An Investigation of Complex Linear Diophantine Fuzzy Ideals in BCK-Algebras

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Abstract

A complex linear Diophantine fuzzy ($\mathcal{CLDF}$) set extends a linear Diophantine fuzzy set ($\mathcal{LDFS}$) by handling uncertainty with complex-valued membership degrees within a unit disc. In this paper, we combine the notions of $\mathcal{LDFS}$, BCK-algebra, and complex fuzzy set ($\mathcal{CFS}$) to preface and elaborate the innovative concepts of $\mathcal{CLDF}$ subalgebras ($\mathcal{CLDF}-\mathcal{S}ubs$), $\mathcal{CLDF}$ ideals ($\mathcal{CLDF}-\mathcal{I}ds$), $\mathcal{CLDF}$ implicative ideals ($\mathcal{CLDF}-\mathcal{II}ds$), and $\mathcal{CLDF}$ positive implicative ideals ($\mathcal{CLDF}-\mathcal{PII}ds$) in BCK-algebras, and probe their fundamental characteristics. These new notations of certain kinds of algebraic substructures in BCK-algebras serve as a bridge among $\mathcal{CLDFS}$, crisp set, and BCK-algebra and also demonstrate the influence of the $\mathcal{CLDFS}$ on a BCK-algebra. Moreover, we examine some illustrative examples and prevalent features of these innovative notions in detail. Finally, characterizations of these intricate fuzzy structures are given, and related results for ideals, implicative ideals, and positive implicative ideals in the view of $\mathcal{CLDFS}s$ are obtained.

Keywords: BCK-algebra; Complex linear Diophantine fuzzy set; Complex linear Diophantine fuzzy subalgebra; Complex linear Diophantine fuzzy ideal

1 introduction

In mathematics, BCK-algebras are a significant part of algebra, particularly in the domain of logic and algebraic aspects related to non-classical substructural logic.[1–3] BCK-algebras serve as development concepts in different mathematical domains. In substructural logics, BCK-algebras are utilized to model logics that lack certain structural rules, like contraction or weakening found in crisp (classical) logic. In mathematical logic, BCK-algebras provide tools for deliberating about alternative logical frameworks that may be more applicable to specific situations in some branches of computer science, philosophy, and linguistics.

The fuzziness of certain topics in pure and applied mathematics is a ramification of mathematics that deals with reasoning and systems where the notions of membership or truth are not binary (i.e., in or out, true or false). Instead of a strict yes/no or true/false dichotomy, fuzzy mathematics permits numerous levels of truth or membership, which are presented by values in the range, usually between 0 and 1. In 1965, Zadeh[4] enhanced the classical set by implementing the fuzzy set ($\mathcal{FS}$). This is important to provide a mathematical structure for addressing uncertainty and vagueness. For a $\mathcal{FS}$ representing "tall people", the membership function might assign a membership value of 0.8 to a person who is 6 feet tall and 0.3 to someone who is 5 feet 5 inches tall, reflecting the gradual nature of "tallness". The membership "θ" and non-membership "ϑ" functions are utilized to describe the intuitionistic fuzzy set ($\mathcal{IFS}$). The uncertainty model was effectively defined by the $\mathcal{IFS}$ theory by Atanassov[5] in 1983. For particular applications, the $\mathcal{IFS}$ is more useful than $\mathcal{FS}$ because it uses the condition $0 \leq \theta + \vartheta \leq 1$.

To develop a model for imprecise data, the theoretical framework of the Pythagorean fuzzy set ($\mathcal{PFS}$) was proposed by Yager[6] as a generalization of the theory of $\mathcal{IFS}$. $\mathcal{PFS}s$ have values for two functions: the membership "θ" and the non-membership "ϑ" functions. In addition, there is a constraint $0 \leq \theta^2 + \vartheta^2 \leq 1$. After that, many researchers explored an innovative idea known as a q-rung orthopair fuzzy set ($q - \mathcal{ROFS}$) to broaden the scope of $\mathcal{IFS}$ and $\mathcal{PFS}$.[7,8] Riaz et al.[9] suggested the linear Diophantine fuzzy set ($\mathcal{LDFS}$) theory and explored its uses in MADM problems. This notion is a novel way to express vagueness in decision-making. $\mathcal{LDFS}$ is more flexible and reliable than recent concepts, such as $\mathcal{IFS}s$, $\mathcal{PFS}s$, and $q - \mathcal{ROFS}s$, as it contains reference factors with membership and non-membership functions. This notion is a novel way to express uncertainty in decision-making. $\mathcal{LDFS}$ is more flexible and reliable than current concepts, such as $\mathcal{IFS}s$, $\mathcal{PFS}s$, and $q - \mathcal{ROFS}s$ because it contains reference or control factors with membership and non-membership functions. In the decision-making issues, Aydogdu et al.[10] developed the linear Diophantine fuzzy Additive Ratio Assessment (ARAS) method, which combines $\mathcal{LDFS}s$ with the ARAS method for decision-making problems. In this regard, for more extensions of $\mathcal{LDFS}s$ and their applications, Almagrabi et al.[11] designed an MCDM framework to evaluate and prioritize emergency response strategies based on multiple criteria, including severity, urgency, and resource availability. Al-Quran[12] proposed aggregation operators are shown to be effective in MADM problems.

Figure 1 effectively represents the hierarchical generalization from the most fundamental $\mathcal{FS}s$ to the most advanced $\mathcal{LDFS}s$.

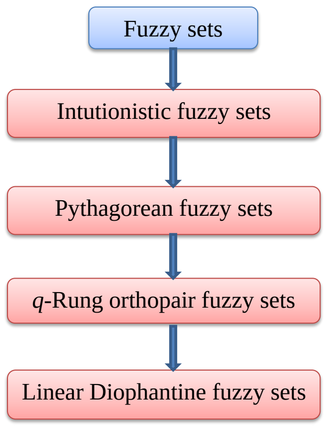

Figure 1: Hierarchical structure of $\mathcal{FS}s$.

Due to the above investigations, Figure 2, (a), (b), (c) and (d), represents the shape of the codomain of $\mathcal{IFS}$, $\mathcal{PFS}$, $q - \mathcal{ROFS}$ and $\mathcal{LDFS}$, respectively.

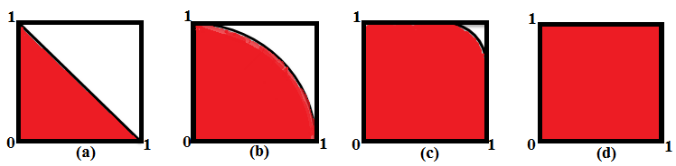

Figure 2: The codomain of $\mathcal{IFS}$, $\mathcal{PFS}$, $q - \mathcal{ROFS}$ and $\mathcal{LDFS}$.

Existing models are insufficient to assess all data, resulting in information loss throughout the process. To address this issue, Ramot et al.[13,14] came up with the idea of complex fuzzy sets ($\mathcal{CFS}s$), which are more advanced than regular $\mathcal{FS}s$ because they allow complex membership values. This makes it possible to represent and analyze uncertainty and ambiguity in complex systems in more complex ways. For example, suppose we are evaluating the quality of a product based on several criteria. In a traditional $\mathcal{FS}$, we might assign a real-valued membership function to represent how well the product meets certain quality standards. For a more nuanced evaluation, we could use a $\mathcal{CFS}$ to include additional dimensions of uncertainty or preferences. Alkouri et al.[15] developed a complex $\mathcal{IFS}$ that is an extension of the standard $\mathcal{CFS}$, where the membership and non-membership functions take complex values rather than real numbers. By using complex $\mathcal{IFSS}$, healthcare professionals can better model the uncertainty and hesitation involved in diagnosing a condition. For instance, the complex membership and non-membership values allow for capturing more nuanced information about the condition?s presence or absence. Also, Ullah et al.[16] proposed complex $\mathcal{PFS}$ to expand the scope of complex $\mathcal{IFS}$ and accommodate complex Pythagorean fuzzy values.

