Dynamics of Predator-Prey Interactions, Analyzing the Effects of Time Delays and Neymark-Saker Bifurcation

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Abstract

The study examines the dynamics of a predator-prey model that includes temporal delays, concentrating on the impact of these delays on system stability and behavior.It delineates criteria for the global stability of the positive equilibrium using a generalized Lyapunov function and the Razumkin-type theorem, emphasizing the significance of temporal delays in biological systems. The research highlights the Neymark-Saker (NS) bifurcation, examining the impact of fractional configurations on this bifurcation and the system's overall dynamic stability. The research utilizes the Lyapunov-Razumihin approach to identify bifurcation points and forecast the system's progression in intricate ecological settings. The research examines the presence of periodic solutions and local stability criteria related to the two delays in predator-prey interactions. Numerical simulations are used to substantiate the theoretical results, specifically for the periodic bifurcation solutions associated with the Neymark-Saker bifurcation.

Keywords: Fixed Points; Bifurcation; Razumkin Theorem; Predator-Prey Dynamics; Time Delays

1 introduction

Dynamic systems employing fractional derivatives signify an evolution from conventional mathematical models that utilize improper derivatives to characterize processes demonstrating memory characteristics and non-local effects. Consequently, these models have arisen as formidable instruments across several domains, including control, physics, biology, and engineering.[2] The examination of dynamical bifurcations, namely the Neymark-Saker (NS) bifurcation, is a significant subject in the analysis of these systems[24].[22] This study examines the correlation between fractional arrangements and their influence on the Neymark-Saker bifurcation by utilizing the Lyapunov-Razumikhin method, which elucidates the impact of time delay on the system's dynamic stability and identifies bifurcation points, particularly concerning the Neymark-Saker bifurcation, indicative of nonlinear dynamic alterations[4].[3] Analyzing these occurrences allows us to forecast the system's evolution in a complex environment, which is crucial for evaluating ecological interactions and the dynamic patterns of species in nature.The examination of stability and bifurcations in delayed and partially transient systems depends on robust mathematical methodologies, notably the Lyapunov-Razumihin method. This methodology offers a theoretical foundation for examining the stability of fixed points in delayed and incomplete systems, use Lyapunov functions to assess the system's behavior around equilibrium points.[27] This method, including the examination of NS bifurcations and predator-prey dynamics illustrating interactions between two species, will be highly beneficial for understanding multi-species interactions. Leslie introduced the renowned Leslie predator-prey model, whereby the predator's carrying capacity is proportionate to the prey population, demonstrating the influence of time delays and fractional order on the development and evolution of quasi-periodic oscillations.[1]

$$\left\{\begin{array}{l}\frac{d\chi}{dT}=\chi\left(a_1-a_2\chi^3\right)-\frac{\mu\chi^2:\mathcal{Y}}{1+\vartheta\chi^2}\\ \frac{d\mathcal{Y}}{dT}=-a_3\mathcal{Y}+\frac{k\mu x^2\mathcal{Y}}{1+\vartheta\chi^2}\end{array}\right. \tag{1}$$

Here, χ and $\mathcal{Y}$ represent the densities of prey and predators, respectively, whereas $a_1, a_2, \mu, \vartheta, a_3, k$ are positive constants. Context: where χ and $\mathcal{Y}$ representthe population densities of prey and predators, respectively, and $a_1, a_2, \mu, \vartheta, a_3, k$ are positive constants. The adequate conditions for the global stability of the positive equilibrium of the system (1) are established utilizing the generalized Lyapunov function and the Razumkin-type theorem. Temporal delays are acknowledged as key components in several biological dynamical systems. They may disturb a steady equilibrium, resulting in population oscillations. Consequently, Shi included the temporal delay related to predator maturity included into the system (1) and formulated the resulting delayed predator-prey model[56].[7]

$$\left\{\begin{array}{l}\frac{d\chi}{dt}=\chi\left(a_1^3-\chi^3\right)-\frac{\chi^2\mathcal{Y}}{1+\vartheta\chi^2}\\ \frac{d\mathcal{Y}}{dt}=\mathcal{Y}\left(-a_2+\frac{a_3\chi^2}{1+\vartheta\chi^2}\right)\end{array}\right. \tag{2}$$

Utilizing the forward Euler method on system (2), we derive the discrete predator-prey system as follows:

$$\left\{\begin{array}{l}\chi\rightarrow\chi+\varrho\chi\left(b_1^3-\chi^3\right)-\frac{\chi^2\mathcal{Y}}{1+\vartheta\chi^2}\\ \mathcal{Y}\rightarrow\mathcal{Y}+\varrho\mathcal{Y}\left(-b_2+\frac{a_3\chi^2}{1+\vartheta\chi^2}\right)\end{array}\right. \tag{3}$$

Constant a_1 signifies the time delay caused by the negative feedback of the prey, whereas constant a_2 indicates the time delay due to the constraints imposed by the adult predator. $a_1\chi^2(t)\mathcal{Y}^2(t)+m\mathcal{Y}^2(t)$ is the response function that delineates the predation behavior of the mature predator towards its prey.The aim of this study is to analyze the influence of the two delays on the system's dynamics. Specific sufficient requirements for the local stability of the positive fixed point and the possibility of periodic solutions through the Neymark-Sacker bifurcation are established concerning the two delays. The normal form technique and central manifold theory are employed to determine the direction and stability of the periodic bifurcation solution of the Neymark-Sacker bifurcation. Numerical simulations are presented to demonstrate the theoretical analysis.[9]

2 The presence and stability of fixed positions

understanding fixed point Stability is essential within the context of a predator-prey system. The fixed points signify equilibrium states when predator and prey populations have reached balance. By examining their stability, we may predict the long-term trends of these ecological systems, therefore improving our comprehension of the factors affecting their overall dynamics. ecosystem[13].[15]

$$\left\{\begin{array}{l}\chi=\chi+\delta\chi\left(b_1^3-\chi^3\right)-\frac{\delta\chi^2\mathcal{Y}}{1+\omega\chi^2}\\ \mathcal{Y}=\mathcal{Y}+\delta\mathcal{Y}\left(-b_2+\frac{b_3\chi^2}{1+\omega\chi^2}\right)\end{array}\right. \tag{4}$$

Assuming $\omega=1$ without loss of generality, the system (1) is simplified to:

$$\left\{\begin{array}{l}\chi=\chi+\delta\chi\left(a_1^3-\chi^3\right)-\frac{\delta\chi^2\mathcal{Y}}{1+\chi^2}\\ \mathcal{Y}=\mathcal{Y}+\delta\mathcal{Y}\left(-a_2+\frac{a_3\chi^2}{1+\chi^2}\right)\end{array}\right. \tag{5}$$

Lemma 1.[21]
(i) The system (5) possesses two fixed points. The points $(0,0)$ and $(a_1,0)$ are applicable for all parameters.
(ii) The system (5) possesses a equilibrium point $(\chi_0,\mathcal{Y}_0)$iff $a_3a_1^2>a_2(1+a_1^2),$ where $\chi_0,\mathcal{Y}_0$ satisfy

$$\begin{cases} a_1^3 - \chi_0^3 = \dfrac{\chi_0 \mathcal{Y}_0}{1+\chi_0^2} \\ a_2 = \dfrac{a_3 \chi_0^2}{1+\chi_0^2} \end{cases}$$

Subsequently, we shall examine The stability of equilibrium points. The stability of an equilibrium point $(\chi, \mathcal{Y})$ is dictated by the magnitudes of the characteristic equation and it is eigenvalues at that specific position.

