Principal L-fuzzy ideals and filters on a trellis

Sarra Boudaoud[1], **Lemnaouar Zedam**[1], **Soheyb Milles**[2,*]
[1]Laboratory of Pure and Applied Mathematics, Faculty of Mathematics and Computer Science, University of M'sila, University Pole, Raod Bordj Bou Arreridj, M'sila 28000, Algeria
[2]Laboratory of Science for Mathematics, Computer Science and Engineering Applications, Department of Mathematics, University Center of Barika, Amdoukal Road, Barika, 05001, Algeria

Emails: sarra.boudaoud@univ-msila.dz; lemnaouar.zedam@univ-msila.dz; soheyb.milles@cu-barika.dz

Abstract

In this paper, we study the notion of principal (crisp) fuzzy ideals (resp. filters) on the setting of trellises (or weakly associative lattices as called by several authors). More specifically, we introduce the notions of L-fuzzy ideals and L-fuzzy filters on a given trellis and provide basic characterizations of these notions based on their weakly associative meet and join operations. We pay particular attention to the kind of principal L-fuzzy ideals (resp. filters) on a given trellis, which are more complicated in the absence of the (associativity) transitivity property.

Keywords: Trellis; Lattice; Fuzzy set; Principal ideal; Principal filter

1 introduction

One of the interesting mathematical structures that consider the non-transitive relations is the non-associative lattice or "trellis" as it is called by Skala.[20,21] To facilitate the meaning, it is a structure like a lattice $(L, \leq, \wedge, \vee)$ without the property of associativity of the operations $\wedge$ and $\vee$ (that's means also the elimination of the property of transitivity of the order relation $\leq$). Fried[10] was among the first mathematicians who discussed the notion of non-associative lattice under the name "tournament-lattice", and he investigated its various properties and characterizations. Skala[20,21] discussed this notion based on the notion of pseudo-order relation, which is a reflexive and antisymmetric relation, but not necessarily transitive. By stipulation of least upper bounds and greatest lower bounds of finite subsets on a given pseudo-ordered set, the result is the structure of trellis or weakly associative lattice as it is called in several other papers.

Since its introduction, it has become omnipresent, Skala[20,21] has extended to this structure several known notions and properties of a lattice. Fried and Grätzer[11] presented some useful and important examples of trellises. Gladstien[12] characterized complete trellises of finite length, and Bhatta and Shashirekha[1] generalized this characterization in terms of joins of cycles. In,[2,3] they discussed some fixed point theorems for pseudo-ordered sets and trellises.

In the fuzzy setting, this notion of trellis was fuzzyfied under the name L-fuzzy complete propelattice,[24] and the Tarski's fixed point theorem for L-fuzzy complete lattice was extended to this non-transitive fuzzy ordered structure. Also, some fixed points theorems on fuzzy pseudo-ordered sets were discussed in.[9,23]

One of the essential tools in various branches of classical mathematics and its applications is the concept of ideal and its dual (a filter). For instance, ideals and filters appear in topology to express completeness and

DOI: https://doi.org/10.54216/IJNS.260212

compactness in metric spaces,[6],[26] in boolean algebra,[22] in the extensive theory of representation of distributive lattices,[7,14].[19] Also in algebraic structures to define congruence relations and quotient algebras,[18].[25]

Generally, the notion of fuzzy ideal (crisp) (resp. filter) has been discussed in detail on a lattice structure; see, e.g., ([4,8,15,17]). In these studies, the property of associativity of the lattice meet and join operations has played a central role in the characterizations of (crisp) fuzzy ideals and filters, in particular, the principal ones.[5,16]

Inspired by the usefulness of the notion of (crisp) fuzzy ideal (resp. filter) on a lattice and by the importance of the notion of trellises introduced by Fried[10] and Skala,[20,21] we study in this paper these notions of (crisp) fuzzy ideals and filters on a trellis. More specifically, we introduce the notion of L-fuzzy down-set (resp. up-set) on a trellis and discuss its interesting properties. Also, we extend the notion of principal (crisp) fuzzy ideal (resp. filter) to the setting of trellises. That is the smallest (crisp) fuzzy ideal (resp. filter) containing a given (crisp) fuzzy singleton. We pay particular attention to what happens in the absence of the (associativity) transitivity property on trellises. Furthermore, under certain conditions, we prove that this notion of principal (crisp) fuzzy ideal (resp. filter) on a trellis coincides with the notion of L-fuzzy down-set (resp. up-set) generated by a (crisp) fuzzy singleton.

The paper is structured as follows: Section 2 revisits essential concepts and properties of pseudo-ordered sets, trellises, fuzzy sets, and related ideas crucial for this paper. Section 3 introduces the concept of L-fuzzy trellises on a crisp trellis, defined as an L-fuzzy set stable under the supremum and infimum of the binary operations $\sqcap$ and $\sqcup$. Additionally, we define L-fuzzy ideals (and filters) on a trellis. Section 4 explores intriguing characterizations of crisp principal ideals (and filters) on a trellis based on the notions of left-transitive (and right-transitive) elements. Section 5 presents the concept of L-fuzzy down-sets (and L-fuzzy up-sets) on a trellis analogously to crisp down-sets (and up-sets), and examines their notable properties. In section 6, we introduce the notion of principal L-fuzzy ideal (resp. principal L-fuzzy filter) on a given trellis, and characterize them via down-sets and up-sets generated by L-fuzzy singletons. Finally, Section 7 offers concluding comments and suggests directions for future research.