Rosenfeld[17] propounded the concept of fuzzy groups as a combination between traditional groups and $\mathcal{FS}s$. Also, Xi[18] and Ahmad[19] implemented the theory of fuzzy BCK/BCI-algebras by merging the ideas of crisp BCK/BCI-algebras and $\mathcal{FS}s$. Jun et al.[20] initiated the integration of fuzzy BCK/BCI-algebra and $\mathcal{IFS}$. Nirmala et al.[21] explained Pythagorean fuzzy BCK-algebras. Muhiuddin et al.[22] applied the concept of $\mathcal{LDFS}s$ in BCK/BCI-algebras. They proposed the ideas of subalgebras and commutative ideals based on $\mathcal{LDFS}s$. Kamaci[23] established $\mathcal{LDF}$ algebraic structures as a new area of research, combining fuzzy set theory and $\mathcal{LD}$ equations. Yousafzai et al.[24] provided an alternative definition of $\mathcal{CLDFS}$ and applied it in information theory. As a combination between $\mathcal{CFS}$ and $\mathcal{LDFS}$, Kamaci[25] propounded $\mathcal{CLDFS}s$, and then applied it to medical diagnosis. Guan et al.[26] for the first time applied the theory of $\mathcal{CLDFS}s$ to algebraic structures, especially AG-groupoids. By extending the work of $\mathcal{CLDF}$ AG-groupoids and previous works on fuzzy algebraic structures, the notion of $\mathcal{CLDF}$ subalgebras and $\mathcal{CLDF}$ (implicative and positive implicative) ideals in BCK-algebras is presented and studied. To establish the originality of this structure, Figure 3 demonstrates a novel hybrid extension of $\mathcal{LDF}$ BCK-algebras and complex fuzzy BCK-algebras known as $\mathcal{CLDF}$ BCK-algebras.

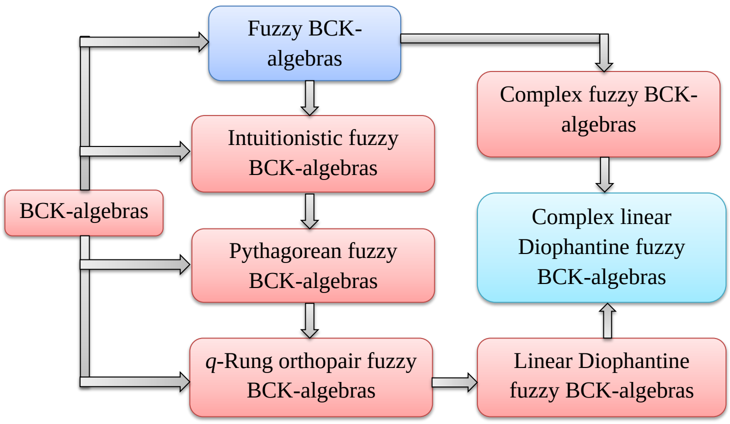

Figure 3: Hierarchical structure of $\mathcal{CLDF}$ BCK-algebras.

The rest of the manuscript is systematized as follows: In Section 2, we provide elementary definitions related to BCK-algebra and $\mathcal{CLDFS}$s. In Section 3, we define $\mathcal{CLDFS}ubs$ and $\mathcal{CLDF}-\mathcal{I}ds$ in BCK-algebras, provide some instances and look into their features. In Section 4, we define the term of $\mathcal{CLDF}-\mathcal{II}d$ of BCK-algebras. Then, We discuss relations between $\mathcal{CLDF}-\mathcal{I}ds$ and $\mathcal{CLDF}-\mathcal{PII}ds$ in BCK-algebras. In Section 5, we introduce the $\mathcal{CLDF}-\mathcal{PII}d$ for BCK-algebras. Then, we study certain dominant features of this concept in detail. Finally, the conclusions and some future studies of the current paper are proposed in Section 6. Table 1 lists the acronyms used in the study article.

2 Preliminaries

BCK-algebras[3] are types of algebraic structures used in the study of non-classical logics, particularly in the context of certain types of implication algebras. These algebras generalize certain aspects of set theory, logic

Table 1: Acronyms List.

Acronyms	Representations
$\mathcal{FS}(s)$	Fuzzy set(s)
$\mathcal{CFS}(s)$	Complex fuzzy set(s)
$\mathcal{IFS}(S)$	Intuitionistic fuzzy set(s)
$\mathcal{PFS}(s)$	Pythagorean fuzzy set(s)
$q-\mathcal{ROFS}(s)$	q-Rung orthopair fuzzy set(s)
$\mathcal{LDFS}(s)$	Linear Diophantine fuzzy set(s)
$\mathcal{CLDF}$	Complex linear Diophantine fuzzy
$\mathcal{CLDFS}(s)$	Complex linear Diophantine fuzzy set(s)
$\mathcal{CLDF}-\mathcal{S}ub(s)$	$\mathcal{CLDF}$ subalgebra(s)
$\mathcal{CLDF}-\mathcal{I}d(s)$	$\mathcal{CLDF}$ ideal(s)
$\mathcal{CLDF}-\mathcal{II}d(s)$	$\mathcal{CLDF}$ implicative ideal(s)
$\mathcal{CLDF}-\mathcal{PII}d(s)$	$\mathcal{CLDF}$ positive implicative ideal(s)

and have applications in some areas, such as theoretical computer science and mathematical logic.

A BCK-algebra is a structure $(\mathcal{U};\between,0)$ consisting of a non-void set $\mathcal{U}$, a binary operation $\between$ on $\mathcal{U}$, and a constant $0\in\mathcal{U}$, satisfying the following axioms: $\forall u_{d_0}, v_{d_1}, w_{d_2}\in\mathcal{U}$

(I_1) $((u_{d_0}\between v_{d_1})\between(u_{d_0}\between w_{d_2}))\between(w_{d_2}\between v_{d_1})=0,$

(I_2) $(u_{d_0}\between(u_{d_0}\between v_{d_1}))\between v_{d_1}=0,$

$(I_3) u_{d_0}\between u_{d_0}=0,$

(I_4) $u_{d_0}\between v_{d_1}=0$ and $v_{d_1}\between u_{d_0}=0\Rightarrow u_{d_0}=v_{d_1},$

(I_5) $u_{d_0}\between 0\Rightarrow u_{d_0}=0,$

(I_6) $0\between u_{d_0}=0.$

A non-void $\mathcal{I}$ of $\mathcal{U}$ is referred to a *subalgebra* of $\mathcal{U}$ if $u_{d_0}\between v_{d_1}\in\mathcal{I}, \forall u_{d_0}, v_{d_1}\in\mathcal{I}$.

