma_9, \varsigma_{10})}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\xi(\varsigma_{10}, \varsigma_{11})}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\psi(\varsigma_{11}, \varsigma_{12})}\right)\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+12}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\daleth(\varsigma_{12}, \varsigma_{13})}\right) * \dots\\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^m p^{n+4m}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\dots\delta(\varsigma_{4m-4}, \varsigma_{4m+1})\delta(\varsigma_4 m, \varsigma_{4m+1})}\right)
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
&\mathfrak{I}(\varsigma_n, \varsigma_{n+4m+1}, \iota) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_{4m+1}, \frac{\iota}{p^n}\right)\\
&\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^n \daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+1}\varphi(\varsigma_1, \varsigma_2)}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+2}\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+3}\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+4}\delta(\varsigma_4, \varsigma_{4m+1})\daleth(\varsigma_4, \varsigma_5)}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+5}\delta(\varsigma_4, \varsigma_{4m+1})\varphi(\varsigma_5, \varsigma_6)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+6}\delta(\varsigma_4, \varsigma_{4m+1})\xi(\varsigma_6, \varsigma_7)}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+7}\delta(\varsigma_4, \varsigma_{4m+1})\psi(\varsigma_7, \varsigma_8)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+8}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\daleth(\varsigma_8, \varsigma_9)}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+9}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\varphi(\varsigma_9, \varsigma_{10})}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\xi(\varsigma_{10}, \varsigma_{11})}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\psi(\varsigma_{11}, \varsigma_{12})}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+12}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\daleth(\varsigma_{12}, \varsigma_{13})}\right) \diamond \dots\\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^m p^{n+4m}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\dots\delta(\varsigma_{4m-4}, \varsigma_{4m+1})\delta(\varsigma_4 m, \varsigma_{4m+1})}\right)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&\mathfrak{D}(\varsigma_n, \varsigma_{n+4m+1}, \iota) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_{4m+1}, \frac{\iota}{p^n}\right)\\
&\leq \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^n \daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+1}\varphi(\varsigma_1, \varsigma_2)}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+2}\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+3}\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+4}\delta(\varsigma_4, \varsigma_{4m+1})\daleth(\varsigma_4, \varsigma_5)}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+5}\delta(\varsigma_4, \varsigma_{4m+1})\varphi(\varsigma_5, \varsigma_6)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+6}\delta(\varsigma_4, \varsigma_{4m+1})\xi(\varsigma_6, \varsigma_7)}\right)
\end{aligned}
$$

$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+7}\delta(\varsigma_4, \varsigma_{4m+1})\psi(\varsigma_7, \varsigma_8)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+8}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\daleth(\varsigma_8, \varsigma_9)}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+9}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\varphi(\varsigma_9, \varsigma_{10})}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\xi(\varsigma_{10}, \varsigma_{11})}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\psi(\varsigma_{11}, \varsigma_{12})}\right)$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+12}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\daleth(\varsigma_{12}, \varsigma_{13})}\right) \diamond \ldots$$
$$\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^m p^{n+4m}\delta(\varsigma_4, \varsigma_{4m+1})\delta(\varsigma_8, \varsigma_{4m+1})\delta(\varsigma_{12}, \varsigma_{4m+1})\ldots\delta(\varsigma_{4m-4}, \varsigma_{4m+1})\delta(\varsigma_4 m, \varsigma_{4m+1})}\right) \quad (40)$$

Furthermore, from (18), (19), (24), (25), (31) and (32), we can obtain

$$\begin{aligned}
\mathfrak{A}(\varsigma_0, \varsigma_6, \iota) &\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^4\delta(\varsigma_4, \varsigma_6)}\right), \\
\mathfrak{I}(\varsigma_0, \varsigma_6, \iota) &\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^4\delta(\varsigma_4, \varsigma_6)}\right) \text{ and} \\
\mathfrak{D}(\varsigma_0, \varsigma_6, \iota) &\leq \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^4\delta(\varsigma_4, \varsigma_6)}\right) \quad (41)
\end{aligned}$$

In similar manner, we can deduce

$$\begin{aligned}
\mathfrak{A}(\varsigma_0, \varsigma_{10}, \iota) &\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_1, \varsigma_2, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right) \\
&* \mathfrak{A}\left(\varsigma_2, \varsigma_3, \frac{\iota}{5\xi(\varsigma_2, \varsigma_3)}\right) * \mathfrak{A}\left(\varsigma_3, \varsigma_4, \frac{\iota}{5\psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_4, \varsigma_{10}, \frac{\iota}{5\delta(\varsigma_4, \varsigma_{10})}\right) \\
&\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^4\delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_4, \varsigma_5)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^5\delta(\varsigma_4, \varsigma_9)\varphi(\varsigma_5, \varsigma_6)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^6\delta(\varsigma_4, \varsigma_9)\xi(\varsigma_6, \varsigma_7)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^7\delta(\varsigma_4, \varsigma_9)\psi(\varsigma_7, \varsigma_8)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_2, \frac{\iota}{(5)^2p^8\delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_8, \varsigma_{10})}\right), \\
\mathfrak{I}(\varsigma_0, \varsigma_{10}, \iota) &\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_1, \varsigma_2, \frac{\iota}{5\varphi(\varsigma_1, \varsigma_2)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_2, \varsigma_3, \frac{\iota}{5\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{I}\left(\varsigma_3, \varsigma_4, \frac{\iota}{5\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_4, \varsigma_{10}, \frac{\iota}{5\delta(\varsigma_4, \varsigma_{10})}\right) \\
&\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5\daleth(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p\varphi(\varsigma_1, \varsigma_2)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^2\xi(\varsigma_2, \varsigma_3)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^3\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^4\delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_4, \varsigma_5)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^5\delta(\varsigma_4, \varsigma_9)\varphi(\varsigma_5, \varsigma_6)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^6\delta(\varsigma_4, \varsigma_9)\xi(\varsigma_6, \varsigma_7)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2p^7\delta(\varsigma_4, \varsigma_9)\psi(\varsigma_7, \varsigma_8)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_2, \frac{\iota}{(5)^2p^8\delta(\varsigma_4, \varsigma_9)\daleth(\varsigma_8, \varsigma_{10})}\right),
\end{aligned}$$

$$\begin{aligned}\mathfrak{D}(\varsigma_0,\varsigma_{10},\iota)\leq \mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5\varphi(\varsigma_1,\varsigma_2)}\right)\\ \diamond\mathfrak{D}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5\xi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{D}\left(\varsigma_3,\varsigma_4,\frac{\iota}{5\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{D}\left(\varsigma_4,\varsigma_{10},\frac{\iota}{5\delta(\varsigma_4,\varsigma_{10})}\right)\\ \leq \mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p\varphi(\varsigma_1,\varsigma_2)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^2\xi(\varsigma_2,\varsigma_3)}\right)\\ \diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^3\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^4\delta(\varsigma_4,\varsigma_9)\urcorner(\varsigma_4,\varsigma_5)}\right)\\ \diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^5\delta(\varsigma_4,\varsigma_9)\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^6\delta(\varsigma_4,\varsigma_9)\xi(\varsigma_6,\varsigma_7)}\right)\\ \diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^7\delta(\varsigma_4,\varsigma_9)\psi(\varsigma_7,\varsigma_8)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_2,\frac{\iota}{(5)^2p^8\delta(\varsigma_4,\varsigma_9)\urcorner(\varsigma_8,\varsigma_{10})}\right)\end{aligned}\tag{42}$$

We obtain for each $m = 1, 2, 3, \ldots,$

$$\begin{aligned}\mathfrak{A}(\varsigma_0,\varsigma_{4m+2},\iota)\geq \mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)*\mathfrak{A}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5\varphi(\varsigma_1,\varsigma_2)}\right)\\ *\mathfrak{A}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5\xi(\varsigma_2,\varsigma_3)}\right)*\mathfrak{A}\left(\varsigma_3,\varsigma_4,\frac{\iota}{5\psi(\varsigma_3,\varsigma_4)}\right)*\mathfrak{A}\left(\varsigma_4,\varsigma_{4m+2},\frac{\iota}{5\delta(\varsigma_4,\varsigma_{4m+2})}\right)\\ \geq \mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p\varphi(\varsigma_1,\varsigma_2)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^2\xi(\varsigma_2,\varsigma_3)}\right)\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^3\psi(\varsigma_3,\varsigma_4))}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^4\delta(\varsigma_4,\varsigma_{4m+2})\urcorner(\varsigma_4,\varsigma_5)}\right)\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^5\delta(\varsigma_4,\varsigma_{4m+2})\varphi(\varsigma_5,\varsigma_6)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^6\delta(\varsigma_4,\varsigma_{4m+2})\xi(\varsigma_6,\varsigma_7)}\right)\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^7\delta(\varsigma_4,\varsigma_{4m+2})\psi(\varsigma_7,\varsigma_8)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^8\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_8,\varsigma_9)}\right)\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^10\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\xi(\varsigma_{10},\varsigma_{11})}\right)\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^11\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\psi(\varsigma_{11},\varsigma_{12})}\right)\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^12\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_{12},\varsigma_{13})}\right)*\ldots\\ *\mathfrak{A}\left(\varsigma_0,\varsigma_2,\frac{\iota}{(5)^mp^4m\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\ldots\delta(\varsigma_{4m-4},\varsigma_{4m+2})\delta(\varsigma_4m,\varsigma_{4m+2})}\right)\end{aligned}$$

