Fuzzy Bounded Linear Operators on Fuzzy Anti-Normed Spaces

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Abstract

The primary goal of this paper is to study and introduce fuzzy anti-normed linear spaces, as well as, some additional properties concerning these spaces. From this point of view, some theoretical results are obtained; for example, it was proved that the space of all linear and fuzzy bounded operators over fuzzy anti-normed linear spaces is fuzzy complete. Moreover, some additional theoretical results are stated and proved.

Keywords: Fuzzy bounded operator; Fuzzy anti-normed linear space; Fuzzy anti-norm for bounded operator; Fuzzy anti-metric spaces

1. Introduction

Finite dimensional fuzzy normed linear spaces were first described by Felbin in 1992 [1] and then he proved the completeness of the fuzzy normed space's and finite dimensional subspaces, as well as, Felbin proved that the fuzzy norm and fuzzy equivalency are the same. In 1994, Cheng and Mordeson [2] studied fuzzy normed linear spaces and fuzzy linear operators and then in the same year they described the algebraic structure and continuity of the fuzzy linear operator of a linear space. A theorem of common fixed-point in M-fuzzy metric spaces for a certain property is stated and proved by Sedghi and Shobe in 2006 [3]. The fuzzy norms of strongly and weakly fuzzy bounded linear operators are defined by Bag and Samanta in 2008 [4], which are used as the fundamental concept for establishing the fuzzy Hahn-Banach theorem while the concept of 2-fuzzy 2-normed linear spaces based on α-level 2-norm sets are defined and extended by Somasundaram and Beaula in 2009 [5]. Janfada et al. [6] introduced some further features of strongly and weakly fuzzy bounded linear operators back in 2011. Reddy examined the FANLS in 2011 and reported a number of findings, including completeness, boundedness, etc., [7].

As a continuation of the topics mentioned above and within the general trends of fuzzy set theory, Jebrial and Samanta [8] presented the notion of fuzzy anti-normed linear space (FANLS) in 2010. Later on, in 2012 [9], Dina et al. modified the definition of fuzzy anti-norm, which is termed as fuzzy ant-norm with relative to a t-conorm. In 2018 [10], Beaula and Mariya generalized the concept of fuzzy anti-norm for 2-fuzzy 2-anti normed linear space and studying the continuity and some of its other kinds. An article presented in 2018 by Kočinac including the finite dimensional fuzzy anti-normed linear space topological properties are studied, including anti-convergence, statistical fuzzy anti-convergence and boundedness [11]. Then after fuzzy anti- inner product spaces and their properties are studied by different researchers, such as Sinha in 2019 [12], Ali and Hussein in 2020 [13].

The main theme of this article is to introduce fuzzy anti-normed linear operators, which will provide the basis for constructing operator FANLSs. In addition, some additional related properties are presented for completeness purpose.

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2. Basic Definition

In order to make the fundamental concepts and the obtained results of this article as understandable to the reader as possible, as well as, self-contained. In this section some, but not all, of the basic concepts of the research topics will be introduced [15-22].

Definition (2.1), [1]: Suppose that Y represents a linear space over a real field $\mathbb{K}$. If a fuzzy subset of $Y \times \mathbb{R}$ satisfies the following requirements, it is referred to as a fuzzy anti-norm (AN) on Y. If for all $y, u \in Y$:

1. $AN(y, \alpha) = 1, \forall \alpha \in \mathbb{R}$, with $\alpha \leq 0$.
2. For all $\alpha \in \mathbb{R}$, with $\alpha > 0$, $AN(y, \alpha) = 0$ if and only if $x = 0$.
3. For all $\alpha \in \mathbb{R}$, with $\alpha > 0$, $AN(\sigma y, t) = AN\left(y, \frac{t}{|\sigma|}\right), \forall \alpha \in \mathbb{K}, \sigma \neq 0$.
4. For all $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$ and $\beta > 0$, $AN(u + y, \alpha + \beta) \leq \max\{AN(u, \alpha), AN(y, \beta)\}$.
5. $AN(y, \alpha)$ is non-increasing function of $\alpha \in \mathbb{R}$ and $\lim_{\alpha \to \infty} AN(y, \alpha) = 0$.

Next, over the linear space X, the pair (X, AN) is referred to as (FANLS).

Example (2.2), [7]: Let $(X, \|.\|)$ be a normed space. Define:

$$AN(x, t) = \begin{cases} \frac{\|x\|}{t + \|x\|} \text{ if } t > 0 \\ 1 \text{ if } t \leq 0 \end{cases}$$

Then (X, AN) is FANLS.

Definition (2.3), [14]: Assume that (X, AN) is a FANLS and (y_k) be a sequence in Y. We say that (y_k) is convergent sequence if there exists $y \in Y$, satisfies $AN(y_k - y, \alpha) < \theta \ \forall \alpha > 0$ and $0 < \theta < 1, k \geq N$, or $AN(y_k - y, \alpha) \longrightarrow 0$, as $k \longrightarrow \infty$ or $\lim_{k \to \infty} AN(y_k, \alpha) = y$.