A non-void $\mathcal{I}$ of $\mathcal{U}$ is referred to an *ideal* of $\mathcal{U}$ (see[1,2]) if it meets:

$$0\in\mathcal{I} \text{ and } (\forall u_{d_0},\ v_{d_1}\in\mathcal{I})\ (u_{d_0}\between v_{d_1}\in\mathcal{I}\wedge v_{d_1}\in\mathcal{I}\Rightarrow u_{d_0}\in\mathcal{I}). \tag{1}$$

A non-void $\mathcal{I}$ of $\mathcal{U}$ is referred to an *implicative ideal* of $\mathcal{U}$ if it meets:

$$0\in\mathcal{I} \text{ and } (\forall u_{d_0},\ v_{d_1}, w_{d_2}\in\mathcal{I})(((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})\in\mathcal{I}\wedge w_{d_2}\in\mathcal{I}\Rightarrow u_{d_0}\in\mathcal{I}). \tag{2}$$

A non-void $\mathcal{I}$ of $\mathcal{U}$ is referred to a *positive implicative ideal* of $\mathcal{U}$ if it meets:

$$0\in\mathcal{I} \text{ and } (\forall u_{d_0},\ v_{d_1}, w_{d_2}\in\mathcal{I})\ ((u_{d_0}\between v_{d_1})\between w_{d_2}\in\mathcal{I}\wedge v_{d_1}\between w_{d_2}\in\mathcal{I}\Rightarrow u_{d_0}\between w_{d_2}\in\mathcal{I}). \tag{3}$$

Definition 2.1 ([4])**.** Let $\mathcal{U}$ be a universal discourse set. Then, a $\mathcal{FS}$ F in $\mathcal{U}$ is an object form

$$F=\{(u_{d_0},(\mu_F(u_{d_0})))\ :\ u_{d_0}\in\mathcal{U}\},$$

where $\mu_F(u_{d_0})\in[0,1]$ represents the grade of members of $u_{d_0}\in\mathcal{U}$ into the set F.

Definition 2.2 ([14])**.** Let $\mathcal{U}$ represent a universal discourse set. Then, a $\mathcal{CFS}$ $\mathcal{F}$ in $\mathcal{U}$ is an object form

$$\mathcal{F}=\{(u_{d_0},(\mu_{\mathcal{F}}(u_{d_0})))\ :\ u_{d_0}\in\mathcal{U}\},$$

where $\mu_{\mathcal{F}}(u_{d_0})=\mu_{\mathcal{F}}(u_{d_0})e^{i\theta_{\mathcal{F}}(u_{d_0})}$ represents the grade of complex-valued members of $u_{d_0}\in\mathcal{U}$ into the set $\mathcal{C}$.

Definition 2.3 ([9]). Let $\mathcal{U}$ represent a universal discourse set. Then, a $\mathcal{LDFS}$ $\mathcal{L}$ in $\mathcal{U}$ is an object form

$$\mathcal{L} = \{(u_{d_0}, (\mu_{\mathcal{L}}(u_{d_0}), \lambda_{\mathcal{L}}(u_{d_0})), (\gamma^{\mathcal{L}}(u_{d_0}), \delta^{\mathcal{L}}(u_{d_0}))) \; : \; u_{d_0} \in \mathcal{U}\},$$

where $\mu_{\mathcal{L}}, \lambda_{\mathcal{L}}, \gamma^{\mathcal{L}}, \delta^{\mathcal{L}} \in [0,1]$, respectively, represent the grades of members, non-members, and references parameters of $u_{d_0} \in \mathcal{U}$ into the set $\mathcal{L}$ with the conditions $0 \leqslant \gamma^{\mathcal{L}} + \delta^{\mathcal{L}} \leqslant 1$ and $0 \leqslant \gamma^{\mathcal{L}}\mu_{\mathcal{L}} + \delta^{\mathcal{L}}\lambda_{\mathcal{L}} \leqslant 1$.

Definition 2.4 ([24]). Let $\mathcal{U}$ be a universal discourse set. Then, a $\mathcal{CLDFS}$ $\tilde{\mathcal{L}}$ in $\mathcal{U}$ is an object form

$$\tilde{\mathcal{L}} = \{(u_{d_0}, (\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})}), (\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}), \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}))) \; : \; u_{d_0} \in \mathcal{U}\},$$

where $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})}$ and $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})}$ are complex-valued membership and non-membership functions, respectively such that

$$|\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})}|, |\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})}|, \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}), \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \in [0,1],$$
$$0 \leqslant \widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant 2\pi, \text{ and } 0 \leqslant \widehat{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant 2\pi,$$

satisfying

$$|\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})}|\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) + |\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})}|\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant 1,$$
$$\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) + \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant 1, \text{ and } 0 \leqslant \widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}) + \widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant 2\pi, \forall u_{d_0} \in \mathcal{U}.$$

3 Complex Linear Diophantine Fuzzy Ideals

In this section, we apply the $\mathcal{CLDFS}s$ to subalgebras and ideals of BCK-algebras. We originate the notions of a $\mathcal{CLDF}-\mathcal{S}ub$ and a $\mathcal{CLDF}-\mathcal{I}d$, and discuss their characterizations.

Definition 3.1. A $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ of $\mathcal{U}$ is called a $\mathcal{CLDF}-\mathcal{S}ub$ if $\forall u_{d_0}, v_{d_1} \in \mathcal{U}$:

$(CL1)$ $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})}$,

$(CL2)$ $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})}$,

$(CL3)$ $\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})$,

$(CL4)$ $\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1})$.

Example 3.2. Consider a BCK-algebra $\mathcal{U} = \{0, u_{d_0}, v_{d_1}, w_{d_2}\}$ with Table 2:

Table 2: Cayley's table for $\between$-operation

$\between$	0	u_{d_0}	v_{d_1}	w_{d_2}
0	0	0	0	0
u_{d_0}	u_{d_0}	0	0	u_{d_0}
v_{d_1}	v_{d_1}	u_{d_0}	0	v_{d_1}
w_{d_2}	w_{d_2}	w_{d_2}	w_{d_2}	0

Now, define a $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ on $\mathcal{U}$ as:

$$\tilde{\mathcal{L}}(\eth) = \begin{cases} <0.75e^{i0.5\pi}, 0.05e^{i0.7\pi}>, <0.85, 0.05> & \text{if } \eth = 0, \\ <0.55e^{i0.6\pi}, 0.05e^{i0.5\pi}>, <0.65, 0.05> & \text{if } \eth = u_{d_0}, \\ <0.05e^{i0.1\pi}, 0.15e^{i0.7\pi}>, <0.05, 0.15> & \text{if } \eth = \{v_{d_1}, w_{d_2}\}. \end{cases}$$

It is easy to demonstrate that $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF}-\mathcal{S}ub$ of $\mathcal{U}$.