$$\begin{aligned}\mathfrak{I}(\varsigma_0,\varsigma_{4m+2},\iota)\leq \mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{I}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5\varphi(\varsigma_1,\varsigma_2)}\right)\\ \diamond\mathfrak{I}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5\xi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{I}\left(\varsigma_3,\varsigma_4,\frac{\iota}{5\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{I}\left(\varsigma_4,\varsigma_{4m+2},\frac{\iota}{5\delta(\varsigma_4,\varsigma_{4m+2})}\right)\\ \leq \mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p\varphi(\varsigma_1,\varsigma_2)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^2\xi(\varsigma_2,\varsigma_3)}\right)\\ \diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^3\psi(\varsigma_3,\varsigma_4))}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^4\delta(\varsigma_4,\varsigma_{4m+2})\urcorner(\varsigma_4,\varsigma_5)}\right)\\ \diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^5\delta(\varsigma_4,\varsigma_{4m+2})\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^6\delta(\varsigma_4,\varsigma_{4m+2})\xi(\varsigma_6,\varsigma_7)}\right)\\ \diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^7\delta(\varsigma_4,\varsigma_{4m+2})\psi(\varsigma_7,\varsigma_8)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^8\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_8,\varsigma_9)}\right)\\ \diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^10\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\xi(\varsigma_{10},\varsigma_{11})}\right)\\ \diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^11\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\psi(\varsigma_{11},\varsigma_{12})}\right)\end{aligned}$$

$$
\begin{aligned}
&\diamond \mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^12\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_{12},\varsigma_{13})}\right)\diamond\dots\\
&\diamond \mathfrak{I}\left(\varsigma_0,\varsigma_2,\frac{\iota}{(5)^mp^4m\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+2})\delta(\varsigma_4 m,\varsigma_{4m+2})}\right)\\
&\mathfrak{D}(\varsigma_0,\varsigma_{4m+2},\iota)\le \mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5\varphi(\varsigma_1,\varsigma_2)}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5\xi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{D}\left(\varsigma_3,\varsigma_4,\frac{\iota}{5\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{D}\left(\varsigma_4,\varsigma_{4m+2},\frac{\iota}{5\delta(\varsigma_4,\varsigma_{4m+2})}\right)\\
&\le \mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p\varphi(\varsigma_1,\varsigma_2)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^2\xi(\varsigma_2,\varsigma_3)}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^3\psi(\varsigma_3,\varsigma_4))}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^4\delta(\varsigma_4,\varsigma_{4m+2})\urcorner(\varsigma_4,\varsigma_5)}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^5\delta(\varsigma_4,\varsigma_{4m+2})\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^6\delta(\varsigma_4,\varsigma_{4m+2})\xi(\varsigma_6,\varsigma_7)}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^7\delta(\varsigma_4,\varsigma_{4m+2})\psi(\varsigma_7,\varsigma_8)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^8\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_8,\varsigma_9)}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^10\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\xi(\varsigma_{10},\varsigma_{11})}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^11\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\psi(\varsigma_{11},\varsigma_{12})}\right)\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^12\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_{12},\varsigma_{13})}\right)\diamond\dots\\
&\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_2,\frac{\iota}{(5)^mp^4m\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+2})\delta(\varsigma_4 m,\varsigma_{4m+2})}\right)
\end{aligned}\tag{43}
$$

Now, using (21), (27), and (35), we deduce that

$$
\begin{aligned}
&\mathfrak{A}(\varsigma_n,\varsigma_{n+4m+2},\iota)\ge\mathfrak{A}\left(\varsigma_0,\varsigma_{4m+2},\frac{\iota}{p^n}\right)\\
&\ge\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^n\urcorner(\varsigma_0,\varsigma_1)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+1}\varphi(\varsigma_1,\varsigma_2)}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+2}\xi(\varsigma_2,\varsigma_3)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+3}\psi(\varsigma_3,\varsigma_4)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+4}\delta(\varsigma_4,\varsigma_{4m+2})\urcorner(\varsigma_4,\varsigma_5)}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+5}\delta(\varsigma_4,\varsigma_{4m+2})\varphi(\varsigma_5,\varsigma_6)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+6}\delta(\varsigma_4,\varsigma_{4m+2})\xi(\varsigma_6,\varsigma_7)}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+7}\delta(\varsigma_4,\varsigma_{4m+2})\psi(\varsigma_7,\varsigma_8)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+8}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\urcorner(\varsigma_8,\varsigma_9)}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+9}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\varphi(\varsigma_9,\varsigma_{10})}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+10}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\xi(\varsigma_{10},\varsigma_{11})}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+11}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\psi(\varsigma_{11},\varsigma_{12})}\right)\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^{n+12}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\urcorner(\varsigma_{12},\varsigma_{13})}\right)*\dots\\
&*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^mp^{n+4m}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+2})\delta(\varsigma_4 m,\varsigma_{4m+2})}\right)\\
&\mathfrak{I}(\varsigma_n,\varsigma_{n+4m+2},\iota)\le\mathfrak{I}\left(\varsigma_0,\varsigma_{4m+2},\frac{\iota}{p^n}\right)\\
&\le\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^n\urcorner(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+1}\varphi(\varsigma_1,\varsigma_2)}\right)
\end{aligned}
$$

$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+2}\xi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+3}\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+4}\delta(\varsigma_4,\varsigma_{4m+2})\daleth(\varsigma_4,\varsigma_5)}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+5}\delta(\varsigma_4,\varsigma_{4m+2})\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+6}\delta(\varsigma_4,\varsigma_{4m+2})\xi(\varsigma_6,\varsigma_7)}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+7}\delta(\varsigma_4,\varsigma_{4m+2})\psi(\varsigma_7,\varsigma_8)}\right)\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+8}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\daleth(\varsigma_8,\varsigma_9)}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+9}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\varphi(\varsigma_9,\varsigma_{10})}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+10}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\xi(\varsigma_{10},\varsigma_{11})}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+11}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\psi(\varsigma_{11},\varsigma_{12})}\right)$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^{n+12}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\daleth(\varsigma_{12},\varsigma_{13})}\right)\diamond\dots$$
$$\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^mp^{n+4m}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+2})\delta(\varsigma_4m,\varsigma_{4m+2})}\right)$$

$$\mathfrak{D}\left(\varsigma_n,\varsigma_{n+4m+2},\iota\right)\leq\mathfrak{D}\left(\varsigma_0,\varsigma_{4m+2},\frac{\iota}{p^n}\right)$$
$$\leq\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n}\daleth(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+1}\varphi(\varsigma_1,\varsigma_2)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+2}\xi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+3}\psi(\varsigma_3,\varsigma_4)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+4}\delta(\varsigma_4,\varsigma_{4m+2})\daleth(\varsigma_4,\varsigma_5)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+5}\delta(\varsigma_4,\varsigma_{4m+2})\varphi(\varsigma_5,\varsigma_6)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+6}\delta(\varsigma_4,\varsigma_{4m+2})\xi(\varsigma_6,\varsigma_7)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+7}\delta(\varsigma_4,\varsigma_{4m+2})\psi(\varsigma_7,\varsigma_8)}\right)\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+8}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\daleth(\varsigma_8,\varsigma_9)}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+9}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\varphi(\varsigma_9,\varsigma_{10})}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+10}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\xi(\varsigma_{10},\varsigma_{11})}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+11}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\psi(\varsigma_{11},\varsigma_{12})}\right)$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^4p^{n+12}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\daleth(\varsigma_{12},\varsigma_{13})}\right)\diamond\dots$$
$$\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^mp^{n+4m}\delta(\varsigma_4,\varsigma_{4m+2})\delta(\varsigma_8,\varsigma_{4m+2})\delta(\varsigma_{12},\varsigma_{4m+2})\dots\delta(\varsigma_{4m-4},\varsigma_{4m+2})\delta(\varsigma_4m,\varsigma_{4m+2})}\right)\quad(44)$$