The Jacobian matrix J of the system (5) at the equilibrium point.t $(\chi, \mathcal{Y})$ is represented as

$$J(\chi, \mathcal{Y}) = \begin{pmatrix} 1 + \Delta\left(a_1^3 - 4\chi^3\right) - \frac{2\Delta\chi\mathcal{Y}}{(1+\chi^2)^2} & -\frac{\Delta\chi^2}{1+\chi^2} \\ \frac{2\Delta\Delta_3\chi\mathcal{Y}}{(1+\chi^2)^2} & 1 + \Delta\left(-a_2 + \frac{a_3\chi^2}{1+\chi^2}\right) \end{pmatrix},$$

and we have

$$J(0,0) = \begin{pmatrix} 1 + \Delta a_1^3 & 0 \\ 0 & 1 - \Delta a_2 \end{pmatrix}$$

$$J\left(a_1, 0\right) = \begin{pmatrix} 1 - 3\Delta a_1^3 & -\frac{\Delta a_1^2}{1+a_1^2} \\ 0 & 1 + \Delta\left(-a_2 + \frac{a_3 a_1^2}{1+a_1^2}\right) \end{pmatrix}$$

$$J(\chi, \mathcal{Y}) = \begin{pmatrix} 1 + \Delta\left(a_1^3 - 4\chi_0^3\right) - \frac{2\Delta x_0\mathcal{Y}_0}{\left(1+\chi_0^2\right)^2} & -\frac{\Delta\chi_0^2}{1+\chi_0^2} \\ \frac{2\Delta a_3\chi_0\mathcal{Y}_0}{\left(1+\chi_0^2\right)^2} & 1 + \Delta\left(-a_2 + \frac{a_3\chi_0^2}{1+\chi_0^2}\right) \end{pmatrix}$$

To investigate the stability of the equilibrium points of system (2), We initially offer a lemma as follows:

Lemma 2.[22] Define $G(\alpha) = \alpha^2 + B_1\alpha + C_1$. Assume that $G(1) > 0$, and α_1, α_2 denoted the two roots of $G(\alpha) = 0$, then
(1) $|\alpha_1| < 1$,$|\alpha_2| < 1$ iff $G(-1) > 0$ and $C_1 < 1$;
(2) $|\alpha_1| < 1$, $|\alpha_2| > 1$ (or $|\alpha_1| > 1$, $|\alpha_2| < 1$) iff $G(-1) < 0$;
(3) $|\alpha_1| > 1$,$|\alpha_2| > 1$ iff $G(-1) > 0$, $C_1 < 1$;
(4) $\alpha_1 = -1$ and $|\alpha_2| \neq 1$ iff $G(-1) = 0$ and $B_1 \neq 0, 2$;
(5) α_1 , α_2 are complex and $|\alpha_1| = 1$,$|\alpha_2| = 1$ iff $B_1^2 - 4C_1 < 0$ and $C_1 = 1$.

Let α_1 and α_2 denoted the Jacobian matrix J with it is characteristic equation has two roots J. The stationary point $(\chi, \mathcal{Y})$ is designated as a sink if $|\alpha_1| < 1$ and $|\alpha_2| < 1$, and the sink exhibits local asymptotic stability. $(\chi, \mathcal{Y})$ is designated as a source if $|\alpha_1| > 1$ and $|\alpha_2| > 1$, indicating that the source is locally unstable. $(\chi, \mathcal{Y})$ is designated as a saddle if $|\alpha_1| > 1$ and $|\alpha_2| < 1$, or if $|\alpha_1| < 1$ and $|\alpha_2| > 1$. $(\chi, \mathcal{Y})$ is termed non-hyperbolic if either $|\alpha_1| = 1$ or $|\alpha_2| = 1$.

From Lemma 2, we may derive the subsequent results:
Proposition 3. The eigenvalues at the equilibrium point $(0, 0)$ are $\alpha_1 = 1 + \Delta a_1^3, \alpha_2 = 1 - \Delta a_2$.
(I) If $0 < \Delta a_2 < 2$then the equilibrium point $(0, 0)$ is a saddle ;
(II) if $\Delta a_2 > 2$ then equilibrium point $(0, 0)$ is a source ;
(III) if $\alpha a_2 = 2$ then equilibrium point $(0, 0)$ is non-hyperbolic .

Proposition 4.[23] The eigenvalues of the equilibrium point $(a_1, 0)$ are $\alpha_1 = 1-3\alpha a_1^3, \alpha_2 = 1+\alpha\left(-a_2 + \frac{a_3 a_1^2}{1+a_1^2}\right)$.
(I) The equilibrium point $(a_1, 0)$ is a saddle if $0 < \Delta a_1^3 < \frac{2}{3}$;
(II) The equilibrium point is denoted $(a_1, 0)$ is a source if $\Delta a_1^3 > \frac{2}{3}$;
(III) The equilibrium point $(a_1, 0)$ is non-hyperbolic if $\Delta a_1^3 = \frac{2}{3}$.

the Jacobian matrix J of the system (2) has the characteristic equation it is evaluated at $(\chi_0, \mathcal{Y}_0)$ may be expressed as

$$\alpha^2 + p(\chi_0, \mathcal{Y}_0)\alpha + q(\chi_0, \mathcal{Y}_0) = 0 \tag{6}$$

where

$$p(\chi_0, \mathcal{Y}_0) = -2 - \left(-a_2 + a_1^3 - 4\chi_0^3 + \frac{a_3\chi_0^2}{1+\chi_0^2} - \frac{2\chi_0\mathcal{Y}_0}{(1+\chi_0^2)^2}\right)\Delta$$

$$\begin{aligned} q(\chi_0, \mathcal{Y}_0) =& 1 + \left(-a_2 + a_1^3 - 4\chi_0^3 + \frac{a_3\chi_0^2}{1+x_0^2} - \frac{2\chi_0\mathcal{Y}_0}{(1+\chi_0^2)^2}\right)\Delta \\ &+ \left(-a_1^3a_2 + 4a_2\chi_0^3 + \frac{a_1^4\chi_0^2}{1+\chi_0^2} - \frac{4\chi_0^5}{1+\chi_0^2} + \frac{2a_2\chi_0\mathcal{Y}_0}{(1+\chi_0^2)^2}\right)\Delta^2 \end{aligned}$$