Lemma 3.3. *If* $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ *is a* $\mathcal{CLDF}-\mathcal{S}ub$ *of* $\mathcal{U}$*, then*

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(0)}} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(0)}} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(0) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(0) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}),
\end{aligned}$$

for all $u_{d_0} \in \mathcal{U}$*.*

Proof. Let $u_{d_0} \in \mathcal{U}$. Then, we have

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(0)}} &= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0}\between u_{d_0})}}\\
&\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}}\\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(0)}} &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0}\between u_{d_0})}}\\
&\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}}\\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}},
\end{aligned}$$

$$\widehat{\gamma^{\tilde{\mathcal{L}}}}(0) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}),$$

and

$$\widetilde{\delta^{\tilde{\mathcal{L}}}}(0) = \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) = \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}).$$

Therefore,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(0)}} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(0)}} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(0) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(0) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}).
\end{aligned}$$

□

Definition 3.4. A $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ of $\mathcal{U}$ is called a $\mathcal{CLDF}-\mathcal{I}d$ if $\forall\, u_{d_0}, v_{d_1} \in \mathcal{U}$:

(CL5)
$$\begin{pmatrix}
\widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(0)}} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(0)}} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0})
\end{pmatrix},$$

(CL6)
$$\begin{pmatrix}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0}\between v_{d_1})}} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0}\between v_{d_1})}} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}}
\end{pmatrix},$$

(CL7)
$$\begin{pmatrix}
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1})
\end{pmatrix}.$$

Table 3: Cayley's table for $\between$-operation

$\between$	0	ℓ	u_{d_0}	v_{d_1}	w_{d_2}
0	0	0	0	0	0
ℓ	ℓ	0	ℓ	0	0
u_{d_0}	u_{d_0}	u_{d_0}	0	0	0
v_{d_1}	v_{d_1}	v_{d_1}	v_{d_1}	0	0
w_{d_2}	w_{d_2}	w_{d_2}	w_{d_2}	v_{d_1}	0

Example 3.5. Consider a BCK-algebra $\mathcal{U} = \{0, \ell, u_{d_0}, v_{d_1}, w_{d_2}\}$ defined by Table 3:
Now, define a $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}} e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}} e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ on $\mathcal{U}$ as:

$$\tilde{\mathcal{L}}(\eth) = \begin{cases} <0.8e^{i0.5\pi}, 0.1e^{i0.6\pi}>, <0.75, 0.05> & \text{if } \eth = 0, \\ <0.6e^{i0.6\pi}, 0.3e^{i0.6\pi}>, <0.55, 0.25> & \text{if } \eth = \ell, \\ <0.4e^{i0.4\pi}, 0.5e^{i0.6\pi}>, <0.35, 0.45> & \text{if } \eth = u_{d_0}, \\ <0.2e^{i0.6\pi}, 0.7e^{i0.2\pi}>, <0.15, 0.65> & \text{if } \eth = v_{d_1}, \\ <0.2e^{i0.4\pi}, 0.7e^{i0.5\pi}>, <0.15, 0.65> & \text{if } \eth = w_{d_2}. \end{cases}$$

It is straightforward to demonstrate that $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}} e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}} e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ is a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$.

Lemma 3.6. *Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}} e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}} e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$ and $u_{d_0}, v_{d_1} \in \mathcal{U}$ such that $u_{d_0} \leqslant v_{d_1}$. Then,*

$$\begin{pmatrix} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}), \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}). \end{pmatrix}.$$

Proof. Let $u_{d_0}, v_{d_1} \in \mathcal{U}$ such that $u_{d_0} \leqslant v_{d_1}$. Then,

$$\begin{aligned} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})}} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}} \\ &= \widehat{\mu_{\tilde{\mathcal{L}}}}(0) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(0)}} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}} \\ &= \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}}, \end{aligned}$$

$$\begin{aligned} \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})}} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}} \\ &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(0)}} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}} \\ &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}}, \end{aligned}$$

$$\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})$$

and

$$\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) = \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) = \widehat{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).$$

Therefore, $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}}$, $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}}$, $\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})$ and $\widehat{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})$. $\square$

Lemma 3.7. *Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}} e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}} e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$ and $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$ such that $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$. Then,*

$$\begin{pmatrix} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(u_{d_0})}} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(w_{d_2})}} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widehat{\theta_{\tilde{\mathcal{L}}}(v_{d_1})}}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(u_{d_0})}} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(w_{d_2})}} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1}) e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}(v_{d_1})}}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}), \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}). \end{pmatrix}.$$

Proof. Let $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$ such that $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$. Then we have

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}(v_{d_1}) \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})
\end{aligned}$$

and

$$\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).$$

□

Theorem 3.8. *Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$. Then, $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$ implies*

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}),
\end{aligned}$$

and

$$\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}).$$

Proof. Let $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$, $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$. Then, $(u_{d_0} \between v_{d_1}) \between w_{d_2} = 0$. Now, since $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$, then

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}),
\end{aligned}$$

and

$$\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}).$$

□

Theorem 3.9. *If $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$, then, for all $u_{d_0}, u_{d_1}, u_{d_2}, \cdots, u_{d_n} \in \mathcal{U}$,*

$$\prod_{i=1}^{n} u_{d_0} \between u_{d_i} = 0 \Rightarrow \begin{cases}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_1})} \wedge \cdots \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_n})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_n})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_1})} \vee \cdots \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_n})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_n})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_2}) \wedge \cdots \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_n}), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_2}) \vee \cdots \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_n}),
\end{cases} \tag{4}$$

where $\prod_{i=1}^{n} u_{d_0} \between u_{d_i} = (\cdots((u_{d_0} \between u_{d_1}) \between u_{d_2}) \between \cdots) \between u_{d_n}$.

Proof. The proof can be found on n. Let $\tilde{\mathcal{L}}$ be a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$. For $n = 2$, the condition (4) is valid, as demonstrated by Theorem 3.8. Assume that, for $n = k$, $\tilde{\mathcal{L}}$ fulfills (4), that is, for all $u_{d_0}, u_{d_1}, u_{d_2}, \cdots, u_{d_k} \in$

$\mathcal{U}$, $\prod_{i=1}^{k} u_{d_0} \between u_{d_i} = 0$ implies

$$\begin{aligned}
\widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0})} &\geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_1})} \wedge \cdots \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_k})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_k})},\\
\widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_0})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_0})} &\leqslant \widetilde{\lambda}_{\tilde{\mathcal{L}}}(v_{d_1})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(v_{d_1})} \vee \cdots \vee \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_k})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_k})},\\
\widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_0}) &\geqslant \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_1}) \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_2}) \wedge \cdots \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_k}),\\
\widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_0}) &\leqslant \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_1}) \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_2}) \vee \cdots \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_k}).
\end{aligned}$$

Let $u_{d_0}, u_{d_1}, u_{d_2}, \cdots, u_{d_k}, u_{d_{k+1}} \in \mathcal{U}$ such that $\prod_{i=1}^{k+1} u_{d_0} \between u_{d_i} = 0$. Then,

$$\begin{aligned}
\widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})} &\geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_2})} \wedge \cdots \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})},\\
\widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(x \between u_{d_1})} &\leqslant \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_2})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_2})} \vee \cdots \vee \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})},\\
\widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1}) &\geqslant \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_2}) \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_3}) \wedge \cdots \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_{k+1}}),\\
\widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1}) &\leqslant \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_2}) \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_3}) \vee \cdots \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_{k+1}}).
\end{aligned}$$

Since $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$, it follows from Definition 3.4 (CL6) and (CL7) that

$$\begin{aligned}
\widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0})} &\geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})} \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_1})}\\
&\geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_1})} \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_2})} \wedge \cdots, \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})},\\
\widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_0})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_0})} &\leqslant \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1})} \vee \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_1})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_1})}\\
&\leqslant \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_1})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_1})} \vee \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_2})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_2})} \vee \cdots \vee \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(u_{d_{k+1}})},\\
\widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_0}) &\geqslant \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1}) \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_1})\\
&\geqslant \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_1}) \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_2}) \wedge \cdots \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_{k+1}}),\\
\widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_0}) &\leqslant \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_0} \between u_{d_1}) \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_1})\\
&\leqslant \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_1}) \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_2}) \vee \cdots \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_{k+1}}).
\end{aligned}$$