Definition (2.4), [14]: Assume that (X, AN) is a FANLS and (y_k) be a sequence in Y. We say that (y_k) is a Cauchy sequence (CS) if $AN(y_m - y_n, \alpha) < \theta, \forall \alpha > 0$ and $0 < \theta < 1$, $m, n \geq N$.

Definition (2.5), [10]: Assume that X and Y are two FANLSs and $\tilde{L} : X \longrightarrow Y$ is an operator. We say $\tilde{L}$ is fuzzy-anti-continuous at $x_0 \in X$ if $\forall\, 0 < \alpha < 1, \exists\, 0 < \beta < 1$, $AN(x - x_0, t) < \beta$ implies $AN(\tilde{L}x - \tilde{L}x_0, t) < \alpha, \forall t > 0$.

Theorem (2.6): If (X, AN_X) and (Y, AN_Y) are two FANLSs, then $(X \times Y, AN_{X \times Y})$ is FANLS when $AN_{X \times Y}((u, v), \alpha) = \max\{AN_X(u, \alpha), AN_Y(v, \alpha)\}, \forall (u, v) \in X \times Y$ and $\alpha > 0$.

Proof:

1. For all $t \leq 0$, $AN_{X \times Y}((u, v), \alpha) = 1$, since $AN_X(u, \alpha) = 1$ and $AN_Y(v, \alpha) = 1$.

2. For all $t > 0$, $AN_{X \times Y}((u, v), \alpha) = 0$

$\Leftrightarrow \max\{AN_X(u, \alpha), AN_Y(v, \alpha)\} = 0$

$\Leftrightarrow AN_X(u, \alpha) = 0$ and $AN_Y(v, \alpha)\} = 0$

$\Leftrightarrow u = 0$ and $v = 0$

$\Leftrightarrow (u, v) = (0, 0)$.

3. $AN_{X \times Y}(\sigma(u, v), \alpha) = AN_{X \times Y}((\sigma u, \sigma v), t)$

$= \max\{AN_X(\sigma u, \alpha), AN_Y(\sigma v, \alpha)\}$

$= \max\{AN_X(u, \frac{\alpha}{|\sigma|}), AN_Y(v, \frac{\alpha}{|\sigma|})\} = AN_{X \times Y}((u, v), \frac{\alpha}{|\sigma|})$

$\forall \alpha > 0$ and $\forall \sigma \in \mathbb{R}$ with $\sigma \neq 0$.

4. For all $\alpha, \beta > 0$, $AN_{X \times Y}((u_1, v_1) + (u_2, v_2), \alpha + \beta) \leq \max\{AN_{X \times Y}((u_1, v_1), \alpha), AN_{X \times Y}((u_2, v_2), \beta)$;

$AN_{X \times Y}((u_1, v_1) + (u_2, v_2), \alpha + \beta) = AN_{X \times Y}((u_1 + u_2, v_1 + v_2), \alpha + \beta)$

$= \max\{AN_X(u_1 + u_2, \alpha + \beta), AN_Y(v_1 + v_2, \alpha + \beta)\}$

$= \max\{AN_X(u_1, \alpha), AN_X(u_2, \beta), AN_Y(v_1, \alpha), AN_Y(v_2, \beta)\}$

$$= \max\{AN_{X\times Y}((u_1, v_1), \alpha), AN_{X\times Y}((u_2, v_2), \beta);$$

5. $AN_{X\times Y}((u, v), \alpha)$ is non-increasing function of α since $AN_X(u, \alpha)$ and $AN_Y(v, \alpha)\}$ are non-increasing function of t. Also $\lim\limits_{\alpha\to\infty} AN_{X\times Y}((u, v), \alpha) = 0$ since $\lim\limits_{\alpha\to\infty} AN_X(u, \alpha) = 0$ and $\lim\limits_{\alpha\to\infty} AN_Y(v, \alpha) = 0$. Hence $(X\times Y, AN_{X\times Y})$ is FANLS.

In a similar way, we can prove the next two corollaries:

Corollary (2.7): If (X_1, AN_1), (X_2, AN_2), …, (X_k, AN_k) are k-FANLSs, then (X, AN_X) is FANLS where $X = X_1 \times X_2 \times \cdots \times X_k$ and $AN_X((x_1, x_2, \dots, x_k), \alpha) = \max\{AN_1(x_1, \alpha), AN_2(x_2, \alpha), \dots, AN_k(x_k, \alpha)\}, \forall\, u = (x_1, x_2, \dots, x_k) \in X$ and $\alpha > 0$.

Corollary (2.8): If (X, AN) is FANLS, then (X^k, AN_k) is FANLS, where $AN_k((x_1, x_2, \dots, x_k), t) = \max\{AN(x_1, t), AN(x_2, t), \dots, AN(x_k, t)\}, \forall\, x = (x_1, x_2, \dots, x_k) \in X^k$ and $t > 0$.

Definition (2.9), [9]: The FANLS (X, AN) is complete if every Cauchy sequence in X converge to a vector in X.