Let

$$\xi = -a_2 + a_1^3 - 4\chi_0^3 + \frac{a_3\chi_0^2}{1+\chi_0^2} - \frac{2\chi_0\mathcal{Y}_0}{(1+\chi_0^2)^2}, \quad \varpi = -a_1^3a_2 + 4a_2\chi_0^3 + \frac{a_1^4\chi_0^2}{1+\chi_0^2} - \frac{4\chi_0^5}{1+\chi_0^2} + \frac{2a_2\chi_0 y_0}{(1+\chi_0^2)^2},$$

then equation. (3) can be written as

$$\alpha^2 - (2+\xi\Delta)\alpha + \left(1 + \xi\Delta + \varpi\Delta^2\right) = 0$$

And let $F(\alpha) = \alpha^2 - (2+\xi\Delta)\alpha + \left(1 + \xi\Delta + \varpi\Delta^2\right)$, then we have

$$F(1) = \varpi\Delta^2 > 0, F(-1) = 4 + 2\xi\Delta + \varpi\Delta^2.$$

Proposition 5.[25] Let $(\chi_0, \mathcal{Y}_0)$ be positive equilibrium point of system (2):
(I) the positive equilibrium point $(\chi_0, \mathcal{Y}_0)$ constitutes a sink if any of the following requirements is satisfied:
(I.1) $-2\sqrt{\varpi} \le \xi < 0$ and $0 < \Delta < -\frac{\xi}{\varpi}$;
(2.I) $\xi < -2\sqrt{\varpi}$ and $0 < \Delta < \frac{-\xi - \sqrt{\xi^2 - 4\varpi}}{\varpi}$;
(II) the positive fixed point $(\chi_0, \mathcal{Y}_0)$ is a source if any of the following requirements are met:
(1.II) $-2\sqrt{\varpi} \le \xi < 0$ and $\Delta > -\frac{\xi}{\varpi}$;
(2.II) $\xi < -2\sqrt{\varpi}$ and $\Delta > \frac{-\xi + \sqrt{\xi^2 - 4\varpi}}{\varpi}$;
(3,II) $\xi \ge 0$;
(III) the positive fixed point $(\chi_0, \mathcal{Y}_0)$ is a saddle if the following condition holds: $\xi < -2\sqrt{\varpi}$ and $\frac{-\xi - \sqrt{\xi^2 - 4\varpi}}{\varpi} < \Delta < \frac{-\xi + \sqrt{\xi^2 - 4\varpi}}{\varpi}$;
(IV) the positive fixed point $(\chi_0, \mathcal{Y}_0)$ is classified as non-hyperbolic if any of the following requirements are satisfied::
(IV.1) $\xi < -2\sqrt{\varpi}$ and $\delta = \frac{-\xi \pm \sqrt{\xi^2 - 4\varpi}}{\varpi}$ and $\Delta \neq -\frac{2}{\xi}, -\frac{4}{\xi}$;
(IV.2) $-2\sqrt{\varpi} < \xi < 0$ and $\delta = -\frac{\Delta}{\varpi}$.

According to Lemma 2, if condition (IV.2) of Proposition 4 is satisfied, then one eigenvalue of the positive fixed point $(\chi_0, \mathcal{Y}_0)$ is -1, whereas the other is neither 1 nor -1.

Assume

$$F_{B1}=\left\{(a_1,a_2,a_3,\Delta):\Delta=\frac{-\xi-\sqrt{\xi^2-4\varpi}}{\varpi},\xi<-2\sqrt{\varpi},a_1,a_2,a_3,\Delta>0\right\}$$

or

$$F_{B2}=\left\{(a_1,a_2,a_3,\Delta):\Delta=\frac{-\xi+\sqrt{\xi^2-4\varpi}}{\varpi},\xi<-2\sqrt{\varpi},a_1,a_2,a_3,\Delta>0\right\},$$

The flip bifurcation at the fixed point $(\chi_0,\mathcal{Y}_0)$ will transpire if the parameters fluctuate within the narrow vicinity of F_{B1} or F_{B2}.

Permit

$$H_B=\left\{(a_1,a_2,a_3,\Delta):\Delta=-\frac{\xi}{\varpi},-2\sqrt{\varpi}<\xi<0,a_1,a_2,a_3,\Delta>0\right\}$$

3 Fractional Razumikhin-Lyapunov Stability Theorem for Time-Delay Systems

Examine a nonlinear fractional time-delay system.

$$ {}_{t_0}^{a}D_t^p\chi(t)=g\left(t,\chi_t\right), \tag{7}$$

where $\chi(t)\in\mathbb{R}^n, 0<q\leq 1$, and $g:\mathbb{R}\times C\rightarrow\mathbb{R}^n$. To determine the future development of the state, it is crucial to provide the initial state variables $\chi(t)$ throughout a duration of r, namely from t_0-r to t_0.

$$\chi_{t_0}=\psi, \tag{8}$$

where $\psi\in C$ is given. In other words $\chi\left(t_0+\Theta\right)=\psi(\Theta),-r\leq\Theta\leq 0$.

Definition 1.[8]Assume that $g(t,0)=0$ for all $t\in\mathbb{R}$. The solution $\chi=0$ of (7) is deemed steady if, for any $t_0\in\mathbb{R}$ and $\epsilon>0$, there exists a $\Delta=\Delta(\epsilon,t_0)$ such that $\|\psi\|<\Delta$ ensures $\|\chi_t(t_0,\psi)\|<\epsilon$ for $t\geq t_0$. The solution $\chi=0$ of (1) is classified as asymptotically stable if it is stable and there exists a $\Delta_a=\Delta_a\left(t_0\right)>0$ such that $\|\psi\|<\Delta_a$ guarantees that $\chi_t\left(t_0,\psi\right)\rightarrow 0$ as $t\rightarrow\infty$. The solution $\chi=0$ is deemed uniformly stable if the parameter δ in the definition is not contingent upon t_0. The solution $\chi=0$ of (7) is uniformly asymptotically stable if it is uniformly stable and there exists a $\Delta_a>0$ such that, for any $\zeta>0$, there exists a $T(\zeta)$ satisfying $\|\varphi\|<\delta_a$ implies that $\|\chi_t\left(t_0,\psi\right)\|<\epsilon$ for $t\geq t_0+T(\zeta)$ for every $t_0\in\mathbb{R}$.