□

Theorem 3.10. *Let* $\tilde{\mathcal{L}} = (<\widehat{\mu}_{\tilde{\mathcal{L}}}e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}}, \widetilde{\lambda}_{\tilde{\mathcal{L}}}e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}}>, <\widehat{\gamma}^{\tilde{\mathcal{L}}}, \widetilde{\delta}^{\tilde{\mathcal{L}}}>)$ *be a* $\mathcal{CLDF} - \mathcal{I}d$ *of* $\mathcal{U}$*. Then,*

$$\begin{aligned}
\widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})} &\geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})} \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})},\\
\widetilde{\lambda}_{\tilde{\mathcal{L}}}(\lambda_0 \between v_{d_1})e^{i\widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})} &\leqslant \widetilde{\lambda}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})} \vee \widehat{\mu}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})},\\
\widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1}) &\geqslant \widehat{\gamma}^{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2}) \wedge \widehat{\gamma}^{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}),\\
\widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1}) &\leqslant \widetilde{\delta}^{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2}) \vee \widetilde{\delta}^{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}),
\end{aligned}$$

for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$.

Proof. It is worth noting that $((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2})) \leqslant (w_{d_2} \between v_{d_1})$ for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$. It follows from Lemma 3.6 that

$$\begin{aligned}
\widehat{\mu}_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}))} &\geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})},\\
\widetilde{\lambda}_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}))e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}))} &\leqslant \widetilde{\lambda}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})e^{i\widetilde{\varphi}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})},\\
\widehat{\gamma}^{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}) &\geqslant \widehat{\gamma}^{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}),\\
\widetilde{\delta}^{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}) &\leqslant \widetilde{\delta}^{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}).
\end{aligned}$$

By Definition 3.4 (CL6) and (CL7) that,

$$\begin{aligned}
&\widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1})}\\
&\qquad \geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}))e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}))} \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})}\\
&\qquad \geqslant \widehat{\mu}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})} \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})}\\
&\qquad = \widehat{\mu}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})} \wedge \widehat{\mu}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})e^{i\widehat{\theta}_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1})},
\end{aligned}$$

$$
\begin{aligned}
&\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})}\\
&\qquad \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between(u_{d_0}\between w_{d_2}))} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})}\\
&\qquad \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2}\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})}\\
&\qquad = \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2}\between v_{d_1})},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1}) &\geqslant \gamma_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2})) \wedge \gamma_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})\\
&\geqslant \gamma_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}) \wedge \gamma_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})\\
&= \gamma_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2}) \wedge \gamma_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}),
\end{aligned}
$$

$$
\begin{aligned}
\delta_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1}) &\leqslant \delta_{\tilde{\mathcal{L}}}((u_{d_0} \between v_{d_1}) \between (u_{d_0} \between w_{d_2})) \vee \delta_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})\\
&\leqslant \delta_{\tilde{\mathcal{L}}}(u_{d_0} \between v_{d_1}) \vee \delta_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2})\\
&= \delta_{\tilde{\mathcal{L}}}(u_{d_0} \between w_{d_2}) \vee \delta_{\tilde{\mathcal{L}}}(w_{d_2} \between v_{d_1}).
\end{aligned}
$$

This completes the proof. □

Theorem 3.11. *Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$. Then,*

$$
\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(u_{d_0}\between v_{d_1}))} &> \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(u_{d_0}\between v_{d_1}))} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1})) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}),
\end{aligned}
$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$.

Proof. Let $\tilde{\mathcal{L}}$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$ such that $u_{d_0} \between (u_{d_0} \between v_{d_1}) \leqslant v_{d_1}$. Then, we have

$$
\begin{aligned}
&\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(u_{d_0}\between v_{d_1}))}\\
&\qquad \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between v_{d_1})) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between(u_{d_0}\between v_{d_1}))\between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\theta_{\tilde{\mathcal{L}}}}}(v_{d_1})\\
&\qquad = \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})}\\
&\qquad = \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})},
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(u_{d_0}\between v_{d_1}))}\\
&\qquad \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between v_{d_1})) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between(u_{d_0}\between v_{d_1}))\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})}\\
&\qquad = \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})}\\
&\qquad = \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1})) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between v_{d_1})) \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})\\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1})\\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (u_{d_0} \between v_{d_1})) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between v_{d_1})) \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1})\\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1})\\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).
\end{aligned}
$$

This completes the proof. □

Theorem 3.12. *Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$ satisfying the condition* (4). *Then, $\tilde{\mathcal{L}}$ is a CLDFId of $\mathcal{U}$.*

Proof. It is worth nothing that $(\cdots((0 \between u_{d_0}) \between u_{d_0}) \between \cdots) \between u_{d_0} = 0$ for all $u_{d_0} \in \mathcal{U}$. According to (4),

$$\begin{array}{rcl} \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} & \geqslant & \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} & \leqslant & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) & \geqslant & \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) & \leqslant & \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}). \end{array}$$

Let $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$, $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$. Then,

$$0 = (u_{d_0} \between v_{d_1}) \between w_{d_2} = (\cdots(((u_{d_0} \between v_{d_1}) \between w_{d_2}) \between 0) \between \cdots) \between 0,$$

and so

$$\begin{array}{rcl} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} & \geqslant & \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \\ & = & \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} & \leqslant & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \\ & = & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) & \geqslant & \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \\ & = & \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}), \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) & \leqslant & \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \\ & = & \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}). \end{array}$$

Hence, by Lemma 3.7, we conclude that $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$. □

Theorem 3.13. *Every $\mathcal{CLDF}-\mathcal{I}d$ of BCK-algebra $\mathcal{U}$ is a $\mathcal{CLDF}-\mathcal{S}ub$ of $\mathcal{U}$.*

Proof. Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be any $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$ and $u_{d_0}, v_{d_1} \in \mathcal{U}$. Since $(u_{d_0} \between v_{d_1}) \between w_{d_2} = (u_{d_0} \between u_{d_0}) \between v_{d_1} = 0 \between v_{d_1} = 0$, it follows that $u_{d_0} \between v_{d_1} \leqslant u_{d_0}$ in $\mathcal{U}$. Lemma 3.6 asserts that

$$\begin{array}{rcl} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} & \leqslant & \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} & \geqslant & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) & \leqslant & \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}), \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) & \geqslant & \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}). \end{array}$$

Thus, we have

$$\begin{array}{rcl} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} & \geqslant & \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\ & = & \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} & \leqslant & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\ & = & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) & \geqslant & \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\ & = & \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) & \leqslant & \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\ & = & \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}). \end{array}$$

Therefore, $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ is a $\mathcal{CLDF}-\mathcal{S}ub$ of $\mathcal{U}$. □

Remark 3.14. The converse of Theorem 3.13 is not true in general as seen in Example 3.3.

Example 3.15. Consider a BCK-algebra $U = \{0, u_{d_0}, v_{d_1}, w_{d_2}\}$ with Table 4.