Accordingly, we get

$$\mathfrak{A}\left(\varsigma_n,\varsigma_{n+4m+3},\iota\right)\geq\mathfrak{A}\left(\varsigma_0,\varsigma_{4m+3},\frac{\iota}{p^n}\right)$$
$$\geq\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n}\daleth(\varsigma_0,\varsigma_1)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5p^{n+1}\varphi(\varsigma_1,\varsigma_2)}\right)$$
$$*\,\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+2}\xi(\varsigma_2,\varsigma_3)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5p^{n+3}\psi(\varsigma_3,\varsigma_4)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5)^2p^{n+4}\delta(\varsigma_4,\varsigma_{4m+3})\daleth(\varsigma_4,\varsigma_5)}\right)$$
$$*\,\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+5}\delta(\varsigma_4,\varsigma_{4m+3})\varphi(\varsigma_5,\varsigma_6)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+6}\delta(\varsigma_4,\varsigma_{4m+3})\xi(\varsigma_6,\varsigma_7)}\right)$$
$$*\,\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^2p^{n+7}\delta(\varsigma_4,\varsigma_{4m+3})\psi(\varsigma_7,\varsigma_8)}\right)*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+8}\delta(\varsigma_4,\varsigma_{4m+3})\delta(\varsigma_8,\varsigma_{4m+3})\daleth(\varsigma_8,\varsigma_9)}\right)$$
$$*\,\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{(5)^3p^{n+9}\delta(\varsigma_4,\varsigma_{4m+3})\delta(\varsigma_8,\varsigma_{4m+3})\varphi(\varsigma_9,\varsigma_{10})}\right)$$

$$
\begin{aligned}
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10} \delta(\varsigma_4, \varsigma_{4m+3}) \delta(\varsigma_8, \varsigma_{4m+3}) \xi(\varsigma_{10}, \varsigma_{11})}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11} \delta(\varsigma_4, \varsigma_{4m+3}) \delta(\varsigma_8, \varsigma_{4m+3}) \psi(\varsigma_{11}, \varsigma_{12})}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^4 p^{n+12} \delta(\varsigma_4, \varsigma_{4m+3}) \delta(\varsigma_8, \varsigma_{4m+3}) \delta(\varsigma_{12}, \varsigma_{4m+3}) \urcorner(\varsigma_{12}, \varsigma_{13})}\right) * \ldots \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_3, \frac{\iota}{(5)^m p^{n+4m} \delta(\varsigma_4, \varsigma_{4m+3}) \delta(\varsigma_8, \varsigma_{4m+3}) \delta(\varsigma_{12}, \varsigma_{4m+3}) \ldots \delta(\varsigma_{4m-4}, \varsigma_{4m+3}) \delta(\varsigma_4 m, \varsigma_{4m+3})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{A}\left(\varsigma_n, \varsigma_{n+4m+4}, \iota\right) \geq \mathfrak{A}\left(\varsigma_0, \varsigma_{4m+4}, \frac{\iota}{p^n}\right) \\
&\geq \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^n \urcorner(\varsigma_0, \varsigma_1)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+1} \varphi(\varsigma_1, \varsigma_2)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+2} \xi(\varsigma_2, \varsigma_3)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+3} \psi(\varsigma_3, \varsigma_4)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+4} \delta(\varsigma_4, \varsigma_{4m+4}) \urcorner(\varsigma_4, \varsigma_5)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+5} \delta(\varsigma_4, \varsigma_{4m+4}) \varphi(\varsigma_5, \varsigma_6)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+6} \delta(\varsigma_4, \varsigma_{4m+4}) \xi(\varsigma_6, \varsigma_7)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+7} \delta(\varsigma_4, \varsigma_{4m+4}) \psi(\varsigma_7, \varsigma_8)}\right) * \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+8} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \urcorner(\varsigma_8, \varsigma_9)}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+9} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \varphi(\varsigma_9, \varsigma_{10})}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \xi(\varsigma_{10}, \varsigma_{11})}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \psi(\varsigma_{11}, \varsigma_{12})}\right) \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^4 p^{n+12} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \delta(\varsigma_{12}, \varsigma_{4m+4}) \urcorner(\varsigma_{12}, \varsigma_{13})}\right) * \ldots \\
&* \mathfrak{A}\left(\varsigma_0, \varsigma_4, \frac{\iota}{(5)^m p^{n+4m} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \delta(\varsigma_{12}, \varsigma_{4m+4}) \ldots \delta(\varsigma_{4m-4}, \varsigma_{4m+4}) \delta(\varsigma_4 m, \varsigma_{4m+4})}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{I}\left(\varsigma_n, \varsigma_{n+4m+4}, \iota\right) \leq \mathfrak{I}\left(\varsigma_0, \varsigma_{4m+4}, \frac{\iota}{p^n}\right) \\
&\leq \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^n \urcorner(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+1} \varphi(\varsigma_1, \varsigma_2)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+2} \xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+3} \psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+4} \delta(\varsigma_4, \varsigma_{4m+4}) \urcorner(\varsigma_4, \varsigma_5)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+5} \delta(\varsigma_4, \varsigma_{4m+4}) \varphi(\varsigma_5, \varsigma_6)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+6} \delta(\varsigma_4, \varsigma_{4m+4}) \xi(\varsigma_6, \varsigma_7)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+7} \delta(\varsigma_4, \varsigma_{4m+4}) \psi(\varsigma_7, \varsigma_8)}\right) \diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+8} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \urcorner(\varsigma_8, \varsigma_9)}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+9} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \varphi(\varsigma_9, \varsigma_{10})}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \xi(\varsigma_{10}, \varsigma_{11})}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \psi(\varsigma_{11}, \varsigma_{12})}\right) \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^4 p^{n+12} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \delta(\varsigma_{12}, \varsigma_{4m+4}) \urcorner(\varsigma_{12}, \varsigma_{13})}\right) \diamond \ldots \\
&\diamond \mathfrak{I}\left(\varsigma_0, \varsigma_4, \frac{\iota}{(5)^m p^{n+4m} \delta(\varsigma_4, \varsigma_{4m+4}) \delta(\varsigma_8, \varsigma_{4m+4}) \delta(\varsigma_{12}, \varsigma_{4m+4}) \ldots \delta(\varsigma_{4m-4}, \varsigma_{4m+4}) \delta(\varsigma_4 m, \varsigma_{4m+4})}\right),
\end{aligned}
$$

and

$$\begin{aligned}
&\mathfrak{D}\left(\varsigma_n, \varsigma_{n+4m+4}, \iota\right) \leq \mathfrak{D}\left(\varsigma_0, \varsigma_{4m+4}, \frac{\iota}{p^n}\right)\\
&\leq \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^n \urcorner(\varsigma_0, \varsigma_1)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+1}\varphi(\varsigma_1, \varsigma_2)}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+2}\xi(\varsigma_2, \varsigma_3)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{5p^{n+3}\psi(\varsigma_3, \varsigma_4)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+4}\delta(\varsigma_4, \varsigma_{4m+4})\urcorner(\varsigma_4, \varsigma_5)}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+5}\delta(\varsigma_4, \varsigma_{4m+4})\varphi(\varsigma_5, \varsigma_6)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+6}\delta(\varsigma_4, \varsigma_{4m+4})\xi(\varsigma_6, \varsigma_7)}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^2 p^{n+7}\delta(\varsigma_4, \varsigma_{4m+4})\psi(\varsigma_7, \varsigma_8)}\right) \diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+8}\delta(\varsigma_4, \varsigma_{4m+4})\delta(\varsigma_8, \varsigma_{4m+4})\urcorner(\varsigma_8, \varsigma_9)}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+9}\delta(\varsigma_4, \varsigma_{4m+4})\delta(\varsigma_8, \varsigma_{4m+4})\varphi(\varsigma_9, \varsigma_{10})}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+10}\delta(\varsigma_4, \varsigma_{4m+4})\delta(\varsigma_8, \varsigma_{4m+4})\xi(\varsigma_{10}, \varsigma_{11})}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^3 p^{n+11}\delta(\varsigma_4, \varsigma_{4m+4})\delta(\varsigma_8, \varsigma_{4m+4})\psi(\varsigma_{11}, \varsigma_{12})}\right)\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_1, \frac{\iota}{(5)^4 p^{n+12}\delta(\varsigma_4, \varsigma_{4m+4})\delta(\varsigma_8, \varsigma_{4m+4})\delta(\varsigma_{12}, \varsigma_{4m+4})\urcorner(\varsigma_{12}, \varsigma_{13})}\right) \diamond \dots\\
&\diamond \mathfrak{D}\left(\varsigma_0, \varsigma_4, \frac{\iota}{(5)^m p^{n+4m}\delta(\varsigma_4, \varsigma_{4m+4})\delta(\varsigma_8, \varsigma_{4m+4})\delta(\varsigma_{12}, \varsigma_{4m+4})\dots\delta(\varsigma_{4m-4}, \varsigma_{4m+4})\delta(\varsigma_4 m, \varsigma_{4m+4})}\right). \quad (45)
\end{aligned}$$