The Lyapunov-Razumikhin method is an effective instrument for assessing the stability of discrete-time systems with delays and can elucidate the conditions that result in NS splitting. The Neimark-Sacker bifurcation, often termed Neimark-Sacker bifurcation, transpires in discrete-time systems. In a time-delay system, the "state" at time t necessitates a value of $\chi(t)$ within the interval $[t-r,t]$, denoted as χ_t. Consequently, it is reasonable to anticipate that the associated Lyapunov function is represented as $V\left(t,\chi_t\right)$, which relies on χ_t and quantifies the deviation of χ_t from the trivial solution 0[30].[11]

Let $V(t,\psi)$ be differentiable, and let $\chi_t(\iota,\psi)$ denote the solution of (7) at time t with the starting condition $\chi_\tau=\varphi$. We subsequently compute the Caputo derivative of $V\left(t,\chi_t\right)$ about t and assess it at $t=\iota$ as follows:

$$t_0D_t^pV(\iota,\psi)=t_{t_0}D_t^pV\left(t,\chi_t(\iota,\varphi)\right)|_{t=\iota,\chi_t=\psi}=\frac{1}{\Gamma(1-p)}\frac{d}{dt}\left(\int_{t_0}^{t}\frac{V\left(s,\chi_s\right)}{(t-s)^q}ds\right)\bigg|_{t=\iota,\chi_t=\psi},$$

$${}_{t_0}^{c}D_t^pV(\iota,\psi)={}_{t_0}^{c}D_t^pV\left(t,\chi_t(\iota,\psi)\right)\big|_{t=\iota,\chi_t=\psi}=\frac{1}{\Gamma(1-p)}\int_{t_0}^{t}\frac{V'\left(s,\chi_s\right)}{(t-s)^p}ds\bigg|_{t=\iota,\chi_t=\psi}$$

where $0<p<1$.

Theorem 6[12]Assume that $f : \mathbb{R} \times C \to \mathbb{R}^n$ as defined in (7) is a mapping. Mapping from $\mathbb{R}\times$ (bounded sets in C) to bounded sets in $\mathbb{R}^n$, and $\aleph_1, \aleph_2, \aleph_3 : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+ \aleph_1(s)$ and $\aleph_2(s)$ are continuous, non-decreasing functions that are positive for $s > 0$, with $\aleph_1(0) = \aleph_2(0) = 0$, and $\aleph_2$ is monotonically increasing. A continuous function and differentiable exists. $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\aleph_1(\|\chi\|) \leq V(t,\chi) \leq \aleph_2(\|\chi\|), \quad \text{for } t \in \mathbb{R}, \chi \in \mathbb{R}^n$$

The Caputo fractional derivative of V along the solution $\chi(t)$ of (7) is fulfilled.

$${}_{t_0}^{c}D_t^p \nu(t,\chi(t)) \leq -\aleph_3(\|\chi(t)\|), \quad \text{for } 0 < p \leq 1, \text{ at any time } V(t+\Theta,\chi(t+\Theta)) \leq \nu(t,\chi(t))$$

For $0 < p \leq 1$ and $\Theta \in [-r, 0]$, the system (7) exhibits uniform stability.
If, furthermore, $\aleph_3(s) > 0$ for $s > 0$ and$\exists$a continuous non decreasing function $p(s) > s$ for $s > 0$ such that condition (8) is augmented to

$${}_{t_0}^{c}D_t^p V(t,x(t)) \leq -\aleph_3(\|\chi(t)\|), \quad \text{if } V(t+\Theta,\chi(t+\Theta)) \leq p(V(t,\chi(t)))$$

For $0 < q \leq 1$ and $\Theta \in [-r, 0]$, system (7) exhibits uniform asymptotic stability.If moreover $\lim_{s\to\infty} \aleph_1(s) = \infty$, then system (1) is globally uniformly asymptotically stable.

The proof of this theorem is located in references.**?**

Theorem 7:[14] Assuming the conditions of Theorem 6 are met, substituting ${}_{t_0}^{c}D_t^p$ with $t_0 D_t^p$ does not affect the validity of the results concerning this text discusses the concepts of uniform stability, the concepts uniform asymptotic stability, and the concepts global uniform asymptotic stability.

Proof. If $\nu(t,\chi(t)) \geq 0$ that ${}_{t_0}^{c}D_t^p \nu(t, x\chi) \leq t_0 D_t^p \nu(t,\chi(t))$, which implies that ${}_{t_0}^{c}D_t^p \nu\left(\chi_t(\varepsilon)\right) \leq t_0 D_t^p \nu\left(\chi_t(\psi)\right) \leq -\rho\left(\chi_t(\varepsilon)\right)$ for all $t \geq t_0$. Applying the same evidence as in Theorem 6 results in uniform, uniform asymptotic, and global uniform asymptotic stability.

4 Neimark-Sacker bifurcationin Time-Delayed Systems

This section analyzes the NS bifurcation in the time-delayed system (3) at the positive fixed point $(\chi, \mathcal{Y})$. To conduct a comprehensive assessment of the bifurcation analysis . The bifurcation substantially influences the system's dynamics, demonstrating cases where little parameter changes lead to considerable modifications in predator-prey interactions. Moreover, understanding the time-delayed NS bifurcation improves insight into ecosystem dynamics. This understanding subsequently facilitates the formulation of effective conservation and management strategies aimed at ensuring the sustainable coexistence of predator-prey populations. This study begins an analysis of the NS bifurcation at $(\chi, \mathcal{Y})$, based on condition in Lemma 1 . By applying a slight bifurcation parameter close to the critical value[10],[16]

we proceed to investigate the NS bifurcation at $(\chi, \mathcal{Y})$ under condition stated in Theorem 6. By applying a small perturbation $\Delta(|\Delta| \lll 1)$ to the bifurcation parameter around the critical value A_2, the system (5) is changed to

$$\begin{cases} \chi_{n+1} = \chi_n e^{(\nu+\Delta)-\mu\chi_n - c_1\mathcal{Y}_n} \\ \mathcal{Y}_{n+1} = \mathcal{Y}_n e^{d_1 - \frac{c_1\mathcal{Y}_n}{\chi_n}} \end{cases} \tag{9}$$

We transform the fixed point $(\chi, \mathcal{Y})$ to the origin by considering the change of variables $\Xi_n = \chi_n - \frac{(c+\Delta)\aleph}{\nu c_1 + \mu\aleph}, \nu_n = \mathcal{Y}_n - \frac{(\mu+\Delta)d_1}{c_1 d_1 + \nu\aleph}$. As a result, the system (9)is converted into the subsequent form:

$$\left[\begin{array}{c}\Xi_{n+1}\\ \vartheta_{n+1}\end{array}\right]=\left[\begin{array}{cc}\frac{8c_1^2+2\nu_{-1}c_1c_1(1-2\delta)-\nu_1^2\alpha_1^2(1+\delta)}{(2c_1+\nu_1\alpha_1)(4c_1+\nu_1\alpha_1)} & -\frac{c_1\alpha_1(4c_1(1+\delta)+\nu\alpha_1(2+\delta))}{(2c_1+\nu_1\alpha)(4c_1+\nu_1\alpha_1)}\\ \frac{4}{\alpha} & -1\end{array}\right]\left[\begin{array}{c}\Xi_n\\ \vartheta_n\end{array}\right]+\left[\begin{array}{c}H\left(\Xi_n,\vartheta_n\right)\\ K\left(\Xi_n,\vartheta_n\right)\end{array}\right] \tag{10}$$