Definition (2.10): Let $X \neq \emptyset$ and if the fuzzy set $AM\colon X\times X \times \mathbb{R} \longrightarrow [0, 1]$ satisfies for all $y, p, w \in X$:

1. $AM(y, w), \alpha) = 1$ if $\alpha < 0$;

2. For all $\alpha > 0$, $AM(y, w, \alpha) = 0$ if and only if $y = w$;

3. For all $\alpha > 0$, $AM(y, w, \alpha) = AM(w, y, \alpha)$;

4. For all $\alpha > 0, \beta > 0$, $AM(y, w, \alpha + \beta) \leq \max\{AM(y, p, \alpha) + AM(p, w, \beta)\}$

Then (X, AM) is a fuzzy anti-metric space (or simply FAMS).

Example (2.11): Let (X, d) be a metric space, define $AN(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$, then (X, AN) is FAMS.

Proof: It is clear and hence is omitted.

Example (2.12): Let (Y, d) be a metric space, define $AN(w, z, \alpha) = \exp\left[\frac{-d(w,z)}{\alpha}\right]$, then (Y, AN) is FAMS.

Proof: It is clear and hence is omitted.

Theorem (2.13): If (X, AN) is a FANS, then (X, AM_X) is FAMS, where $AM_X(w, z, \alpha) = AN(w - z, \alpha), \forall\, w, z \in X$.

Proof:

1. For all $\alpha < 0$, $AM_X(w, z, \alpha) = 1$.

2. For all $\alpha > 0$, $AM_X(w, z, \alpha) = 0$ if and only if $AN(w - z, \alpha) = 0$ if and only if $w - z = 0$, or equivalently $w = z$.

3. For all $\alpha > 0$, $AM_X(w, z, \alpha) = AN(w - z, \alpha) = AN\left(z - w, \frac{\alpha}{|-1|}\right) = AM_X(z, w, \alpha)$.

4. For all $\alpha > 0$ and $\beta > 0$,

$$AM_X(w, z, \alpha+\beta) = AN(w - z, \alpha+\beta) = AN(w - y + y - z, \alpha+\beta)$$

$$\leq AN(w - y, \alpha) + AN(y - z, \beta) = AM_X(w, y, \alpha) + AN(y, z, \beta)$$

Hence (X, AM_X) is FAMS.

Lemma (2.14): The FAM AM_X induced by the FAN, AN satisfies for all $u, v, z \in X$ and $\sigma \in \mathbb{R}$ with $\sigma \neq 0$:

1. $AM_X(u + z, v + z, t) = AM_X(u, v, t)$.

2. $AM_X(\sigma u, \sigma v, t) = AM_X\left(u, v, \frac{t}{|\sigma|}\right)$.

Theorem (2.15): If (X_1, AN_1) and (X_2, AN_2) are two FANSs, then (X, AN_X) is fuzzy complete FANS if and only if (X_1, AN_1) and (X_2, AN_2) are fuzzy complete metric spaces, where $X = X_1 \times X_2$ and

$$AN_X[(u_1, u_2), t] = \max\{AN_1(u_1, t), AN_2(u_2, t)\}, \forall\, (u_1, u_2) \in X, t > 0.$$

Proof: Let (X_1, AN_1) and (X_2, AN_2) are two fuzzy complete FANS. Let (u_k) be fuzzy Cauchy sequence in X, then $(u_k) = (u_{1k}, u_{2k})$, where $(u_{1k}) \in X_1$ and $(u_{2k}) \in X_2$. Hence $AN_X(u_k - u_m, t) \longrightarrow 0$ as k approaches to ∞ and m approaches to ∞.

Thus, $\max[AN_1(u_{1k} - u_{1m}, t), AN_2(u_{2k} - u_{2m}, t)] \longrightarrow 0$ when k approaches to ∞ and m approaches to ∞.

Hence, $AN_1(u_{1k} - u_{1m}, t) \longrightarrow 0$ in (X_1, AN_1) when k approaches to ∞ and m approaches to ∞ and $AN_2(u_{2k} - u_{2m}, t) \longrightarrow 0$ in (X_2, AN_2), when k approaches to ∞ and m approaches to ∞.

Therefore, (u_{1k}) is fuzzy Cauchy sequence in (X_1, AN_1) and (u_{2k}) is fuzzy Cauchy sequence in (X_2, AN_2), but (X_1, AN_1) and $(X_2, AN_2,)$ are fuzzy complete so there is $u_1 \in X_1$ and $u_2 \in X_2$, such that (u_{1k}) fuzzy converges to $u_1 \in X_1$and (u_{2k}) fuzzy converges to $u_2 \in X_2$. Put $u = (u_1, u_2)$

Then $u \in X$ and (u_k) fuzzy converges to $u \in X$, since

$$AN_X(u_k - u, t) = AN_X[(u_{1k}, u_{2k}) - (u_1, u_2), t]$$

$$= AN_X[(u_{1k} - u_1) + (u_{2k} - u_2), t]$$

$$= \max\{ AN_1[(u_{1k} - u_1), t] + AN_1[(u_{2k} - u_2), t]\}$$

By taking limit to both sides as k approaches to ∞ we have $AN_X(u_k - u, t) \longrightarrow 0$.

Conversely assume that (X, AN_X) is fuzzy complete we will prove that (X_1, AN_1) and (X_2, AN_2) are fuzzy complete. Let (u_{1k}) is fuzzy Cauchy sequence in (X_1, AN_1) and (u_{2k}) is fuzzy Cauchy sequence in (X_2, AN_2).