Furthermore, for every q and from the inequalities (38) - (45), we have

$$\lim_{n\to\infty} \mathfrak{A}\left(\varsigma_n, \varsigma_{n+q}, \iota\right) \geq 1 * 1 * 1 * \cdots * 1 = 1, \quad (46)$$

$$\lim_{n\to\infty} \mathfrak{I}(\varsigma_n, \varsigma_{n+q}, \iota) \leq 0 \diamond 0 \diamond 0 \diamond \dots \diamond 0 = 0, \quad (47)$$

$$\lim_{n\to\infty} \mathfrak{D}(\varsigma_n, \varsigma_{n+q}, \iota) \leq 0 \diamond 0 \diamond 0 \diamond \dots \diamond 0 = 0, \quad (48)$$

as $b(\varsigma_n, \varsigma_{n+q}) < 1/p$ for all $n, q \in \mathbb{N}$ and $p \in (0,1)$ i.e., $\{\varsigma_n\}$ is an orthogonal G-Cauchy sequence in $\mathfrak{X}$. Since $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ is complete, there exists $u \in \mathfrak{K}$ such that $\varsigma_n \to u$ as $n \to \infty$. Now, since $\mathfrak{Y}$ is an $\perp$-continuous mapping, one writes
$\lim \mathfrak{A}(\varsigma_{n+1}, \mathfrak{Y}u, r) = \lim \mathfrak{A}(\mathfrak{Y}\varsigma_n, \mathfrak{Y}u, r) = 1,$
$\lim \mathfrak{I}(\varsigma_{n+1}, \mathfrak{Y}u, r) = \lim \mathfrak{I}(\mathfrak{Y}\varsigma_n, \mathfrak{Y}u, r) = 0,$
$\lim \mathfrak{D}(\varsigma_{n+1}, \mathfrak{Y}u, r) = \lim \mathfrak{D}(\mathfrak{Y}\varsigma_n, \mathfrak{Y}u, r) = 0.$
Now, we investigate that ς is the fixed point of $\mathfrak{Y}$.
By applying equations (46), (47) and (48) and conditions (e), (j) and (o) in definition (3.1), we have

$$\begin{aligned}
\mathfrak{A}\left(\varsigma, \mathfrak{Y}\varsigma, \iota\right) &\geq \mathfrak{A}\left(\varsigma, \varsigma_n, \frac{\iota}{5\urcorner(\varsigma, \varsigma_n)}\right) * \mathfrak{A}\left(\mathfrak{Y}\varsigma_{n-1}, \mathfrak{Y}\varsigma_n, \frac{\iota}{5\varphi(\varsigma_n, \varsigma_{n+1})}\right) * \mathfrak{A}\left(\mathfrak{Y}\varsigma_n, \mathfrak{Y}\varsigma_{n+1}, \frac{\iota}{5\xi(\varsigma_{n+1}, \varsigma_{n+2})}\right)\\
&* \mathfrak{A}\left(\mathfrak{Y}\varsigma_{n+1}, \mathfrak{Y}\varsigma_{n+2}, \frac{\iota}{5\psi(\varsigma_{n+2}, \varsigma_{n+3})}\right) * \mathfrak{A}\left(\mathfrak{Y}\varsigma_{n+2}, \mathfrak{Y}\varsigma, \frac{\iota}{5\delta(\varsigma_{n+3}, \mathfrak{Y}\varsigma)}\right)\\
&\geq \mathfrak{A}\left(\varsigma, \varsigma_n, \frac{\iota}{5\urcorner(\varsigma, \varsigma_n)}\right) * \mathfrak{A}\left(\varsigma_{n-1}, \varsigma_n, \frac{\iota}{5p\varphi(\varsigma_n, \varsigma_{n+1}}\right) * \mathfrak{A}\left(\varsigma_n, \varsigma_{n+1}, \frac{\iota}{5p\xi(\varsigma_{n+1}, \varsigma_{n+2})}\right)\\
&* \mathfrak{A}\left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\iota}{5p\psi(\varsigma_{n+2}, \varsigma_{n+3})}\right) * \mathfrak{A}\left(\varsigma_{n+2}, x, \frac{\iota}{5p\delta(\varsigma_{n+3}, \mathfrak{Y}\varsigma}\right) \text{ and}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{I}\left(\varsigma, \mathfrak{Y}\varsigma, \iota\right) &\leq \mathfrak{I}\left(\varsigma, \varsigma_n, \frac{\iota}{5\urcorner(\varsigma, \varsigma_n)}\right) \diamond \mathfrak{I}\left(\mathfrak{Y}\varsigma_{n-1}, \mathfrak{Y}\varsigma_n, \frac{\iota}{5\varphi(\varsigma_n, \varsigma_{n+1})}\right) \diamond \mathfrak{I}\left(\mathfrak{Y}\varsigma_n, \mathfrak{Y}\varsigma_{n+1}, \frac{\iota}{5\xi(\varsigma_{n+1}, \varsigma_{n+2})}\right)\\
&\diamond \mathfrak{I}\left(\mathfrak{Y}\varsigma_{n+1}, \mathfrak{Y}\varsigma_{n+2}, \frac{\iota}{5\psi(\varsigma_{n+2}, \varsigma_{n+3})}\right) \diamond \mathfrak{I}\left(\mathfrak{Y}\varsigma_{n+2}, \mathfrak{Y}\varsigma, \frac{\iota}{5\delta(\varsigma_{n+3}, \mathfrak{Y}\varsigma)}\right)\\
&\leq \mathfrak{I}\left(\varsigma, \varsigma_n, \frac{\iota}{5\urcorner(\varsigma, \varsigma_n)}\right) \diamond \mathfrak{I}\left(\varsigma_{n-1}, \varsigma_n, \frac{\iota}{5p\varphi(\varsigma_n, \varsigma_{n+1}}\right) \diamond \mathfrak{I}\left(\varsigma_n, \varsigma_{n+1}, \frac{\iota}{5p\xi(\varsigma_{n+1}, \varsigma_{n+2})}\right)\\
&\diamond \mathfrak{I}\left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\iota}{5p\psi(\varsigma_{n+2}, \varsigma_{n+3})}\right) \diamond \mathfrak{I}\left(\varsigma_{n+2}, x, \frac{\iota}{5p\delta(\varsigma_{n+3}, \mathfrak{Y}\varsigma}\right),
\end{aligned}$$

Likewise

$$\begin{aligned}\mathfrak{D}(\varsigma,\mathfrak{Y}\varsigma,\iota) &\leq \mathfrak{D}\left(\varsigma,\varsigma_n,\frac{\iota}{5\daleth(\varsigma,\varsigma_n)}\right)\diamond\mathfrak{D}\left(\mathfrak{Y}\varsigma_{n-1},\mathfrak{Y}\varsigma_n,\frac{\iota}{5\varphi(\varsigma_n,\varsigma_{n+1})}\right)\diamond\mathfrak{D}\left(\mathfrak{Y}\varsigma_n,\mathfrak{Y}\varsigma_{n+1},\frac{\iota}{5\xi(\varsigma_{n+1},\varsigma_{n+2})}\right)\\ &\diamond\mathfrak{D}\left(\mathfrak{Y}\varsigma_{n+1},\mathfrak{Y}\varsigma_{n+2},\frac{\iota}{5\psi(\varsigma_{n+2},\varsigma_{n+3})}\right)\diamond\mathfrak{D}\left(\mathfrak{Y}\varsigma_{n+2},\mathfrak{Y}\varsigma,\frac{\iota}{5\delta(\varsigma_{n+3},\mathfrak{Y}\varsigma)}\right)\\ &\leq \mathfrak{D}\left(\varsigma,\varsigma_n,\frac{\iota}{5\daleth(\varsigma,\varsigma_n)}\right)\diamond\mathfrak{D}\left(\varsigma_{n-1},\varsigma_n,\frac{\iota}{5p\varphi(\varsigma_n,\varsigma_{n+1}}\right)\diamond\mathfrak{D}\left(\varsigma_n,\varsigma_{n+1},\frac{\iota}{5p\xi(\varsigma_{n+1},\varsigma_{n+2})}\right)\\ &\diamond\mathfrak{D}\left(\varsigma_{n+1},\varsigma_{n+2},\frac{\iota}{5p\psi(\varsigma_{n+2},\varsigma_{n+3})}\right)\diamond\mathfrak{D}\left(\varsigma_{n+2},x,\frac{\iota}{5p\delta(\varsigma_{n+3},\mathfrak{Y}\varsigma)}\right).\end{aligned}$$