where

$$H\left(\Xi_n,\vartheta_n\right)=\mu_1\Xi_n^2+\mu_2\Xi_n^2\vartheta_n+\mu_3\Xi_n\vartheta_n^2+\mu_4\vartheta_n^2+\mu_5\vartheta_n^3+\mu_6\Xi_n^3+\mu_7\Xi_n\vartheta_n+O\left((|\Xi_n|+|\vartheta_n|)^4\right)$$

$$G\left(\Xi_n,\vartheta_n\right)=\nu_1\vartheta_n^3+\nu_2\Xi_n^3+\nu_3\Xi_n^2\vartheta_n+\nu_4\Xi_n\vartheta_n^2+O\left((|\nu_n|+|\vartheta_n|)^4\right)$$

$$\mu_1=\frac{\nu\left(-16c^2+4\nu c\alpha_1(-2+\delta)+\nu^2\alpha_1^2\delta\right)}{2(2c+\nu\alpha_1)(4c+\nu\alpha_1)},\mu_2=-\frac{\nu c\left(-16c^2+4\nu c\alpha_1(-2+\delta)+\nu^2\alpha_1^2\Delta\right)}{2(2c+\nu\alpha_1)(4c+\nu\alpha_1)}$$

$$\mu_3=\frac{c^2\left(8c^2+2\nu c\alpha_1(1-2\Delta)-\nu^2\alpha_1^2(1+\delta)\right)}{2(2c+\nu\alpha_1)(4c+\nu\alpha_1)},\mu_4=\frac{c^2\alpha_1(4c(1+\Delta)+\nu\alpha_1(2+\Delta))}{2(2c+\nu\alpha)(4c+\nu\alpha_1)}$$

$$\mu_5=-\frac{c^3\alpha_1(4c(1+\delta)+\nu\alpha_1(2+\delta))}{6(2c+\nu\alpha_1)(4c+\nu\alpha_1)},\mu_6=-\frac{\nu^2\left(-24c^2+\nu^2\alpha_1^2(-1+\delta)+2\nu c\alpha_1(-7+2\delta)\right)}{6(2c+\nu\alpha_1)(4c+\nu\alpha_1)}$$

$$\mu_7=\frac{c\left(-8c^2+\nu^2\alpha_1^2(1+\Delta)+2\nu c\alpha_1(-1+2\Delta)\right)}{(2c+\nu\alpha_1)(4c+\nu\alpha)},\nu_1=\frac{(2c+\nu\alpha_1)^2}{6\left(\frac{4c+2\nu\alpha_1}{4c++\alpha_1\alpha_1}+\Delta\right)^2},$$

$$\nu_2=-\frac{4(2c+\nu\alpha_1)^2}{3\alpha_1^3\left(\frac{4c+2\nu\alpha_1}{4c+\nu\alpha_1}+\Delta\right)^2},\nu_3=\frac{2(2c+\nu\alpha_1)^2}{\alpha_1^2\left(\frac{4c+2\nu\alpha_1}{4c+\nu\alpha_1}+\Delta\right)^2},\nu_4=-\frac{(2c+\nu\alpha_1)^2}{\alpha\left(\frac{4c+2\nu\alpha_1}{4c+\nu\alpha_1}+\Delta\right)^2}.$$

The system (8) characteristic equation of the Jacobian matrix of estimated at origin is

$$\xi^2-\alpha_1(\Delta)\xi+\beta_1(\Delta)=0 \tag{11}$$

where

$$\alpha(\Delta)=-\frac{\nu\alpha_1(4c(1+\Delta)+\nu\alpha_1(2+\Delta))}{(2c+\nu\alpha_1)(4c+\nu\alpha_1)},\beta(\Delta)=\frac{\nu\alpha_1(1+\Delta)+c(2+4\Delta)}{2c+\nu\alpha_1}. \tag{12}$$

The complex solutions for (9) are calculated as:

$$\xi_{1,2}=\frac{\alpha_1(\Delta)}{2}\pm\frac{i}{2}\sqrt{4\beta_1(\Delta)-\alpha_1^2(\Delta)}.$$

Moreover, we obtain

$$\left(\frac{d\left|\xi_1\right|}{d\delta}\right)_{\delta=0}=\left(\frac{d\left|\xi_2\right|}{d\delta}\right)_{\delta=0}=\frac{4c+\nu\alpha}{4c+2\nu\alpha}>0$$

Additionally, it is required that $\xi_{1,2}^k\neq 1$ when $\Delta=0$ for $k=1,2,3,4$, which corresponds to $\alpha_1(0)\neq -2,2,0,1$. We obtain

$$\alpha_1(0)=-\frac{2\nu\alpha_1}{4c+b\alpha_1}<0.$$

Moreover, $\alpha_1(0) = -2$ is equivalent to $c = 0$, which is not possible. Next, to change (8) into normal form at $\Delta = 0$, we use the following transformation:

$$\begin{bmatrix} \Xi_n \\ \vartheta_n \end{bmatrix} = \begin{bmatrix} -\frac{2c\alpha_1}{4c+\nu\alpha} & 0 \\ -\frac{4c}{4c+\nu\alpha_1} & -\frac{1}{2}\sqrt{4-\frac{4\nu^2\alpha_1^2}{(4c+\nu\alpha_1)^2}} \end{bmatrix} \begin{bmatrix} \chi_n \\ \mathcal{Y}_n \end{bmatrix}. \tag{13}$$

Upon application of the mapping (12), the system (9) takes the following form:

$$\begin{bmatrix} \chi_{n+1} \\ \mathcal{Y}_{n+1} \end{bmatrix} = \begin{bmatrix} -\frac{b\alpha}{4c+b\alpha} & -\frac{1}{2}\sqrt{4-\frac{4b^2\alpha^2}{(4c+b\alpha)^2}} \\ \frac{1}{2}\sqrt{4-\frac{4b^2\alpha^2}{(4c+b\alpha)^2}} & -\frac{b\alpha}{4c+b\alpha} \end{bmatrix} \begin{bmatrix} \chi_n \\ \mathcal{Y}_n \end{bmatrix} + \begin{bmatrix} \urcorner(\chi_n, \mathcal{Y}_n) \\ \Pi(\chi_n, \mathcal{Y}_n) \end{bmatrix} \tag{14}$$

where

$$\begin{aligned}
\urcorner(\chi_n, \mathcal{Y}_n) &= c_1\chi_n^2\mathcal{Y}_n + c_2\chi_n\mathcal{Y}_n^2 + c_3\chi_n^2 + c_4Y_n^2 + c_5X_n\mathcal{Y}_n + c_6\chi_n^3 + c_7Y_n^3 + O\left((|\chi_n| + |\mathcal{Y}_n|)^4\right), \\
\Pi(v_n, \mathcal{Y}_n) &= d_1\chi_n^2\mathcal{Y}_n + d_2v_n\mathcal{Y}_n + d_3\chi_n\mathcal{Y}_n^2 + d_4\chi_n^2 + d_5\mathcal{Y}_n^2 + d_6\chi_n^3 + d_7\mathcal{Y}_n^3 + O\left((|\chi_n| + |\mathcal{Y}_n|)^4\right),
\end{aligned}$$