Then $AN_1(u_{1k} - u_{1m}, t) \longrightarrow 0$ in (X_1, AN_1) when k approaches to ∞ and m approaches to ∞ and $AN_2(u_{2k} - u_{2m}, t) \longrightarrow 0$ in (X_2, AN_2), when k approaches to ∞ and m approaches to ∞.

Put $(u_k) = (u_{1k}, u_{2k})$ where $(u_{1k}) \in X_1$ and $(u_{2k}) \in X_2$.

Since $AN_X(u_k - u_m, t) = \max\{AN_1(u_{1k} - u_{1m}, t), AN_2(u_{2k} - u_{2m}, t)\} \longrightarrow 0$, when k approaches to ∞ and m approaches to ∞. Hence (u_k) is fuzzy Cauchy sequence in X, but X is fuzzy complete so there is $u = (u_1, u_2) \in X$, such that $AN_X(u_k - u, t)$ fuzzy converges to zero. But:

$$AN_X(u_k - u, t) = AN_X[(u_{1k}, u_{2k}) - (u_1, u_2), t]$$

$$= AN_X[(u_{1k} - u_1) + (u_{2k} - u_2), t]$$

$$= \max\{AN_1[(u_{1k} - u_1), t], AN_2[(u_{2k} - u_2), t]\}$$

So $AN_1[(u_{1k} - u_1), t]$ is fuzzy converges to zero in (X_1, AN_1) as k approaches to ∞ and $AN_2[(u_{2k} - u_2), t]$ fuzzy converges to zero in (X_2, AN_2) as k approaches to ∞.

Hence (u_{1k}) fuzzy converges to $u_1 \in X_1$ as $k \longrightarrow \infty$ and (u_{2k}) fuzzy converges to $u_2 \in X_2$ as k approaches to ∞.

It follows that (X_1, AN_1) and (X_2, AN_2) are fuzzy complete.

In similar way we can prove the next result

Corollary (2.16): If (X_1, AN_1), (X_2, AN_2), …, (X_k, AN_k) are FANS, then (X, AN_X) is fuzzy complete FANS if and only if (X_1, AN_1), (X_2, AN_2), …, (X_k, AN_k) are fuzzy complete, where $X = X_1 \times X_2 \times \ldots \times X_k$ and

$$AN_X[(u_1, u_2, \ldots, u_k), t] = \max\{AN_1(u_1, t), AN_2(u_2, t), \ldots, AN_k(u_k, t)\}$$

for all $(u_1, u_2, \ldots, u_k) \in X$ and $t > 0$.

In similar way we can prove the next result:

Corollary (2.17): If (X, AN) is FANS, then (X^k, AN_k) is fuzzy complete FANS if and only if (X, AN) fuzzy complete where $X^k = X \times X \times \ldots \times X$ (k-times), $k \in \mathbb{N}$ and

$$AN_k[(u_1, u_2, \ldots, u_k), t] = AN(u_1, t) + AN(u_2, t) + \ldots + AN(u_k, t)$$

for all $(u_1, u_2, \ldots, u_k) \in X^k$ and $t > 0$.

Definition (2.18): If (X, AN_X) and (Y, AN_Y) are two FANS. Then:

1. The operator: $X \longrightarrow Y$ is called fuzzy continuous at $\mathcal{u} \in X$. If for all $\alpha \in (0, 1), \exists\, \beta \in (0, 1)$, if $AN_Y[\mathcal{T}(\mathcal{u}) - \mathcal{T}(\mathcal{v}), \sigma] < \alpha$, for any $\mathcal{v} \in X$ satisfying $AN_X(\mathcal{u} - \mathcal{v}, \sigma) < \beta, \sigma > 0$.

2. If (1) is true $\forall\, \mathcal{u} \in X$, then $\mathcal{T}$ is fuzzy continuous on X.

Theorem (2.19): If (X, AN_X), (Y, AN_Y) are FANS. Then the operator : $X \longrightarrow Y$ is fuzzy continuous at $\mathcal{u} \in X$ if and only if whenever $(\mathcal{u}_k)$ is fuzzy converges to $\mathcal{u} \in X$, then $(\mathcal{T}(\mathcal{u}_k))$ is fuzzy converges to $\mathcal{T}(\mathcal{u}) \in Y$.

Proof: Let $(\mathcal{u}_k) \in X$, where $\mathcal{u}_k \longmapsto \mathcal{u}$ and : $X \longrightarrow Y$ is fuzzy continuous at $\mathcal{u} \in X$.

Let $\alpha \in (0, 1)$, then $\exists\ \beta \in (0, 1)$ and $\forall\ v \in X$ satisfying $AN_X(u - v, t) < \beta$, and hence $AN_Y[\mathcal{T}(u) - \mathcal{T}(v), t] < \alpha$. Since $u_k \mapsto u$, then $\exists\ N \in \mathbb{N}$ if $k \geq N$, which implies $AN_X(u_k - u, t) < \beta$.

Therefore $k \geq N$ implies $AN_Y[\mathcal{T}(u_k) - \mathcal{T}(u), t] < \alpha$. Thus $\mathcal{T}(u_k) \longrightarrow \mathcal{T}(u)$.