Letting $n\to\infty$ in the above inequalities, we deduce $\mathfrak{Y}\varsigma=\varsigma$, i.e., ς is a fixed point of $\mathfrak{Y}$. By applying the inequalities (13) and (14), it is easy to show that ς is a unique fixed point of $\mathfrak{Y}$.
Now, for uniqueness, let $\omega\in\mathfrak{K}$ be another fixed point for $\mathfrak{Y}$ and let there exist $\iota>0$ such that $\mathfrak{A}(u,\omega,\iota)\neq 1,\mathfrak{I}(u,\omega,\iota)\neq 0,\mathfrak{D}(u,\omega,\iota)\neq 0$. We can obtain

$$\varsigma_0\perp u,\varsigma_0\perp\omega. \tag{49}$$

Since $\mathfrak{Y}$ is an $\perp$-preserving, this implies that -preserving, this implies that

$$\mathfrak{Y}^n\varsigma_0\perp\mathfrak{Y}^n u,\mathfrak{Y}^n\varsigma_0\perp\mathfrak{Y}^n\omega, \tag{50}$$

for all $n\in\mathbb{N}$. From (13), (14) and (15) we can derive

$$\begin{aligned}\mathfrak{A}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n u,\iota)&\geq\mathfrak{A}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n u,k\iota)\geq\mathfrak{A}\left(\varsigma_0,u,\frac{\iota}{k^n}\right)\\ \mathfrak{A}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n\omega,\iota)&\geq\mathfrak{A}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n\omega,k\iota)\geq\mathfrak{A}\left(\varsigma_0,\omega,\frac{\iota}{k^n}\right),\\ \mathfrak{I}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n u,\iota)&\leq\mathfrak{I}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n u,k\iota)\leq\mathfrak{I}\left(\varsigma_0,u,\frac{\iota}{k^n}\right)\\ \mathfrak{I}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n\omega,\iota)&\leq\mathfrak{I}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n\omega,k\iota)\leq\mathfrak{I}\left(\varsigma_0,\omega,\frac{\iota}{k^n}\right),\text{and}\\ \mathfrak{D}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n u,\iota)&\leq\mathfrak{D}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n u,k\iota)\leq\mathfrak{D}\left(\varsigma_0,u,\frac{\iota}{k^n}\right)\\ \mathfrak{D}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n\omega,\iota)&\leq\mathfrak{D}(\mathfrak{Y}^n\varsigma_0,\mathfrak{Y}^n\omega,k\iota)\leq\mathfrak{D}\left(\varsigma_0,\omega,\frac{\iota}{k^n}\right).\end{aligned}$$

We can write

$$\begin{aligned}\mathfrak{A}(u,\omega,\iota)&=\mathfrak{A}(\mathfrak{Y}^n u,\mathfrak{Y}^n\omega,\iota)\\ &\geq\mathfrak{A}\left(\varsigma_0,u,\frac{\iota}{5\daleth(\varsigma_0,u)}\right)\\ &*\mathfrak{A}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\varphi(\varsigma_0,\varsigma_1)}\right)*\mathfrak{A}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5p^2\xi(\varsigma_1,\varsigma_2)}\right)\\ &*\mathfrak{A}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5p^3\psi(\varsigma_2,\varsigma_3)}\right)*\mathfrak{A}\left(\varsigma_3,\omega,\frac{\iota}{5p^4\delta(\varsigma_3,\omega)}\right),\\ \mathfrak{I}(u,\omega,\iota)&=\mathfrak{I}(\mathfrak{Y}^n u,\mathfrak{Y}^n\omega,\iota)\\ &\leq\mathfrak{I}\left(\varsigma_0,u,\frac{\iota}{5\daleth(\varsigma_0,u)}\right)\\ &\diamond\mathfrak{I}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\varphi(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{I}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5p^2\xi(\varsigma_1,\varsigma_2)}\right)\\ &\diamond\mathfrak{I}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5p^3\psi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{I}\left(\varsigma_3,\omega,\frac{\iota}{5p^4\delta(\varsigma_3,\omega)}\right),\\ \mathfrak{D}(u,\omega,\iota)&=\mathfrak{D}(\mathfrak{Y}^n u,\mathfrak{Y}^n\omega,\iota)\\ &\leq\mathfrak{D}\left(\varsigma_0,u,\frac{\iota}{5\daleth(\varsigma_0,u)}\right)\\ &\diamond\mathfrak{D}\left(\varsigma_0,\varsigma_1,\frac{\iota}{5\varphi(\varsigma_0,\varsigma_1)}\right)\diamond\mathfrak{D}\left(\varsigma_1,\varsigma_2,\frac{\iota}{5p^2\xi(\varsigma_1,\varsigma_2)}\right)\\ &\diamond\mathfrak{D}\left(\varsigma_2,\varsigma_3,\frac{\iota}{5p^3\psi(\varsigma_2,\varsigma_3)}\right)\diamond\mathfrak{D}\left(\varsigma_3,\omega,\frac{\iota}{5p^4\delta(\varsigma_3,\omega)}\right).\end{aligned} \tag{51}$$

for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$ and if
$\daleth(\varsigma_0, u) = \varphi(\varsigma_0, \varsigma_1) = \xi(\varsigma_1, \varsigma_2) = \psi(\varsigma_2, \varsigma_3) = \delta(\varsigma_3, \omega)$ then we get
$\mathfrak{A}(u, \omega, \iota) = 1, \mathfrak{I}(u, \omega, \iota) = 0, \mathfrak{D}(u, \omega, \iota) = 0$ for all $\iota > 0$;
Hence $u = \omega$. □

Remark 3.5. The opposite is not true: any Neutrosophic orthogonal Pentagonal controlled metric space is a Neutrosophic Pentagonal controlled metric space

Definition 3.6. Let $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ be an Neutrosophic orthogonal Pentagonal controlled metric space. Following that, a sequence $\{\varsigma_n\}$ is considered to be G-convergent to $\mathfrak{x}$, where $\varsigma, \{\varsigma_n\} \in \mathfrak{K}$ if and only if $\lim_{n\to\infty} \mathfrak{A}(\varsigma_n, \varsigma, \iota) = 1$ $\lim_{n\to\infty} \mathfrak{I}(\varsigma_n, \varsigma, \iota) = 0$ and $\lim_{n\to\infty} \mathfrak{D}(\varsigma_n, \varsigma, \iota) = 0$, for any $n > 0$ and for all $\iota > 0$.

Definition 3.7. Let $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ be an Neutrosophic orthogonal Pentagonal controlled metric space. Following that, a sequence $\{\varsigma_n\}$ is considered to be G-Cauchy sequence with $\{\varsigma_n\} \in \mathfrak{K}$ if and only if $\lim_{n\to\infty} \mathfrak{A}(\varsigma_n, \varsigma_{n+m}, \iota) = 1$, $\lim_{n\to\infty} \mathfrak{I}(\varsigma_n, \varsigma_{n+m}, \iota) = 0$ and $\lim_{n\to\infty} \mathfrak{D}(\varsigma_n, \varsigma_{n+m}, \iota) = 0$, for any $m > 0$ and $\iota > 0$.

Definition 3.8. Let $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ be a Pentagonal controlled metric space that is Neutrosophic orthogonal; it is G-complete if and only if all of the G-Cauchy sequences are convergent.

Definition 3.9. $\mathfrak{Y} : \mathfrak{K} \to \mathfrak{K}$ is $\perp$-continuous at $\mathfrak{x} \in \mathfrak{K}$ in a Neutrosophic orthogonal Pentagonal controlled metric space $(\mathfrak{K}, \mathfrak{A}, \mathfrak{I}, \mathfrak{D}, \star, \diamond, \perp)$ if for each orthogonal sequence $\{\varsigma_n\}$ in $\mathfrak{X}$ so that if $\lim_{n\to\infty} \mathfrak{A}(\mathfrak{x}_n, \mathfrak{x}, \iota)$ exists and is finite for all $\iota > 0$, then $\lim_{n\to\infty} \mathfrak{A}(\mathfrak{Y}\mathfrak{x}_n, \mathfrak{Y}\mathfrak{x}, \iota)$ again exists and is finite for all $\iota > 0$. Additionally, $\mathfrak{Y}$ is $\perp$-continuous if $\mathfrak{Y}$ is $\perp$-continuous at each $\mathfrak{x} \in \mathfrak{Z}$. Also $\mathfrak{Y}$ is $\perp$-preserving if $\mathfrak{Y}\mathfrak{x} \perp \mathfrak{Y}\omega$; hence, $\mathfrak{x} \perp \omega$.

Remark 3.10. The limit of a convergent sequence in a Neutrosophic orthogonal Pentagonal controlled metric space does not have to be unique.

Remark 3.11. The convergent sequence in a Neutrosophic orthogonal Pentagonal controlled metric space does not have to be a Cauchy sequence.

Definition 3.12. Let $\mathfrak{K}$ is a Neutrosophic orthogonal Pentagonal controlled metric space, then we define an open ball $\mathfrak{B}(\varsigma, r, \beta)$ with centre ς, radius $r, 0 < r < 1$ and $\beta > 0$ as follows:
$\mathfrak{B}(\varsigma, r, \beta) = \{\Theta \in \mathfrak{K} : \mathfrak{A}(\varsigma, \Theta, \beta) > 1 - r\}, \mathfrak{B}(\varsigma, r, \beta) = \{\Theta \in \mathfrak{K} : \mathfrak{I}(\varsigma, \Theta, \beta) < r\}$, and the corresponding topology that has been defined as $\tau_{\mathfrak{p}\mathfrak{x}} = \{\mathfrak{D} \subset \mathfrak{X} : \mathfrak{B}(\varsigma, r, \beta) \subset \mathfrak{D}\}$.