Table 4: Cayley's table for $\between$-operation

$\between$	0	u_{d_0}	v_{d_1}	w_{d_2}
0	0	0	0	0
u_{d_0}	u_{d_0}	0	u_{d_0}	0
v_{d_1}	v_{d_1}	v_{d_1}	0	0
w_{d_2}	w_{d_2}	w_{d_2}	w_{d_2}	0

Now define a $\mathcal{CLDFS}\ \tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ on $\mathcal{U}$ as:

$$\tilde{\mathcal{L}}(\eth) = \begin{cases} (< 0.65e^{i0.1\pi}, 0.25e^{i0.1\pi} >, < 0.75, 0.05 >) & \text{if } \eth = 0, \\ (< 0.45e^{i0.2\pi}, 0.25e^{i0.2\pi} >, < 0.55, 0.15 >) & \text{if } \eth = u_{d_0}, \\ (< 0.25e^{i0.3\pi}, 0.25e^{i0.1\pi} >, < 0.35, 0.25 >) & \text{if } \eth = v_{d_1}, \\ (< 0.55e^{i0.4\pi}, 0.55e^{i0.4\pi} >, < 0.65, 0.35 >) & \text{if } \eth = w_{d_2}. \end{cases}$$

It is easy to verify that $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ is a $\mathcal{CLDF} - \mathcal{S}ub$ of $\mathcal{U}$, but it is not a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$ because

$$0.45e^{i0.2\pi} = \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \ngeqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} = 0.55e^{i0.4\pi}.$$

Theorem 3.16. *Let* $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ *be a* $\mathcal{CLDF} - \mathcal{S}ub$ *of* $\mathcal{U}$*. Then,* $\tilde{\mathcal{L}}$ *is a* $\mathcal{CLDF} - \mathcal{I}d \Leftrightarrow\ \forall\, u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$ *such that* $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$ *implies*

$$\begin{aligned} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}), \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}). \end{aligned}$$

Proof. ($\Rightarrow$) Follows from Lemma 3.7.

($\Leftarrow$) Let $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ be a $\mathcal{CLDF} - \mathcal{S}ub$ of $\mathcal{U}$ such that $\forall\, u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$, $u_{d_0} \between v_{d_1} \leqslant w_{d_2}$ implies

$$\begin{aligned} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}). \end{aligned}$$

As $u_{d_0} \between (u_{d_0} \between w_{d_2}) \leqslant w_{d_2}$, so by hypothesis

$$\begin{aligned} \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}(u_{d_0}) &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})}, \\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})}, \\ \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}), \\ \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}). \end{aligned}$$

Moreover, Lemma 3.3 asserts that

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(0) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(0) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0})
\end{aligned}$$

$\forall u_{d_0} \in \mathcal{U}$. Therefore, $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ is a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$. □

4 Complex linear Diophantine fuzzy implicative ideals

Here, we apply the $\mathcal{CLDFS}s$ to implicative ideals of BCK-algebras. We originate the notion of a $\mathcal{CLDF}-\mathcal{II}d$, and discuss some properties of it. Then, in the context of BCK-algebras, we discuss the correspondence between $\mathcal{CLDF}-\mathcal{I}ds$ and $\mathcal{CLDF}-\mathcal{II}ds$.

Definition 4.1. A $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ in $\mathcal{U}$ is called $\mathcal{CLDF}-\mathcal{II}d$ of $\mathcal{U}$ if it satisfies $(CL5)$ and for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$:

$(CL8)$ $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}$,

$(CL9)$ $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \wedge \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}$,

$(CL10)$ $\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2})$,

$(CL11)$ $\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2})$.

Example 4.2. Consider a BCK-algebra $\mathcal{U} = \{0, u_{d_0}, v_{d_1}, w_{d_2}, x_{d_3}\}$ with Table 5:
Now, define a $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ on $\mathcal{U}$ as:

Table 5: Cayley's table for $\between$-operation

$\between$	0	u_{d_0}	v_{d_1}	w_{d_2}	x_{d_3}
0	0	0	0	0	0
u_{d_0}	u_{d_0}	0	u_{d_0}	0	0
v_{d_1}	v_{d_1}	v_{d_1}	0	0	0
w_{d_2}	w_{d_2}	w_{d_2}	u_{d_0}	0	0
x_{d_3}	x_{d_3}	w_{d_2}	x_{d_3}	u_{d_0}	0

$$\tilde{\mathcal{L}}(\eth) = \begin{cases}
<0.8e^{i0.7\pi}, 0.1e^{i0.8\pi}>, <0.65, 0.28> & \text{if } \eth = 0,\\
<0.8e^{i0.9\pi}, 0.1e^{i0.7\pi}>, <0.65, 0.28> & \text{if } \eth = u_{d_0},\\
<0.8e^{i0.3\pi}, 0.1e^{i0.2\pi}>, <0.65, 0.28> & \text{if } \eth = v_{d_1},\\
<0.4e^{i0.6\pi}, 0.5e^{i0.2\pi}>, <0.35, 0.58> & \text{if } \eth = w_{d_2},\\
<0.4e^{i0.4\pi}, 0.5e^{i0.5\pi}>, <0.35, 0.58> & \text{if } \eth = x_{d_3}.
\end{cases}$$

It is straightforward to demonstrate that $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ is a $\mathcal{CLDF}-\mathcal{II}d$ of $\mathcal{U}$.

Lemma 4.3. *Every $\mathcal{CLDF}-\mathcal{II}d$ of $\mathcal{U}$ is order preserving.*

Proof. Let $\tilde{\mathcal{L}}$ be a $\mathcal{CLDF}-\mathcal{IId}$ of $\mathcal{U}$. Take $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$, $u_{d_0} \leqslant v_{d_1}$. Then,

$$
\begin{array}{rcl}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} & \geqslant & \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (w_{d_2} \between u_{d_0})) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between (w_{d_2} \between u_{d_0})) \between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (w_{d_2} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (w_{d_2} \between u_{d_0}))} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widehat{\mu_{\tilde{\mathcal{L}}}}(0 \between (w_{d_2} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0 \between (w_{d_2} \between u_{d_0}))} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} & \leqslant & \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (w_{d_2} \between u_{d_0})) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between (w_{d_2} \between u_{d_0})) \between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (w_{d_2} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (w_{d_2} \between u_{d_0}))} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0 \between (w_{d_2} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0 \between (w_{d_2} \between u_{d_0}))} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
& = & \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) & \geqslant & \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (w_{d_2} \between u_{d_0})) \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (w_{d_2} \between u_{d_0})) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widehat{\gamma^{\tilde{\mathcal{L}}}}(0 \between (w_{d_2} \between u_{d_0})) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) & \leqslant & \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (w_{d_2} \between x)) \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between (w_{d_2} \between u_{d_0})) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widetilde{\delta^{\tilde{\mathcal{L}}}}(0 \between (w_{d_2} \between u_{d_0})) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
& = & \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).
\end{array}
$$

□

Theorem 4.4. *Any $\mathcal{CLDF}-\mathcal{IId}$ of $\mathcal{U}$ must be a $\mathcal{CLDF}-\mathcal{Id}$ of $\mathcal{U}$.*

Proof. Let $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ be a $\mathcal{CLDF}-\mathcal{IId}$ of $\mathcal{U}$. Then, for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$,

$$
\begin{array}{rcl}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} & \geqslant & \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} & \leqslant & \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) & \geqslant & \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) & \leqslant & \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}).
\end{array}
$$