and

$$c_1 = 8\sqrt{2}c^2\frac{(c(2c+b\alpha))^{3/2}}{(4c+b\alpha)^3}, c_2 = -\frac{4bc^3\alpha(2c+b\alpha)}{(4c+b\alpha)^3}, c_3 = \frac{4c^2(2c+b\alpha)}{(4c+b\alpha)^2}$$

$$\begin{aligned}
&c_4 = -\frac{4c^2(2c+\mu\alpha)}{(4c+\mu\alpha)^2}, c_5 = -\frac{2\sqrt{2}bc\alpha\sqrt{c(2c+\mu\alpha)}}{(4c+\mu\alpha)^2}, c_6 = \frac{2c^2(2c+\mu\alpha)^2(8c+\mu\alpha)}{3(4c+\mu\alpha)^3}, \\
&c_7 = -\frac{1}{48}c^2\left(4-\frac{4\mu^2\alpha^2}{(4c+\mu\alpha)^2}\right)^{3/2}, d_1 = -\frac{16c^4(2c+\mu\alpha)}{(4c+\mu\alpha)^3}, d_2 = \frac{4\mu c^2\alpha}{(4c+\mu\alpha)^2}, \\
&d_3 = \frac{4\sqrt{2}\mu c^3\alpha\sqrt{c(2c+\mu\alpha)}}{(4c+\mu\alpha)^3}, d_4 = -\frac{4\sqrt{2}c^2\sqrt{c(2c+\mu\alpha)}}{(4c+\mu\alpha)^2}, d_5 = \frac{4\sqrt{2}c^2\sqrt{c(2c+\mu\alpha)}}{(4c+\mu\alpha)^2}, \\
&d_6 = -\frac{2}{3}\sqrt{2}c\frac{(c(2c+\mu\alpha))^{3/2}(8c+\mu\alpha)}{(4c+\mu\alpha)^3}, d_7 = \frac{c\left(160c^4+176\mu c^3\alpha+72b^2c^2\alpha^2+14\mu^3c\alpha^3+b^4\alpha^4\right)}{3(4c+\mu\alpha)^3}.
\end{aligned}$$

The map (13) can undergo NS bifurcation if the following quantity is nonzero:

$$L = \left(-\operatorname{Re}\left(\frac{(1-2\xi_1)\,\xi_2^2}{1-\xi_1}\tau_{20}\tau_{11}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2 + \operatorname{Re}(\xi_2\tau_{21})\right)_{\delta=0}, \tag{15}$$

where

$$\begin{aligned}
\tau_{20} &= \frac{1}{8}\left(\urcorner_{\chi_n\chi_n} - \urcorner_{\mathcal{Y}_n\mathcal{Y}_n} + 2\Pi_{\chi_n\mathcal{Y}_n} + i\left(\Pi_{X_n\chi_n} - \Pi_{\mathcal{Y}\mathcal{Y}_n\mathcal{Y}_n} - 2\urcorner_{\chi_n\mathcal{Y}_n}\right)\right), \\
\tau_{11} &= \frac{1}{4}\left(\urcorner_{\chi_n\chi_n} + \urcorner_{\mathcal{Y}_n\mathcal{Y}_n} + i\left(\Pi_{X_nX_n} + \Pi_{\mathcal{Y}_n\mathcal{Y}_n}\right)\right), \\
\tau_{02} &= \frac{1}{8}\left(\urcorner_{\chi_nX_n} - \urcorner_{\mathcal{Y}_n\mathcal{Y}_n} - 2\Pi_{\chi_n\mathcal{Y}_n} + i\left(\Pi_{X_n\chi_n} - \Pi_{\mathcal{Y}_n\mathcal{Y}_n} + 2\urcorner_{\chi_n\mathcal{Y}_n}\right)\right), \\
\tau_{21} &= \frac{1}{16}\left(\urcorner_{\chi_n\chi_n\chi_n} + \urcorner_{\chi_n\mathcal{Y}_n\mathcal{Y}_n} + \Pi_{\chi_n\chi_n\mathcal{Y}_n} + \Pi_{\mathcal{Y}_n\mathcal{Y}_n\mathcal{Y}_n} + i\left(\Upsilon_{\chi_n\chi_n\chi_n} + \Pi_{\chi_n\mathcal{Y}_n\mathcal{Y}_n} - \urcorner_{\chi_n\chi_n\mathcal{Y}_n} - \urcorner_{\mathcal{Y}_n\mathcal{Y}_n\mathcal{Y}_n}\right)\right).
\end{aligned}$$

Consequently, the outcome obtained The analysis presented above is as follows:Theorem 7 Assume that condition on lemma 1 is satisfied. If L, as stated in (15), assumes a nonzero value, then system (9) under goes NS bifurcation at a_2 provided that a varies within a proximate vicinity of $a_2 = \frac{d(cd+b\aleph)}{cd^2+b(-1+d)\aleph}$. Moreover, when L is negative (or positive), the NS bifurcation observed in system (9) at E_2 is classified as supercritical (subcritical), resulting in a singular closed invariant curve emanating from a_2, which is attracting (repelling)[18][19].[20]

5 Numerical Examples

This part presents simulations numerically to corroborate the theoretical analysis derived in Sections 2 and 3. Let $\mu = 1$, $\mu_1 = 4$, $\mu_2 = 3$, $m = 8$, $s = 1.5$, $s_1 = 0.25$, $s_2 = 0.15$, $\sigma = 0.5$. We now consider the specific instance of system (3):

$$\begin{cases} \frac{d\chi(t)}{dt} = 1.5\chi(t) - \chi(t)\chi(t-\kappa_1) - \frac{4x^2(t)\mathcal{Y}_2(t)}{1+8x^2(t)} \\ \frac{d\mathcal{Y}_1(t)}{dt} = \frac{3x^2(t-\kappa_2)\mathcal{Y}_2(t-\kappa_2)}{1+8x^2(t-\tau_2)} - 0.25\mathcal{Y}_1(t) - 0.5\mathcal{Y}_1(t) \\ \frac{d\mathcal{Y}_2(t)}{dt} = 0.5\mathcal{Y}_1(t) - 0.15\mathcal{Y}_2(t) \end{cases} \tag{16}$$

It possesses a distinct positive equilibrium $C^*(0.4320, 0.4720, 1.4401)$. It is evident that $m_2+n_2 = 1.1196 > 0$ and $(m_2+n_2)(m_1+n_1+p_1) = 0.2212 > m_0+p_0 = 0.096$, therefore confirming that the requirement (H_1) is satisfied. For $\kappa_1 \neq 0$ and $\kappa_2 = 0$, we find that $e_{22}^2 - 3e_{21} = 1.1868 > 0$, $v_{1*} = 0.2107$, and $f_1(v_{1*}) = -0.0038 < 0$, thus confirming that the condition (H_{21}) is satisfied. Subsequently, we obtain $\omega_{10} = 0.4798$ and $\kappa_{10} = 1.6403$. Additionally, $f_1'(v_{1*}) = 0.1679 > 0$.The condition H_{22} is fulfilled. Theorem 6 states that the positive fixed $C^*(0.4320, 0.4720, 1.4401)$ is asymptotically stable for $\kappa_1 \in [0, \kappa_{10})$. Figure 1 depicts the corresponding waveform and phase diagram. When the delay κ_1 surpasses the critical threshold κ_{10}, the positive equilibrium $C^*(0.4320, 0.4720, 1.4401)$ will become unstable, leading to a Neimark-Sacker bifurcation, from which a series of periodic solutions will emerge from the positive equilibrium $C^*(0.4320, 0.4720, 1.4401)$. This phenomenon is demonstrated by the simulations numerically presented in Fig. 2.