For the converse let $u_k \mapsto u$ in X implies $\mathcal{T}(u_k) \mapsto \mathcal{T}(u)$ in Y.

To prove that $: X \longrightarrow Y$ is fuzzy continuous at $u \in X$. If $: X \longrightarrow Y$ is not fuzzy continuous at $u \in X$, then $\forall\ \beta, \beta \in (0, 1), \exists\ w \in X$ satisfying $AN_X(u - w, t) < \beta$, but $AN_Y[\mathcal{T}(u) - \mathcal{T}(w), t] \geq \alpha$.

Now $\forall\ k \in N, \exists\ u_k$ in X, such that $AN_X(u_k - u, t) < \frac{1}{k}$, but $AN_Y[\mathcal{T}(u_k) - \mathcal{T}(u), t] \geq \alpha$.

That is $u_k \mapsto u$ in X but $\mathcal{T}(u_k) \mapsto \mathcal{T}(u)$. Thus $\mathcal{T}$ must be fuzzy continuous at $u \in X$.

Theorem (2.20): Let (X, AN_X), (Y, AN_Y) and (Z, AN_Z) be FANS and if $\mathcal{T} : X \longrightarrow Y$ and $\mathcal{S} : Y \longrightarrow W$ are fuzzy continuous operators then $\mathcal{S} \circ \mathcal{T} : X \longrightarrow Z$ is a fuzzy continuous operator.

Proof: If (u_k) is fuzzy sequence which is converge to $u \in X$, then by Theorem 2.19, $(\mathcal{T} u_k)$ is fuzzy converge sequence to $(u) \in Y$ and using Theorem 2.19 again, we have $\mathcal{S}(\mathcal{T} u_k)$ is a fuzzy convergent sequence to $\mathcal{S}(u) \in Z$ or $(\mathcal{S} \circ \mathcal{T})(u_k)$ is fuzzy convergent sequence to $(\mathcal{S} \circ \mathcal{T})(u) \in Z$.

Thus $\mathcal{S} \circ \mathcal{T}$ is fuzzy continuous.

Theorem (2.21): If (X, AN) is FANS, then the function $u \longmapsto AN(u, t)$ is a fuzzy continuous function from (X, AN) onto $(\mathbb{R}, |.|)$.

Proof: Let $(u_k) \in X$ with $u_k \longmapsto u$ so that $\lim\limits_{k \to \infty} AN(u_k - u, t) = 0$. Now:

$$|AN(u_k, t) - AN(u, t)]\,| \leq AN(u_k - u, t)$$

and upon taking the limit to the both sides of the above inequality as $k \longrightarrow \infty$, implies to:

$$\lim_{k \to \infty} |AN(u_k, t) - AN(u, t)\,| \leq \lim_{k \to \infty} AN(u_k - u, t) = 0$$

So, $AN(u_k, t) \longrightarrow AN(u, t)$.

Hence $u \longmapsto AN(u)$ is a fuzzy continuous function from (X, AN) onto $(\mathbb{R}, |.|)$.

Theorem (2.22): Let (X, AN) be FANS, then:

1. The addition of two fuzzy continuous functions is a fuzzy continuous function.

2. The scalar multiplication is fuzzy continuous function.

Proof:

1. If $u_k \longmapsto u$ and $v_k \longmapsto v$, then as $k \longrightarrow \infty$

$$AN[(u_k + v_k) - (u + v), t+s] \leq \max\{AN(u_k - u, t), AN(v_k - v, s)\}$$

taking the limit to the both sides as $k \longrightarrow \infty$, getting:

$$AN[(u_k + v_k) - (u + v), t + s] \leq \max\{0, 0\} = 0$$

Hence the function $(u, v) \longmapsto (u + v)$ is fuzzy continuous function.

2. If $u_k \longmapsto u$ and $\alpha_k \longmapsto \alpha$, then:

$$AN[\alpha_k u_k - \alpha u, 2t] = \mathfrak{N}[\alpha_k u_k - \alpha_k u + \alpha_k u - \alpha u, 2t]$$
$$\leq \max\{AN[\alpha_k(u_k - u), t], AN[(\alpha_k - \alpha)u, t]\}$$

Now:

$$AN[\alpha_k u_k - \alpha u, 2t] \leq \max\left\{AN\left(u_k - u, \frac{t}{|\alpha_k|}\right), AN\left(u, \frac{t}{|\alpha_k - \alpha|}\right)\right\}$$

Taking limit to both sides as $k \longrightarrow \infty$

$$AN[\alpha_k u_k - \alpha u] \leq \max\{0, 0\} = 0$$

This proves the scalar multiplication is fuzzy continuous function.

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Proposition (2.23): The operator $\mathcal{T}$ of FANS (X, AN_X) onto FANS (Y, AN_Y) is fuzzy continuous at a point $u \in X$ if and only if $\forall\ 0 < \alpha < 1, \exists\ 0 < \beta < 1$, such that, $fb(u, \beta) \subseteq \mathcal{T}^{-1}[fb(\mathcal{T}(u), \alpha)]$.