Example 3.13. Let $\mathfrak{K} = \mathbb{Z} = \mathfrak{M} \cup \mathfrak{B}$, where $\mathfrak{M} = \{-1, -2, -3, \dots\} \cup \{0, 1\}$ and $\mathfrak{B} = \{2, 3, 4, \dots\}$. Define a binary relation $\perp$ by $\varsigma \perp \Theta \Leftrightarrow \varsigma + \Theta \geq 0$. Define $\mathfrak{A}, \mathfrak{I}, \mathfrak{D} : \mathfrak{K} \times \mathfrak{K} \times [0, \infty) \to [0, 1]$ by

$$\mathfrak{A}(\varsigma, \Theta, \iota) = \frac{\iota}{\iota + \max\{\varsigma, \Theta\}}, \mathfrak{I}(\varsigma, \Theta, \iota) = \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}}, \mathfrak{D}(\varsigma, \Theta, \iota) = \frac{\max\{\varsigma, \Theta\}}{\iota} \tag{52}$$

for all $\iota > 0$ and $\varsigma, \Theta \in \mathfrak{Z}$ with a continuous t-norm $*$ and t-conorm defined by: $\iota_1 * \iota_2 = \iota_1 \cdot \iota_2$ and $\iota_1 \diamond \iota_2 = \iota_1 \cdot \iota_2$. Define $\daleth, \varphi, \xi, \psi, \delta : \mathfrak{Z} \times \mathfrak{Z} \to [1, \infty)$ by

$$\daleth(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \max\{\varsigma, \Theta\} & \text{otherwise.} \end{cases}$$

$$\varphi(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \frac{1}{\min\{|\varsigma|, |\Theta|\}} & \text{otherwise.} \end{cases}$$

$$\xi(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \frac{1}{\max\{\varsigma, \Theta\}} & \text{otherwise.} \end{cases}$$

$$\psi(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \min\{|\varsigma|, |\Theta|\} & \text{otherwise.} \end{cases}$$

$$\delta(\varsigma, \Theta) = \begin{cases} 1, & \text{if } \varsigma, \Theta \in \mathfrak{M}, \\ \max\{\varsigma + \Theta, |\varsigma - \Theta|\} & \text{otherwise.} \end{cases}$$

Then, $\mathfrak{K}$ is a Neutrosophic Orthogonal Pentagonal Controlled Metric Space ($\mathfrak{NOPCMS}$). Observe that $\lim_{r\to\infty} \mathfrak{A}(\varsigma, \Theta, \iota) = 1$, $\lim_{r\to\infty} \mathfrak{I}(\varsigma, \Theta, \iota) = 0$ and $\lim_{r\to\infty} \mathfrak{D}(\varsigma, \Theta, \iota) = 0$.
Now, we define $\mathfrak{Y} : \mathfrak{K} \to \mathfrak{K}$ by

$$\mathfrak{Y}\varsigma = \begin{cases} \frac{\varsigma}{2}, & \text{if } \varsigma \in \mathfrak{A}, \\ 1, & \text{if } \varsigma \in \mathfrak{B}, \end{cases} \tag{53}$$

for all $\varsigma \in \mathfrak{K}$.

Proof. Observe that if $\varsigma \perp \Theta$, then clearly $\mathfrak{Y}\varsigma \perp \mathfrak{Y}\Theta$. To prove that the contraction is orthogonal for $k \in \left[\left(\frac{1}{2}\right), 1\right)$, there are some cases:
(1) If $\varsigma, \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = \varsigma/2$ and $\mathfrak{Y}\Theta = \Theta/2$. We have
$\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{A}\left(\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{k\iota}{k\iota + \max\left\{\left(\frac{\varsigma}{2}\right), \left(\frac{\mathfrak{d}}{2}\right)\right\}} \geq \frac{\iota}{\iota + \max\{\varsigma, \Theta\}} = \mathfrak{A}(\varsigma, \omega, \iota)$.
2) If $\varsigma, \Theta \in \mathfrak{B}$, then $\mathfrak{Y}\varsigma = 1$ and $\mathfrak{Y}\omega = 1$. In this case,
$\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{A}(1, 1, k\iota) = \frac{k\iota}{k\iota + \max\{1,1\}} \geq \frac{\iota}{\iota + \max\{\varsigma, \omega\}} = \mathfrak{A}(\varsigma, \omega, \iota)$.
3) If $\varsigma \in \mathfrak{A}$ and $\omega \in \mathfrak{B}$, then $\varsigma = \varsigma/2$ and $\mathfrak{Y}\omega = 1$. Here,
$\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{A}\left(\frac{\varsigma}{2}, 1, k\iota\right) = \frac{k\iota}{k\iota + \max\left\{\left(\frac{\varsigma}{2}\right), 1\right\}} \geq \frac{r}{r + \max\{x, \omega\}} = \mathfrak{A}(\varsigma, \Theta, \iota)$.
4) If $\varsigma \in \mathfrak{B}$ and $\Theta \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = 1$ and $\mathfrak{Y}\Theta = \omega/2$. This implies that
$\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{A}\left(1, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{k\iota}{k\iota + \max\left\{1, \left(\frac{\mathfrak{d}}{2}\right)\right\}} \geq \frac{\iota}{\iota + \max\{\varsigma, \Theta\}} = \mathfrak{A}(\varsigma, \Theta, \iota)$.
5) If $\varsigma, \Theta \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = \varsigma/2$ and $\mathfrak{Y}\Theta = \Theta/2$. We have
$\mathfrak{I}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{I}\left(\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{\max\left\{\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}\right\}}{\mathfrak{K}\iota + \max\left\{\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}\right\}} \leq \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}} = \mathfrak{I}(\varsigma, \Theta, \iota)$.
6) If $\varsigma, \omega \in \mathfrak{B}$, then $\mathfrak{Y}\varsigma = 1$ and $\mathfrak{Y}\omega = 1$. In this case,
$\mathfrak{I}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{I}(1, 1, k\iota) = \frac{\max\{1,1\}}{\mathfrak{K}\iota + \max\{1,1\}} \leq \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}} = \mathfrak{I}(\varsigma, \Theta, \iota)$.
7) If $\varsigma \in \mathfrak{A}$ and $\omega \in \mathfrak{B}$, then $\varsigma = \varsigma/2$ and $\mathfrak{Y}\omega = 1$. Here,
$\mathfrak{I}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{I}\left(\frac{\varsigma}{2}, 1, k\iota\right) = \frac{\max\left\{\frac{\varsigma}{2}, 1\right\}}{\mathfrak{K}\iota + \max\left\{\frac{\varsigma}{2}, 1\right\}} \leq \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}} = \mathfrak{I}(\varsigma, \Theta, \iota)$.
8) If $\varsigma \in \mathfrak{B}$ and $\omega \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = 1$ and $\mathfrak{Y}\omega = \omega/2$. This implies that
$\mathfrak{I}(\mathfrak{Y}\mathfrak{v}, \mathfrak{Y}\Theta, k\iota) = \mathfrak{I}\left(1, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{\max\left\{1, \frac{\mathfrak{d}}{2}\right\}}{\mathfrak{K}\iota + \max\left\{1, \frac{\mathfrak{d}}{2}\right\}} \leq \frac{\max\{\varsigma, \Theta\}}{\iota + \max\{\varsigma, \Theta\}} = \mathfrak{I}(\varsigma, \Theta, \iota)$.
9) If $\varsigma, \omega \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = \varsigma/2$ and $\mathfrak{Y}\omega = \omega/2$. We have
$\mathfrak{D}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{D}\left(\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{\max\left\{\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}\right\}}{\iota} \leq \frac{\max\{\varsigma, \Theta\}}{\iota} = \mathfrak{D}(\varsigma, \Theta\iota)$.
10) If $\varsigma, \omega \in \mathfrak{B}$, then $\mathfrak{Y}\varsigma = 1$ and $\mathfrak{Y}\omega = 1$. In this case,
$\mathfrak{D}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{D}(1, 1, k\iota) = \frac{\max\{1,1\}}{\iota} \leq \frac{\max\{\varsigma, \Theta\}}{\iota} = \mathfrak{D}(\varsigma, \Theta, \iota)$.
11) If $\varsigma \in \mathfrak{A}$ and $\omega \in \mathfrak{B}$, then $\varsigma = \varsigma/2$ and $\mathfrak{Y}\omega = 1$. Here,
$\mathfrak{D}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{D}\left(\frac{\varsigma}{2}, 1, k\iota\right) = \frac{\max\left\{\frac{\varsigma}{2}, 1\right\}}{\mathfrak{K}\iota} \leq \frac{\max\{\varsigma, \omega\}}{\iota} = \mathfrak{D}(\varsigma, \Theta, \iota)$.
12) If $\varsigma \in \mathfrak{B}$ and $\omega \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = 1$ and $\mathfrak{Y}\omega = \omega/2$. This implies that
$\mathfrak{D}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \mathfrak{D}\left(1, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{\max\left\{1, \frac{\mathfrak{d}}{2}\right\}}{\mathfrak{K}\iota} \leq \frac{\max\{\varsigma, \Theta\}}{\iota} = \mathfrak{D}(\varsigma, \Theta, \iota)$.
Hence, it is an $\perp$-contraction. Now, we show that it is not a contraction. Let $\varsigma, \Theta \in \mathfrak{A}$, then $\mathfrak{Y}\varsigma = \varsigma/2$ and $\mathfrak{Y}\omega = \Theta/2$. Here