5.1 Figures

FIG 3 :$C^*$is locally asymptotically stable for $\kappa = 1.28$ For $\kappa_1 = \kappa_2 = \kappa > 0$, it follows that $\omega_0 = 0.3559$ and $\kappa_0 = 1.0219$. As the delay κ escalates from zero to κ_0, the positive equilibrium $C^*(0.4330, 0.4620, 1.5401)$ maintains its asymptotic stability. Subsequent to the delay κ above the critical threshold κ_0, the positive equilibrium $C^*(0.4330, 0.4620, 1.5401)$ will exhibit instability, resulting in a Neimark-Sacker bifurcation. This trait is demonstrated by the numerical simulations shown in Figure 3.

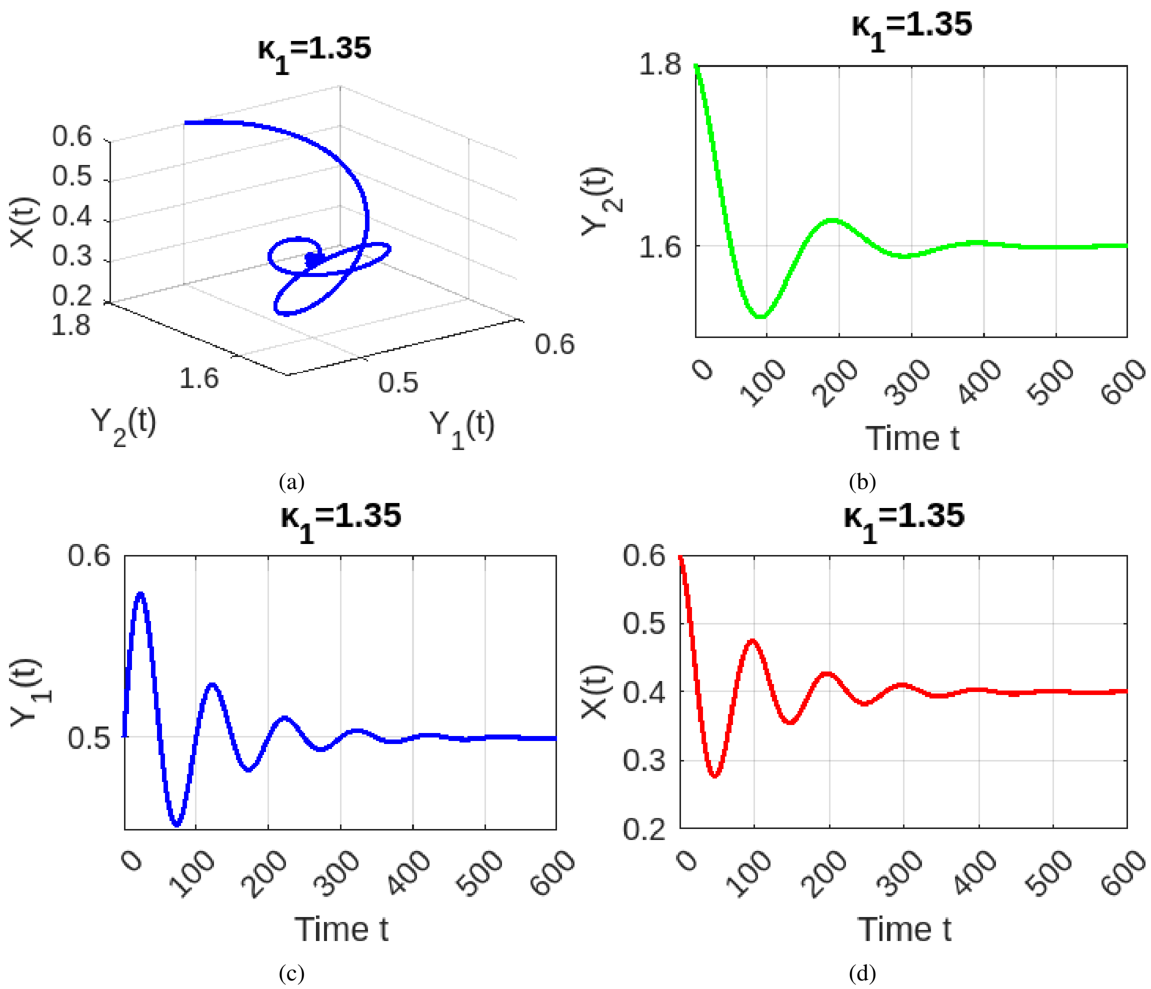
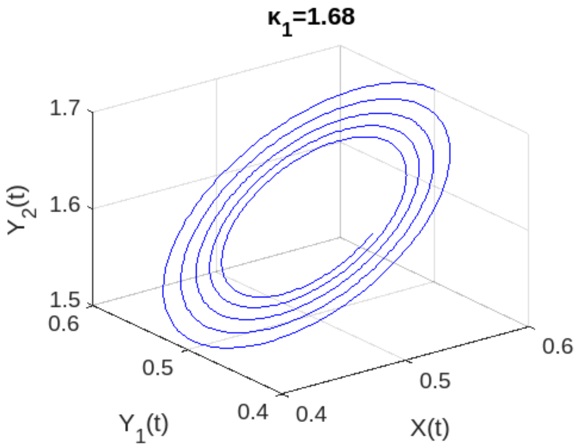
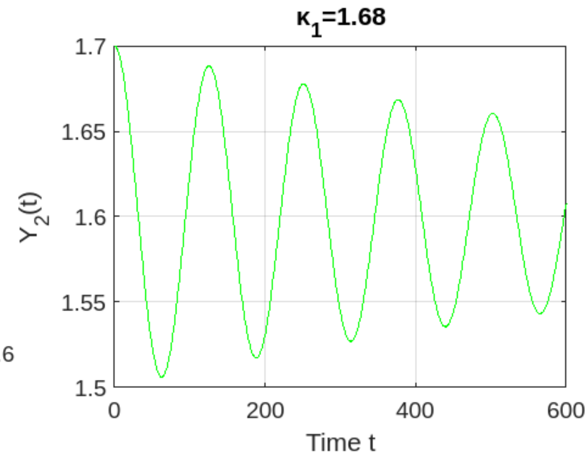