Proof: The operator $\mathcal{T} : X \longrightarrow Y$ is fuzzy continuous at $u \in X$ if and only if $\forall\ 0 < \alpha < 1, \exists\ 0 < \beta < 1$, such that $AN_Y\ [\mathcal{T}(u) - \mathcal{T}(w),\ t] < \alpha$, for all w satisfying $AN_X(u - w, t) < \beta$ that is $w \in fb(u, \beta)$ implies $\mathcal{T}(w) \in fb(\mathcal{T}(u), \alpha)$ or $\mathcal{T}[fb(u, \beta)] \subseteq fb(\mathcal{T}(u), \alpha)]$.

Thus $fb(u, \beta) \subseteq \mathcal{T}^{-1}[fb(\mathcal{T}(u), \alpha)]$.

Theorem (2.24): If (X, AN_X) and (Y, AN_Y) are FANS, then the following statements are equivalent:

(i) The operator $: X \longrightarrow Y$ is fuzzy continuous on X.

(ii) $\mathcal{T}^{-1}(\mathcal{P})$ is fuzzy open in X for all fuzzy open subset $\mathcal{P}$ of Y.

(iii) $\mathcal{T}^{-1}(D)$ is fuzzy closed in X for all fuzzy closed subset D of Y.

Proof:(i)$\Longrightarrow$(ii). Let $\mathcal{T}$:X→Y be fuzzy continuous on X and let $\mathcal{P} \subset Y$ be fuzzy open. If $\mathcal{T}^{-1}(\mathcal{P})$ is empty or $\mathcal{T}^{-1}(\mathcal{P})$ equal to X then (ii) is satisfied. Thus for $\mathcal{T}^{-1}(\mathcal{P})$ is not empty and $\mathcal{T}^{-1}(\mathcal{P})$ not equal to X. Assume that $p \in \mathcal{T}^{-1}(\mathcal{P})$. Then $\mathcal{T}(p) \in \mathcal{P}$. Thus $\exists\ \alpha \in (0, 1)$ satisfying $fb(\mathcal{T}(p), \alpha) \subseteq \mathcal{P}$. Since $\mathcal{T}$ is fuzzy continuous at p, using 2.23 for this α, $\exists\ \beta \in (0, 1)$ satisfying $fb(p, \beta) \subseteq \mathcal{T}^{-1}\ [fb(\mathcal{T}(p), \alpha]) \subseteq \mathcal{T}^{-1}(\mathcal{P})$. Thus $\mathcal{T}^{-1}(\mathcal{P})$ is fuzzy open in X.

(ii)$\Longrightarrow$(iii). If $D \subset Y$ is a fuzzy closed then $Y - D \subset Y$ is fuzzy open so that $\mathcal{T}^{-1}(Y - D) \subset X$ is fuzzy open by (ii) But $\mathcal{T}^{-1}(Y - D) = X - \mathcal{T}^{-1}(D)$ so $\mathcal{T}^{-1}(D) \subset X$ is fuzzy closed.

(iii)$\Longrightarrow$(i). If $D \subset Y$ is fuzzy closed then $Y - D \subset Y$ is fuzzy open. Let $\mathcal{T}(x) \in (Y - D)$, for each $0 < \alpha < 1$ since the fuzzy ball $fb(\mathcal{T}(x), \alpha)$ is fuzzy open set in Y and so $\mathcal{T}^{-1}[fb(\mathcal{T}(x), \alpha)]$ is fuzzy open in X. Whenever $x \in \mathcal{T}^{-1}[cfb(\mathcal{T}(x), \alpha)]$ it implied that we can find $0 < \beta < 1$ with $fb(x, \beta) \subseteq \mathcal{T}^{-1}[fb(\mathcal{T}(x), \alpha)]$.

Hence $\mathcal{T}$ is fuzzy continuous of X by Proposition 2.23.

3. Fuzzy Bounded Linear Operators over FANLS

In this section, the main results related to this article are stated and proved. These results can be considered the fundamental cornerstone for future studies on this topic.

Definition (3.1): If (X, AN_X) and (Y, AN_Y) are two FANLS and $\tilde{L}: X \longrightarrow Y$ is an operator, then $\tilde{L}$ is called fuzzy bounded (FB) operator if for some $t > 0$ and $0 < c < 1$, with:

$$AN_Y(\tilde{L}x, t) \leq c\ AN_X(x)$$

Remark (3.2): If (X, AN_X) and (Y, AN_Y) are FANLS, then the set of all bounded linear operators from X to Y and it is shortened as $B^*(X, Y)$. It is clear that $B^*(X, Y)$ is a linear space over the field $\mathbb{K}$.

Theorem (3.3): The triple $(B^*(X, Y), OAN)$ is FANLS, where $OAN : B^*(X, Y) \times \mathbb{R} \longrightarrow [0, 1]$ is a fuzzy set defined by: $OAN(\tilde{L}, t) = \sup AN(\tilde{L}x, t)$ for all $x \in X$ with $AN(x) = 1$ and for all $t > 0$.