$$\begin{aligned}
\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) &= \mathfrak{A}\left(\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{k\iota}{k\iota + \max\left\{\left(\frac{\varsigma}{2}\right), \left(\frac{\mathfrak{d}}{2}\right)\right\}} \\
\mathfrak{I}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) &= \mathfrak{I}\left(\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{\max\left\{\left(\frac{\varsigma}{2}\right), \left(\frac{\mathfrak{d}}{2}\right)\right\}}{k\iota + \max\left\{\left(\frac{\varsigma}{2}\right), \left(\frac{\mathfrak{d}}{2}\right)\right\}} \\
\mathfrak{D}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) &= \mathfrak{D}\left(\frac{\varsigma}{2}, \frac{\mathfrak{d}}{2}, k\iota\right) = \frac{\max\left\{\left(\frac{\varsigma}{2}\right), \left(\frac{\mathfrak{d}}{2}\right)\right\}}{k\iota}
\end{aligned} \tag{54}$$

Let $\varsigma = \Theta = -2, k = \left(\frac{9}{10}\right)$ and $\iota = 10$, so
$\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) = \frac{9}{9 + \max\{-1, -1\}} \leq \frac{10}{10 + \max\{-2, -2\}} = \mathfrak{A}(\varsigma, \Theta, r)$,
which implies $\mathfrak{A}(\mathfrak{Y}\varsigma, \mathfrak{Y}\Theta, k\iota) \leq \mathfrak{A}(\varsigma, \Theta, \iota)$. This is wrong.
If $\lim_{n\to\infty} \mathfrak{A}(\varsigma_n, \varsigma, \iota)$, $\lim_{r\to\infty} \mathfrak{I}(\varsigma_n, \varsigma, \iota)$ and $\lim_{r\to\infty} \mathfrak{D}(\varsigma_n, \varsigma, \iota)$,
is finite and exists, then also $\lim_{n\to\infty} \mathfrak{A}(\mathfrak{Y}\varsigma_n, \mathfrak{Y}\varsigma, r)$, $\lim_{n\to\infty} \mathfrak{I}(\mathfrak{Y}\varsigma_n, \mathfrak{Y}\varsigma, r)$,
$\lim_{n\to\infty} \mathfrak{D}(\mathfrak{Y}\varsigma_n, \mathfrak{Y}\varsigma, r)$ is finite and exists. This implies that it is $\perp$-continuous.
Also, observe that $\lim_{n\to\infty} \mathfrak{D}(\varsigma_n, \Theta)$, $\lim_{n\to\infty} \mathfrak{D}(\Theta, \varsigma_n)$, $\lim_{n\to\infty} \mathfrak{W}(\varsigma_n, \Theta)$, $\lim_{n\to\infty} \mathfrak{W}(\Theta, \varsigma_n)$,
$\lim_{n\to\infty} \mathfrak{E}(\varsigma_n, \Theta)$, $\lim_{n\to\infty} \mathfrak{E}(\Theta, \varsigma_n)$, $\lim_{n\to\infty} R(\varsigma_n, \Theta)$, $\lim_{n\to\infty} R(\Theta, \varsigma_n)$,
$\lim_{n\to\infty} T(\varsigma_n, \Theta)$, $\lim_{n\to\infty} T(\Theta, \varsigma_n)$ are all exist and are finite.
All circumstances of Theorem (3.4) are fulfilled and 0 is the unique fixed point of $\mathfrak{Y}$. □

4 An Application to a Traffic flow network Using Integral Equation

The existence and uniqueness of a solution to a Fuzzy Fredholm type integral equation of second kind are investigated in this section using Theorem (3.4).
A Fredholm integral equations of second kind would be considered with the context of an orthogonality pentagonal controlled fuzzy metric space may be represented as follows:

$$f(x) = \lambda \int_a^b k(x,t)\varphi(t)dt + g(x)$$

Suppose we want to Model traffic density $f(x)$ in such a Neutrosophic fuzzy metric space, to enforce the orthogonality in this space with five control functions.

The application of the Fredholm integral equations of the second kind in traffic flow networks provides a powerful mathematical framework for analyzing and optimizing traffic systems. By modelling interactions between vehicles, road capacities and external conditions. This equation enables the prediction of traffic patterns and the identification of equilibrium states. Its versatility lies in solving problems related to congestion management, route optimization and system efficiency.

Let $X = C([e,g],\mathbb{R})$ be the set of all continuous real - valued functions defined on $[e,g]$. Now, we consider the fuzzy Fredhlom-type integral equation of the second kind:

$$\varsigma(n) = \varphi(c) + \mu \int_e^g (n,c)\varsigma(n)dc for n, c \in [e,g]. \tag{55}$$

where $\beta > 0, \varphi(c)$ is a fuzzy function of $c \in [e,g]$ and $\mathfrak{Y} \in \mathfrak{K}$. Define $\mathfrak{A}$ by

$$\mathfrak{A}(\varsigma(n),\Theta(n),\iota) = \sup_{n\in[e,g]} \frac{\iota}{\iota + \max\{\varsigma(n),\Theta(n)\}},$$
$$\mathfrak{I}(\varsigma(n),\Theta(n),\iota) = \inf_{n\in[e,g]} \frac{\max\{\varsigma(n),\Theta(n)\}}{\iota + \max\{\varsigma(n),\Theta(n)\}},$$
$$\mathfrak{D}(\varsigma(n),\Theta(n),\iota) = \inf_{n\in[e,g]} \frac{\max\{\varsigma(n),\Theta(n)\}}{\iota},$$

for $\varsigma, \Theta \in \mathfrak{K}$ and $\iota > 0$, with a $\mathfrak{CTN}$ $*$ defined by $r_1 * r_2 = r_1 \cdot r_2$ and $\mathfrak{CTC}$ by $r_1 \diamond r_2 = r_1 \cdot r_2$.
Define $\daleth, \varphi, \xi, \psi, \delta : \mathfrak{K} \times \mathfrak{K} \to [1,\infty)$ by

$$\daleth(\varsigma,\Theta) = \begin{cases} 1, & \text{if } \varsigma,\Theta \in \mathfrak{M}, \\ \max\{\varsigma,\Theta\}, & \text{otherwise} \end{cases}$$
$$\varphi(\varsigma,\Theta) = \begin{cases} 1, & \text{if } \varsigma,\Theta \in \mathfrak{M}, \\ \frac{1}{\min\{|\varsigma|,|\Theta|\}}, & \text{otherwise} \end{cases}$$
$$\xi(\varsigma,\Theta) = \begin{cases} 1, & \text{if } \varsigma,\Theta \in \mathfrak{M}, \\ \frac{1}{\max\{\varsigma,\Theta\}}, & \text{otherwise} \end{cases}$$
$$\psi(\varsigma,\Theta) = \begin{cases} 1, & \text{if } \varsigma,\Theta \in \mathfrak{M}, \\ \min\{|\varsigma|,|\Theta|\}, & \text{otherwise} \end{cases}$$
$$\delta(\varsigma,\Theta) = \begin{cases} 1, & \text{if } \varsigma,\Theta \in \mathfrak{M}, \\ \max\{\varsigma+\Theta,|\varsigma-\Theta|\}, & \text{otherwise} \end{cases}$$

Then, $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},*,\Delta,\perp)$ be a G-complete $\mathfrak{NODCMS}$.

Theorem 4.1. *Assume that* $\max\{(n,c)\varsigma(n),(n,c)\}\Theta(n) \leq \{\varsigma(n),\Theta(n)\}$ *for* $\varsigma,\Theta \in \mathfrak{Z}, k \in (0,1)$, *and* $\forall n, c \in [e,g]$. *Also consider* $\int_e^g dc = g - e \leq k < 1$. *Let* $\mathfrak{Y} : \mathfrak{K} \to \mathfrak{K}$ *be*
(i) $\perp$*-preserving*
(ii) $\perp$*-contraction*
(iii) $\perp$*-continuous*
Then, the fuzzy Fredholm-type integral equation of second kind in equation (55) has a unique solution.