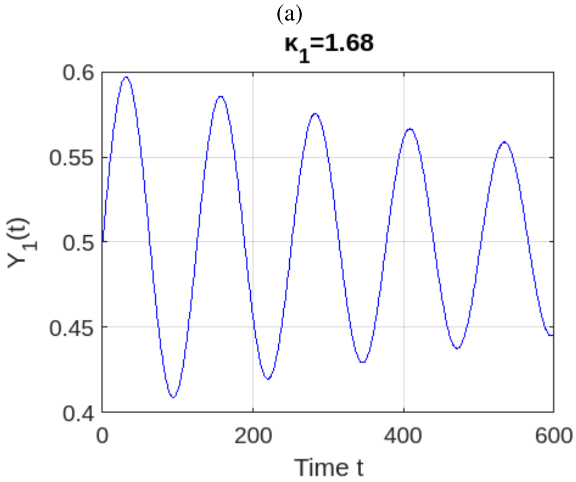
Figure 1: C^* is locally asymptotically stable when κ= 1.35



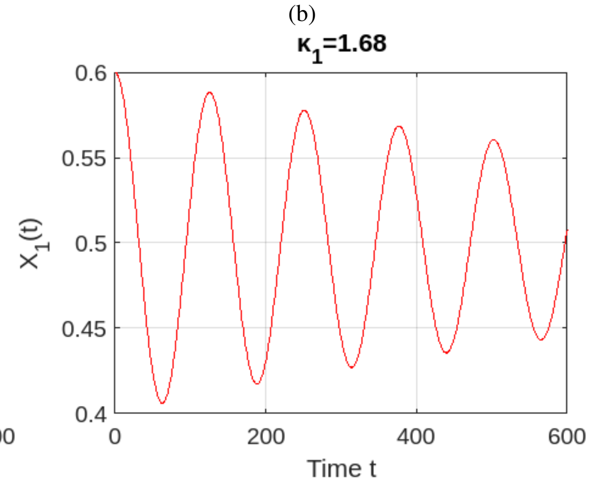
(a)



(b)

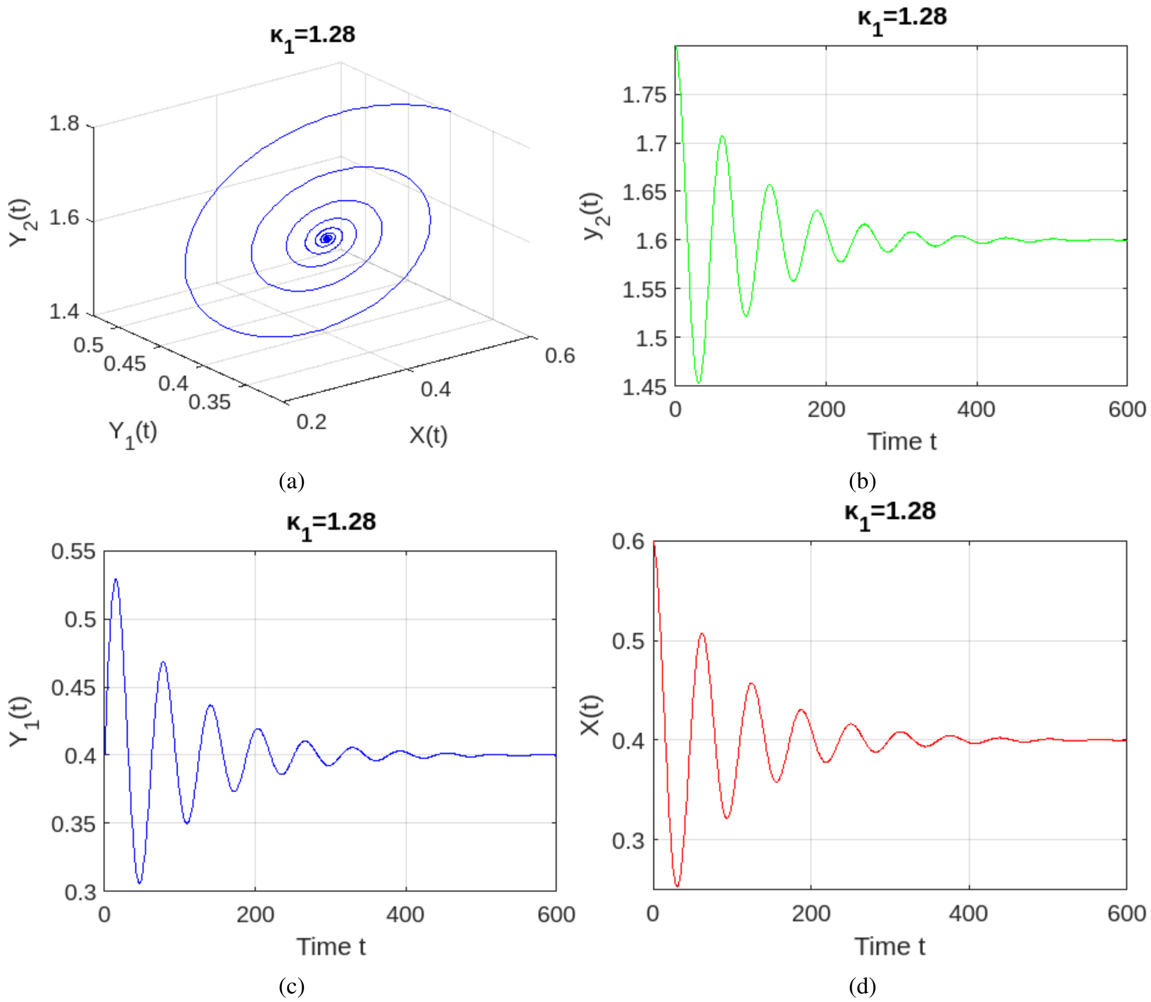


(c)



(d)

Figure 2: C^* is unstable when κ= 1.68



(a) (b) (c) (d)

Figure 3: $C^{*}$is locally asymptotically stable for $\kappa = 1.28$

6 Conclusion

The study underscores the significance of mathematical modeling in ecological research, particularly in elucidating the complexities of predator-prey interactions influenced by time delays, which substantially impact the dynamics of these systems, affecting stability and resulting in intricate oscillatory behaviors. Neimark-Saker bifurcation analysis uncovered critical points, indicating that minor parameter alterations might result in significant behavioral shifts in the system, implying the potential for periodic solutions[28][29].[26] Criteria for local stability of positive equilibrium were defined, offering a basis for forecasting the system's behavior under varying situations. Robust mathematical techniques, like the Lyapunov-Razumihin approach, were employed to analyze stability and bifurcations in delayed systems, becoming indispensable for ecological modeling. The findings from this research can improve conservation and management techniques, augmenting our comprehension of species relationships and ecosystem dynamics[?].[?]

Author contributions All authors provided critical feedback, helped shape the research, analysis , manuscript, designed the model and the computational framework and analyzed the data.
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Competing interests The authors declare that they have no competing interests.
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