Proof:

1. For all $t \leq 0$, $OAN(\tilde{L}, t) = 1$ since $AN(\tilde{L}x, t) = 1$, for all $x \in X$ with $AN(x) = 1$.
2. For all $t \in \mathbb{R}$, with $t \geq 0$, $OAN(\tilde{L}, t) = 0$ if and only if $\sup AN(\tilde{L}x, t) = 0, \forall\ x \in X$ satisfying $AN(x) = 1$ if and only if $AN(\tilde{L}x, t) = 0, \forall\ x \in X$ satisfying $AN(x) = 1$, which is equivalent to $\tilde{L}x = 0, \forall\ x \in X$ if and only if $\tilde{L} = 0$.
3. For all $t \geq 0$, with $t \in \mathbb{R}$, $OAN(\alpha\tilde{L}, t) = \sup AN(\alpha\tilde{L}x, t), \forall\ x \in X$ satisfying $AN(x) = 1$, with $\alpha \in \mathbb{K}$ and $\alpha \neq 0$. $OAN(\alpha\tilde{L}, t) = \sup AN\left(\tilde{L}x, \frac{t}{|\alpha|}\right), \forall\ x \in X$ satisfying $AN(x) = 1$ with $\alpha \in \mathbb{K}, \alpha \neq 0$.

$$OAN(\alpha\tilde{L}, t) = OAN\left(\tilde{L}, \frac{t}{|\alpha|}\right), \forall\ x \in X \text{ satisfying } AN(x) = 1, \text{ with } \alpha \in \mathbb{K}, \alpha \neq 0.$$

4. $OAN(\tilde{L} + \tilde{T}, t + s) \leq \max\{OAN(\tilde{L}, t), OAN(\tilde{T}, s)\}, \forall\ \tilde{L}, \tilde{T} \in B^*(X, Y)$.

Thus, for all $x \in X$ satisfying $AN(x) = 1$ and hence:

$$\begin{aligned} OAN(\tilde{L} + \tilde{T}(x), t + s) &= \sup AN(\tilde{L} + \tilde{T})(x), t + s) \\ &= \sup AN(\tilde{L}\,x + \tilde{T}x), t + s) \\ &\leq \max\{\sup AN(\tilde{L}x, t), \sup AN(\tilde{T}x, s)\} \end{aligned}$$

$$\leq \max\{\mathrm{OAN}(\tilde{L}, t), \mathrm{OAN}(\tilde{T}, s)\}$$

5. $\mathrm{OAN}(\tilde{L}, p)$ is non-increasing function of $p \in \mathbb{R}$ and $\lim_{t\to\infty} \mathrm{OAN}(\tilde{L}, t) = 0$, since $\lim_{t\to\infty} \mathrm{AN}(x, t) = 0$.

Example (3.4): Let $(B(X, Y), \|.\|)$ be a normed space, then $(B^{*}(X, Y), \mathrm{OAN})$ is FANLS, where:

$$\mathrm{OAN}(\tilde{L}, t) = \begin{cases} \frac{\|\tilde{L}\|}{t+\|\tilde{L}\|}, & for\ t > 0 \\ 1, & for\ t \leq 0 \end{cases}$$

Proof:

1. For all $t \leq 0$, by the definition given above, we obtain $\mathrm{OAN}(\tilde{L}, t) = 1$.

2. For all $t > 0$, $\mathrm{OAN}(\tilde{L}, t) = 0$, which is equivalent to $\|\tilde{L}\| = 0$, i.e., $\tilde{L} = 0$.

3. For all $t > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, then:

$$\mathrm{OAN}(\alpha\tilde{L}, t) = \frac{\|\alpha\tilde{L}\|}{t+\|\alpha\tilde{L}\|}$$

$$= \frac{|\alpha|\|\tilde{L}\|}{t+|\alpha|\|\tilde{L}\|}$$

$$= \frac{\|\tilde{L}\|}{\frac{t}{|\alpha|}+\|\tilde{L}\|}$$

$$= \mathrm{OAN}\left(\tilde{L}, \frac{t}{|\alpha|}\right)$$

4. For all $t, s > 0$, then $(t+s) > 0$ and hence we have to prove that:

$$\mathrm{OAN}(\tilde{L} + \tilde{T}, t + s) \leq \max\{\mathrm{OAN}(\tilde{L}, t), \mathrm{OAN}(\tilde{T}, s)\}$$

Now:

$$\mathrm{OAN}(\tilde{L} + \tilde{T}, t + s) - \mathrm{OAN}(\tilde{L}, t) = \frac{\|\tilde{L}+\tilde{T}\|}{t+s+\|\tilde{L}+\tilde{T}\|} - \frac{\|\tilde{L}\|}{t+\|\tilde{L}\|}$$

$$= \frac{\|\tilde{L}+\tilde{T}\|(t+\|\tilde{L}\|) - \|\tilde{L}\|(t+s+\|\tilde{L}+\tilde{T}\|)}{(t+s+\|\tilde{L}+\tilde{T}\|)(t+\|\tilde{L}\|)}$$

$$\leq \frac{\|\tilde{L}\|}{t+\|\tilde{L}\|}$$

$$= \mathrm{OAN}(\tilde{L}, t)$$

Similarly, $\mathrm{OAN}(\tilde{L} + \tilde{T}, t + s) - \mathrm{OAN}(\tilde{T}, s) \leq \mathrm{OAN}(\tilde{T}, s)$

Hence $\mathrm{OAN}(\tilde{L} + \tilde{T}, t + s) \leq \max\{\mathrm{OAN}(\tilde{L}, t), \mathrm{OAN}(\tilde{T}, s)\}$.