Putting $v_{d_1} = u_{d_0}$ and $w_{d_2} = v_{d_1}$,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between u_{d_0})) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between(u_{d_0}\between u_{d_0}))\between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between 0) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between 0)\between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between u_{d_0})) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between(u_{d_0}\between u_{d_0}))\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}((x \between 0) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between 0)\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1})} \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between u_{d_0})) \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between 0) \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (u_{d_0} \between u_{d_0})) \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between 0) \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1}).
\end{aligned}$$

Hence, $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$. □

Theorem 4.5. *If $\mathcal{U}$ is an implicative BCK-algbera, then every $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$ is a $\mathcal{CLDF} - \mathcal{II}d$.*

Proof. Let $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ be a $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$ and since $\mathcal{U}$ is an implicative BCK-algebra, $u_{d_0} = u_{d_0} \between (v_{d_1} \between u_{d_0})$ for all $u_{d_0}, v_{d_1} \in \mathcal{U}$. Then by Definition 6 (CL6) and (CL7),

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}),
\end{aligned}$$

for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$. Hence, $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF} - \mathcal{II}d$ of $\mathcal{U}$. □

Theorem 4.6. *If $\mathcal{U}$ is an implicative BCK-algebra, then a $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ of $\mathcal{U}$ is $\mathcal{CLDF} - \mathcal{I}d$ of $\mathcal{U}$ if and only if it is a CLDFII of $\mathcal{U}$.*

Theorem 4.7. *Let $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ be a $\mathcal{CLDF} - \mathcal{II}d$ of $\mathcal{U}$. Then, the set $\mathcal{U}_{\tilde{\mathcal{L}}} = \Big\{u_{d_0} \in \mathcal{U} : \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} = \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)},\ \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} = \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)}, \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(0), \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) = \widetilde{\delta^{\tilde{\mathcal{L}}}}(0)\Big\}$ is an implicative ideal of $\mathcal{U}$.*

Proof. Obviously, $0 \in \mathcal{U}_{\tilde{\mathcal{L}}}$. Let $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}_{\tilde{\mathcal{L}}}$, $(u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \in \mathcal{U}_{\tilde{\mathcal{L}}}$ and $w_{d_2} \in \mathcal{U}_{\tilde{\mathcal{L}}}$. Then,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})} &= \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} = \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})} &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} = \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) &= \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(0), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) &= \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) = \widetilde{\delta^{\tilde{\mathcal{L}}}}(0).
\end{aligned}$$

It follows that

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(w_{d_2}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(0), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(w_{d_2}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(0).
\end{aligned}$$

Combining with Definition 6 (CL5), we obtain

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &= \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &= \widehat{\gamma^{\tilde{\mathcal{L}}}}(0), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &= \widetilde{\delta^{\tilde{\mathcal{L}}}}(0),
\end{aligned}$$

and so $u_{d_0} \in \mathcal{U}_{\tilde{\mathcal{L}}}$. Hence, $\mathcal{U}_{\tilde{\mathcal{L}}}$ is an implicative ideal of $\mathcal{U}$. □

Proposition 4.8. *Let $\tilde{\mathcal{L}} = (< \widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}} >, < \widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}} >)$ be a $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$. Then, the below stated statements are equivalent $\forall u_{d_0}, v_{d_1} \in \mathcal{U}$:*

(i) $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF}-\mathcal{II}d$ of $\mathcal{U}$.

(ii) $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))}$, $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))}$, $\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))$ and $\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))$.

(iii) $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} = \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))}$, $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} = \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))}$, $\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) = \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))$ and $\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) = \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))$.

Proof. $(i) \Rightarrow (ii)$: Since $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF}-\mathcal{IId}$ of $\mathcal{U}$, then we have

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between 0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between 0)} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)} \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between 0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between 0)} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)} \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between 0) \wedge \gamma^{\tilde{L}}(0) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})) \wedge \gamma^{\tilde{L}}(0) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between 0) \vee \delta^{\tilde{L}}(0) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})) \vee \delta^{\tilde{L}}(0) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})),
\end{aligned}$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$.
$(ii) \Rightarrow (iii)$: Since $u_{d_0} \between (v_{d_1} \between u_{d_0}) \leqslant u_{d_0}$. Then, by Theorem 3.8,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))
\end{aligned}$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$. By (ii), we have

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))
\end{aligned}$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$. As a result,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &= \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &= \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))
\end{aligned}$$

$(iii) \Rightarrow (i)$: Since $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF}-\mathcal{Id}$ of $\mathcal{U}$, then Definition 6 $(CL5)$ is satisfied, and

$$\begin{aligned}
&\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))} \\
&\qquad \geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
&\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0}))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between u_{d_0}))} \\
&\qquad \leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between(v_{d_1}\between u_{d_0}))\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
&\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \wedge \gamma^{\tilde{L}}(w_{d_2}), \\
&\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between u_{d_0})) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \vee \delta^{\tilde{L}}(w_{d_2})
\end{aligned}$$

for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$. By (iii), we have

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(w_{d_2})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \wedge \gamma^{\tilde{L}}(w_{d_2}), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1} \between u_{d_0})) \between w_{d_2}) \vee \delta^{\tilde{L}}(w_{d_2})
\end{aligned}$$

for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$. Thus, $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF} - \mathcal{II}d$ of $\mathcal{U}$. □

5 Complex linear Diophantine fuzzy positive implicative ideals

In the current segment, we utilized the $\mathcal{CLDFS}s$ to positive implicative ideals of BCK-algebras. We originate the notion of a $\mathcal{CLDF} - \mathcal{PII}d$, and discuss some characteristics of it. Then, in the context of BCK-algebras, we discuss the correspondence between $\mathcal{CLDF} - \mathcal{I}ds$ and $\mathcal{CLDF} - \mathcal{PII}ds$.

Definition 5.1. A $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ in BCK-algebra $\mathcal{U}$ is called $\mathcal{CLDF} - \mathcal{PII}d$ of $\mathcal{U}$ if it satisfies $(CL5)$ and for all $u_{d_0}, v_{d_1}, w_{d_2} \in \mathcal{U}$:

$(CL12)$ $\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})}$
$$\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})},$$

$(CL13)$ $\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})}$
$$\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1}) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0} \between (v_{d_1}) \between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})},$$

$(CL14)$ $\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) \geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})$,

$(CL15)$ $\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) \leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})$.

Example 5.2. Consider a BCK-algebra $\mathcal{U} = \{0, u_{d_0}, v_{d_1}, w_{d_2}, x_{d_3}\}$ with Table 6:
Now define a $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ on $\mathcal{U}$ as:

Table 6: Cayley's table for $\between$-operation

$\between$	0	u_{d_0}	v_{d_1}	w_{d_2}	x_{d_3}
0	0	0	0	0	0
u_{d_0}	u_{d_0}	0	0	0	0
v_{d_1}	v_{d_1}	v_{d_1}	0	0	0
w_{d_2}	w_{d_2}	w_{d_2}	w_{d_2}	0	0
x_{d_3}	x_{d_3}	x_{d_3}	x_{d_3}	x_{d_3}	0

$$\tilde{\mathcal{L}}(\eth) = \begin{cases} <0.8e^{i0.5\pi}, 0.1e^{i0.9\pi}>, <0.7, 0.1> \text{ if } \eth = 0, \\ <0.7e^{i039\pi}, 0.4e^{i0.1\pi}>, <0.6, 0.3> \text{ if } \eth = u_{d_0}, \\ <0.6e^{i0.7\pi}, 0.4e^{i0.2\pi}>, <0.5, 0.3> \text{ if } \eth = v_{d_1}, \\ <0.4e^{i0.5\pi}, 0.6e^{i0.3\pi}>, <0.3, 0.5> \text{ if } \eth = w_{d_2}, \\ <0.2e^{i0.6\pi}, 0.9e^{i0.4\pi}>, <0.1, 0.8> \text{ if } \eth = x_{d_3}. \end{cases}$$

It is straightforward to verify that $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ is a $\mathcal{CLDF} - \mathcal{PII}d$ of $\mathcal{U}$.