Proof. Define $\mathfrak{Y}:\mathfrak{K}\to\mathfrak{K}$ by $\mathfrak{Y}\varsigma(n)=\varphi(c)+\mu\int_e^g(n,c)e(n)dc$ for all $n,c\in[e,g]$.
(i) Take orthogonality as $\varsigma(n)\perp\Theta(n)\Leftrightarrow\varsigma(n)+\omega(n)\geq 0$. We see that $\varsigma(n)$ and $\mathfrak{Y}\varsigma(n)$ belong to $\mathfrak{K}$. So, if $\varsigma(n)\perp\Theta(n)$, then clearly $\mathfrak{Y}\varsigma(n)\perp\mathfrak{Y}\Theta(n)$.
(ii) Observe that the existence of a fixed point of the operator $\mathfrak{Y}$ is equivalent to the existence of a solution of the Fredholm-type integral Equation (55).
(iii) Note that $\max\{(n,c)\varsigma(n),(n,c)\Theta(n)\}\leq\max\{\varsigma(n),\Theta(n)\}$

$$\Rightarrow\varphi(c)+\mu\int_e^g\max\{\varsigma(n),\Theta(n)\} \tag{56}$$

(iv) Now, for all $\varsigma,\Theta\in\mathfrak{K}$, we have

$$\begin{aligned}
\mathfrak{A}(\mathfrak{Y}\varsigma(n),\mathfrak{Y}\Theta(n),k\iota)&=\sup_{n\in[e,g]}\frac{k\iota}{k\iota+\max\{\mathfrak{Y}\varsigma(n),\mathfrak{Y}\Theta(n)\}},\\
&=\sup_{n\in[e,g]}\frac{k\iota}{k\iota+\max\{(n,c)\varsigma(n)dc,\int_e^g(n,c)\Theta(n)dc\}}\\
&=\sup_{l\in[e,g]}\frac{k\iota}{k\iota+\int_e^g\max\{(n,c)\varsigma(n),(n,c)\Theta(n)\}dc}\\
&=\sup_{l\in[e,g]}\frac{k\iota}{k\iota+\int_e^g\max\{\varsigma(n),\Theta(n)\}dj}\\
&=\sup_{l\in[e,g]}\frac{k\iota}{k\iota+\max\{\varsigma(n),\Theta(n)\}\int_e^g dj}\\
&\geq\sup_{l\in[e,g]}\frac{k\iota}{k\iota+k\max\{\varsigma(n),\Theta(n)\}}\\
&\geq\frac{\iota}{\iota+\max\{\varsigma(n),\Theta(n)}\\
&=\mathfrak{A}(\varsigma(n),\Theta(n),\iota),\\
\mathfrak{I}(\mathfrak{Y}\varsigma(n),\mathfrak{Y}\Theta(n),k\iota)&=\inf_{n\in[e,g]}\frac{k\iota}{k\iota+\max\{\mathfrak{Y}\varsigma(n),\mathfrak{Y}\Theta(n)\}},\\
&=\inf_{n\in[e,g]}\frac{k\iota}{k\iota+\max\{(n,c)\varsigma(n)dc,\int_e^g(n,c)\Theta(n)dc\}}\\
&=\inf_{l\in[e,g]}\frac{k\iota}{k\iota+\int_e^g\max\{(n,c)\varsigma(n),(n,c)\Theta(n)\}dc}\\
&=\inf_{l\in[e,g]}\frac{k\iota}{k\iota+\int_e^g\max\{\varsigma(n),\Theta(n)\}dj}\\
&=\inf_{l\in[e,g]}\frac{k\iota}{k\iota+\max\{\varsigma(n),\Theta(n)\}\int_e^g dj}\\
&\leq\inf_{l\in[e,g]}\frac{k\iota}{k\iota+k\max\{\varsigma(n),\Theta(n)\}}\\
&\leq\frac{\iota}{\iota+\max\{\varsigma(n),\Theta(n)}\\
&=\mathfrak{I}(\varsigma(n),\Theta(n),\iota),\\
\mathfrak{D}(\mathfrak{Y}\varsigma(n),\mathfrak{Y}\Theta(n),k\iota)&=\inf_{n\in[e,g]}\frac{k\iota}{k\iota+\max\{\mathfrak{Y}\varsigma(n),\mathfrak{Y}\Theta(n)\}},\\
&=\inf_{n\in[e,g]}\frac{k\iota}{k\iota+\max\{(n,c)\varsigma(n)dc,\int_e^g(n,c)\Theta(n)dc\}}\\
&=\inf_{l\in[e,g]}\frac{k\iota}{k\iota+\int_e^g\max\{(n,c)\varsigma(n),(n,c)\Theta(n)\}dc}\\
&=\inf_{l\in[e,g]}\frac{k\iota}{k\iota+\int_e^g\max\{\varsigma(n),\Theta(n)\}dj}\\
&=\inf_{l\in[e,g]}\frac{k\iota}{k\iota+\max\{\varsigma(n),\Theta(n)\}\int_e^g dj}
\end{aligned}$$

$$\leq \inf_{\iota\in[e,g]} \frac{k\iota}{k\iota + k\max\{\varsigma(n),\Theta(n)\}}$$
$$\leq \frac{\iota}{\iota + \max\{\varsigma(n),\Theta(n)}$$
$$= \mathfrak{D}(\varsigma(n),\Theta(n),\iota).$$

(v) Hence, $\mathfrak{Y}$ is an $\perp$-contraction.
(vi) Suppose $\{\varsigma_n\}$ is an orthogonal sequence in $\mathfrak{K}$ such that $\{\varsigma_n\}$ converges to $x \in \mathfrak{K}$. Because $\mathfrak{Y}$ is $\perp$-preserving, $\{\mathfrak{Y}\varsigma_n\}$ is an orthogonal sequence for each $n \in \mathbb{D}$. From (ii), we have
$\mathfrak{A}\left(\varsigma(n),\Theta(n),k\iota\right) \geq \mathfrak{A}(\varsigma(n),\Theta(n),\iota)$,
$\mathfrak{I}(\varsigma(n),\Theta(n),k\iota) \leq \mathfrak{I}(\varsigma(n),\Theta(n),\iota)$ and
$\mathfrak{D}(\varsigma(n),\Theta(n),k\iota) \leq \mathfrak{D}(\varsigma(n),\Theta(n),\iota)$
As $\lim_{n\to\infty} \mathfrak{A}(\varsigma(n),\Theta(n),\iota)$, $\lim_{n\to\infty} \mathfrak{I}(\varsigma(n),\Theta(n),\iota)$ and $\lim_{n\to\infty} \mathfrak{D}(\varsigma(n),\Theta(n),\iota)$ is finite and exists, for all $\iota > 0$, it is clear that
$\lim_{n\to\infty} \mathfrak{A}(\varsigma(n),\Theta(n),k\iota)$, $\lim_{n\to\infty} \mathfrak{I}(\varsigma(n),\Theta(n),k\iota)$ and $\lim_{n\to\infty} \mathfrak{D}(\varsigma(n),\Theta(n),k\iota)$ is finite and exists.
Hence, $\mathfrak{Y}$ is $\perp$-continuous.
Consequently, every condition of Theorem (3.4) is met. As the consequence, there is only one fixed point for the operator F. This demonstrates that there's only one solution to the fuzzy Fredholm-type integral equation (4.1). □

Corollary 4.2. *Let* $(\mathfrak{K},\mathfrak{A},\mathfrak{I},\mathfrak{D},*,\diamond,\perp)$ *be a G-complete* $\mathfrak{NODCMS}$*. Define* $\mathfrak{Y}:\mathfrak{K}\to\mathfrak{K}$ *by*
$\mathfrak{Y}\varsigma(n) = \varphi(c) + \mu\int_e^g (n,c)e(n)dc$ *for all* $n, c \in [e,g]$.
Suppose the following conditions hold:
(I) $\max\{(n,c)\varsigma(n),(n,c)\Theta(n)\} \leq \max\{\varsigma(n),\Theta(n)\}$ *for* $x, d \in \mathfrak{K}, k \in (0,1)$ *and* $\forall n, c \in [e,g]$.
(II) $\int_e^g dc = g - e \leq k < 1$.
Then, there is a solution to the integral equation (1) has a solution.

Proof. From Theorem (4.1), we can readily demonstrate it. □

5 Conclusion

The present paper specifies several fixed point theorems and the required criteria for a sequence to be Cauchy from the point perspective of Neutrosophic orthogonal pentagonal controlled fuzzy metric spaces. In turn, we streamlined the proofs of some fixed point theorems using pentagonal controlled fuzzy metric spaces with the well-known contraction conditions via orthogonality. We additionally addressed the Fredholm integral equation and how network traffic flow uses it. We aspire to further enhance our findings in the conceptual frameworks of neutrosophic orthogonal hexagonal controlled fuzzy metric spaces and neutrosophic orthogonal n-controlled metric spaces.

Conflicts of Interest

The authors declare that the publication of this paper has no conflicts of interest.

Author Contribution

Investigation M.R., M.J., and R.S.; methodology, M.R., M.J., and R.S.; supervision, M.R., M.J., and R.S.; writing-original draft, M.R., M.J., and R.S.; writing-review and editing, M.R., M.J., and R.S. All authors contributed to various stages of the study's design and execution and have reviewed and approved the final manuscript for submission.

Data Availability

No data were used to support this study.

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