5. If $t, s > 0$ and $t < s$, then we have to prove that $\mathrm{OAN}(\tilde{T}, s) > \mathrm{OAN}(\tilde{L}, t)$. Now:

$$\mathrm{OAN}(\tilde{T}, s) - \mathrm{OAN}(\tilde{L}, t) = \frac{\|\tilde{T}\|}{s+\|\tilde{T}\|} - \frac{\|\tilde{L}\|}{t+\|\tilde{L}\|}$$

$$= \frac{\|\tilde{T}\|(t+\|\tilde{L}\|) - \|\tilde{L}\|(s+\|\tilde{T}\|)}{(s+\|\tilde{T}\|)(t+\|\tilde{L}\|)} > 0$$

Hence $\mathrm{OAN}(\tilde{L}, t) < \mathrm{OAN}(\tilde{T}, s)$.

Thus $\mathrm{OAN}(\tilde{L}, t)$ is non-increasing

Also, $\lim_{t\to\infty} \mathrm{OAN}(\tilde{L}, t) = \lim_{t\to\infty} \frac{\|\tilde{L}\|}{t+\|\tilde{L}\|} = 0$.

Example (3.5): Let $(B(X, Y), \|.\|)$ be a normed space, then $(B^{*}(X, Y), \mathrm{OAN})$ is FANLS, where:

$$\mathrm{OAN}(\tilde{L}, t) = \begin{cases} 0, & for\ t > \|\tilde{L}\| \\ 1, & for\ t \leq \|\tilde{L}\| \end{cases}$$

Theorem (3.6): If Y is a fuzzy complete FANLS, then ($B^*(X, Y)$, OAN) is fuzzy complete.

Proof: If ($\tilde{L}_k$) is a Cauchy sequence in $B^*(X, Y)$, then for all $\alpha \in (0, 1)$, there exists $N \in \mathbb{N}$, satisfying $OAN[\tilde{L}_j - \tilde{L}_m] < \alpha, \forall\, j, m \geq N$. Thus, for $x \in X$ and $j, m \geq N$

$$AN_Y[\tilde{L}_j x - \tilde{L}_m x] < AN_Y[(\tilde{L}_j - \tilde{L}_m)(x)] < \theta AN_X[x] \qquad \ldots(1)$$

Now, for any fixed x and given $\theta_x \in (0, 1)$ rewrite (9) as:

$$AN_Y[\tilde{L}_j x - \tilde{L}_m x] < AN_Y[(\tilde{L}_j - \tilde{L}_m)(x)] < \theta_x AN_X[x]$$

this implies that ($\tilde{L}_k x$) is a Cauchy sequence in Y and since Y is fuzzy complete, so ($\tilde{L}_k x$) converges to $y \in Y$, that is, $\tilde{L}_k x \mapsto y$.

The vector y depends on $x \in X$ and this defines an operator $\tilde{L} : X \longrightarrow Y$ by $\tilde{L}(u) = y$. The operator $\tilde{L}$ is linear, since:

$$\begin{aligned}\tilde{L}[\alpha z + \beta w] &= \lim_{k\to\infty} \tilde{L}_k[\alpha z + \beta w] \\ &= \alpha \lim_{k\to\infty} \tilde{L}_k[z] + \beta \lim_{k\to\infty} \tilde{L}_k[w] \\ &= \alpha\tilde{L}[z] + \beta\tilde{L}[w]\end{aligned}$$

Using (1) to be holds for all $m \geq j$ and $\tilde{L}_m x \mapsto \tilde{L}x$, when m approaches to ∞ and hence (1) for all $j \geq N$ and $\forall\, x \in X$ will imply to:

$$\begin{aligned}AN_Y\left[\tilde{L}_j x - \tilde{L}x\right] &= AN_Y[\tilde{L}_j x - \lim_{m\to\infty} \tilde{L}_m x] \\ &= \lim_{m\to\infty} AN_Y[\tilde{L}_j x - \tilde{L}_m x]\end{aligned}$$

Hence:

$$AN_Y\left[\tilde{L}_j x - \tilde{L}x\right] < \theta\, AN_X[x] \qquad \ldots(2)$$

Therefore, $\tilde{L}_j - \tilde{L}$ with $j \geq N$ is FB using $\tilde{L}_j$ is FB and so $\tilde{L} = \tilde{L}_j - \left(\tilde{L}_j - \tilde{L}\right)$ is FB, that is, $\tilde{L} \in B^*(X, Y)$ from (2) by taking the supremum over all x with $AN(x) = 1$. This will imply to:

$$OAN\,[\tilde{L}_j - \tilde{L}] < \theta, \forall\, j \geq N$$

Thus $\mathcal{T}_k \to \mathcal{T} \in B^*(X, Y)$.

4. Conclusion

In this paper, we have studied and introduced the fuzzy anti-normed linear spaces, as well as, some additional properties concerning these spaces. From this point of view, some theoretical results are obtained; for example, it was proved that the space of all linear and fuzzy bounded operators over fuzzy anti-normed linear spaces is fuzzy complete. Moreover, some additional theoretical results are stated and proved.

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