Proposition 5.3. *A $\mathcal{CLDF} - \mathcal{I}d$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ of $\mathcal{U}$ is a $\mathcal{CLDF} - \mathcal{PII}d$ if and* *only if*

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}),
\end{aligned}$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$.

Proof. Suppose that a $\mathcal{CLDF}-\mathcal{I}d\ \tilde{\mathcal{L}}$ is a $\mathcal{CLDF}-\mathcal{PII}d$ of $\mathcal{U}$. Then,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1}\between w_{d_2})},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(x \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(x\between w_{d_2})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1}\between w_{d_2})},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2}).
\end{aligned}$$

Substituting $w_{d_2} = v_{d_1}$, we have

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1}\between v_{d_1})},\\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)},\\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1}\between v_{d_1})},\\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)},\\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1} \between v_{d_1})\\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(0)\\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1} \between v_{d_1})\\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(0)\\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}).
\end{aligned}$$

On the other hand, suppose that $\tilde{\mathcal{L}}$ is $\mathcal{CLDF}-\mathcal{I}d$ of $\mathcal{U}$ satisfying the subsequent inequalities

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between v_{d_1})
\end{aligned}$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$. Now, we prove that

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1}\between w_{d_2})},\\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1}\between w_{d_2})},\\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2}),\\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2}).
\end{aligned}$$

Suppose, in contrast, that there exist $u'_{d_0}, v'_{d_1} \in \mathcal{U}$ such that

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u'_{d_0} \between v'_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u'_{d_0}\between v'_{d_1})} &\leqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u'_{d_0}\between v'_{d_1})\between v'_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v'_{d_1} \between v'_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v'_{d_1}\between v'_{d_1})}, \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u'_{d_0}\between v'_{d_1})\between v'_{d_1})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(0)e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(0)}, \\
&= \widehat{\mu_{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u'_{d_0}\between v'_{d_1})\between v'_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u'_{d_0} \between v'_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u'_{d_0} \between v'_{d_1})} &\geqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u'_{d_0}\between v'_{d_1})\between v'_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v'_{d_1} \between v'_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v'_{d_1}\between v'_{d_1})}, \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u'_{d_0}\between v'_{d_1})\between v'_{d_1})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(0)e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(0)}, \\
&= \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u'_{d_0}\between v'_{d_1})\between v'_{d_1})},
\end{aligned}$$

$$\begin{aligned}
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u'_{d_0} \between v'_{d_1}) &\leqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v'_{d_1} \between v'_{d_1}) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(0) \\
&= \widehat{\gamma^{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1}),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u'_{d_0} \between v'_{d_1}) &\geqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v'_{d_1} \between v'_{d_1}) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(0) \\
&= \widetilde{\delta^{\tilde{\mathcal{L}}}}((u'_{d_0} \between v'_{d_1}) \between v'_{d_1})
\end{aligned}$$

which is a contradiction. Therefore,

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} &\geqslant \widehat{\mu_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between w_{d_2})} \wedge \widehat{\mu_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(v_{d_1}\between w_{d_2})}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between w_{d_2})} &\leqslant \widetilde{\lambda_{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between w_{d_2})} \vee \widetilde{\lambda_{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(v_{d_1}\between w_{d_2})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) &\geqslant \widehat{\gamma^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \wedge \widehat{\gamma^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2}), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between w_{d_2}) &\leqslant \widetilde{\delta^{\tilde{\mathcal{L}}}}((u_{d_0} \between v_{d_1}) \between w_{d_2}) \vee \widetilde{\delta^{\tilde{\mathcal{L}}}}(v_{d_1} \between w_{d_2}).
\end{aligned}$$

Thus, $\tilde{\mathcal{L}}$ is a $\mathcal{CLDF} - \mathcal{PII}d$ of $\mathcal{U}$. ☐

Proposition 5.4. *A $\mathcal{CLDFS}$ $\tilde{\mathcal{L}} = (<\widehat{\mu_{\tilde{\mathcal{L}}}}e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}}, \widetilde{\lambda_{\tilde{\mathcal{L}}}}e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}}>, <\widehat{\gamma^{\tilde{\mathcal{L}}}}, \widetilde{\delta^{\tilde{\mathcal{L}}}}>)$ of $\mathcal{U}$ is a $\mathcal{CLDF} - \mathcal{PII}d$ if and only if*

$$\begin{aligned}
\widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &= \widehat{\mu_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between (v_{d_1} \between u_{d_0})))e^{i\widehat{\theta_{\tilde{\mathcal{L}}}}(u_{d_0}\between(v_{d_1}\between(v_{d_1}\between u_{d_0})))}, \\
\widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1})e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}(u_{d_0}\between v_{d_1})} &= \widetilde{\lambda_{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between (v_{d_1} \between u_{d_0})))e^{i\widetilde{\varphi_{\tilde{\mathcal{L}}}}((u_{d_0}\between v_{d_1})\between v_{d_1})}, \\
\widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &= \widehat{\gamma^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between (v_{d_1} \between u_{d_0}))), \\
\widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between v_{d_1}) &= \widetilde{\delta^{\tilde{\mathcal{L}}}}(u_{d_0} \between (v_{d_1} \between (v_{d_1} \between u_{d_0}))),
\end{aligned}$$

for all $u_{d_0}, v_{d_1} \in \mathcal{U}$.

Proof. It is straightforward. ☐

6 Conclusions

A complex linear Diophantine fuzziness of algebraic structures is a novel field that extends complex fuzzy algebraic structures. In our paper, we prefaced and elaborated the innovative concepts of $\mathcal{CLDF} - \mathcal{S}ubs$,

$\mathcal{CLDF}-\mathcal{I}ds$, $\mathcal{CLDF}-\mathcal{II}ds$ and $\mathcal{CLDF}-\mathcal{PII}ds$ in BCK-algebras using the notions of $\mathcal{LDFS}$, BCK-algebra and $\mathcal{CFS}$. We probed their fundamental characteristics. Furthermore, we discussed some of their qualities and looked into their relationships. Our key findings are provided in Sections 3, 4, and 5. Because every $\mathcal{CFS}$ may be seen as a $\mathcal{CLDFS}$, our findings on $\mathcal{CLDF}$-substructures are generalizations of complex fuzzy substructures of BCK-algebras.

In the future, we will utilize this commenced approach to different topics of algebraic structures such as BE-algebras, residuated lattice, near ring, Lie algebras and MV-algebras. Further, we will examine the $\mathcal{CLDF}-\mathcal{S}ubs$, $\mathcal{CLDF}-\mathcal{I}ds$, $\mathcal{CLDF}-\mathcal{II}ds$ and $\mathcal{CLDF}-\mathcal{PII}ds$ in hyperstructures. These foundations not only strengthen the mathematical structure but also pave the way for specific applications in numerous realms.

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