2 Basic concepts

This section serves an introductory purpose. First, we recall some definitions and properties related to pseudo-ordered sets and trellises. Second, we recall the notion of L-fuzzy set and some related operations that will be needed throughout this paper.

2.1 Pseudo-ordered sets and trellises

In this subsection, we recall some basic concepts of pseudo-ordered sets and trellises. More information can be found in.[10,11,20,21]

Definition 2.1. [20] Let X be a nonempty set. A binary relation on X is called a pseudo-order relation and denoted by $\trianglelefteq$, if it is reflexive ($x \trianglelefteq x$, for any $x \in X$) and antisymmetric ($x \trianglelefteq y$ and $y \trianglelefteq x$ implies $x = y$, for any $x, y \in X$).

The couple $(X, \trianglelefteq)$ is called a pseudo-ordered set (psoset, for short).

Example 2.2. Let $X = \{a, b, c, d, e\}$ and the binary relation defined on X as follows $\trianglelefteq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e), (b, c), (b, e), (c, d), (c, e), (d, e)\}$.
The couple $(X, \trianglelefteq)$ is a pseudo-ordered set. We note that $\trianglelefteq$ is not transitive because (b, c) and $(c, d) \in \trianglelefteq$, while $(b, d) \notin \trianglelefteq$.

Remark 2.3. One easily observes that the notion of pseudo-order relation generalizes the notion of order relation since any order relation on a set X is a pseudo-order relation on X and not conversely.

For a given pseudo-ordered set $(X, \trianglelefteq)$ and $\mathcal{A} \subseteq X$, a preorder relation $\lesssim$ (i.e., reflexive and transitive relation) on $\mathcal{A}$ can be defined by: $x \lesssim y$ if and only if there exists a finite sequence $(a_1, ..., a_n)$ of elements of $\mathcal{A}$ such that $x = a_1 \trianglelefteq ... \trianglelefteq a_n = y$, for any $x, y \in \mathcal{A}$.

For a given psoset $(X, \trianglelefteq)$, the notions of boundedness, upper bounds (resp. least upper bounds), and lower bounds (resp. greatest lower bounds) are defined analogously to the corresponding notions on ordered sets.

For a given element x on a given psoset $(X, \trianglelefteq)$, we use the following notations:

(i) $\downarrow x = \{y \in X \ : \ y \trianglelefteq x\}$;

(ii) $\uparrow x = \{y \in X \ : \ x \trianglelefteq y\}$;

(i) $\downharpoonleft x = \{y \in X \ : \ y \lesssim x\}$;

(ii) $\upharpoonleft x = \{y \in X \ : \ x \lesssim y\}$.

Notice that in the case of an ordered set (i.e., $\trianglelefteq$ is transitive), $\downarrow x$ (resp. $\uparrow x$) coincides with $\downharpoonleft x$ (resp. $\upharpoonleft x$).

Similarly to the case of ordered sets and lattices, Skala[20] defined the so-called trellises as a subclass of psosets.

Definition 2.4. [20] Let $(X, \trianglelefteq)$ be a psoset. Then

(i) $(X, \trianglelefteq)$ is called a $\sqcap$-semi-trellis if any two elements $x, y \in X$ have a greatest lower bound, denoted $x \sqcap y$ and called the meet (infimum) of x and y;

(ii) $(X, \trianglelefteq)$ is called a $\sqcup$-semi-trellis if any two elements $x, y \in X$ have a smallest upper bound, denoted $x \sqcup y$ and called the join (supremum) of x and y;

(iii) $(X, \trianglelefteq)$ is called a trellis if it is both $\sqcap$- and $\sqcup$-semi-trellis.

A trellis can also be defined as an algebraic structure (see, e.g., Skala[20]), a set X equipped with two binary operations $\sqcap$ and $\sqcup$ that are idempotent, commutative, satisfy the absorption laws ($x \sqcap (x \sqcup y) = x$ and $x \sqcup (x \sqcap y) = x$, for any $x, y \in X$), and the part-preservation ($x \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x = x \sqcap ((x \sqcup y) \sqcap (x \sqcup z))$, for any $x, y, z \in X$). The pseudo-order relation, the meet, and the join operations are related as follows: $x \trianglelefteq y$ if and only if $x \sqcap y = x$; $x \trianglelefteq y$ if and only if $x \sqcup y = y$. Often, the notation $(X, \trianglelefteq, \sqcap, \sqcup)$ for a given trellis is used.

A bounded trellis is a trellis that additionally has the smallest element 0 and greatest element 1, which satisfy $0 \trianglelefteq x \trianglelefteq 1$, for any $x \in X$.

Remark 2.5. We note that if $\trianglelefteq$ is transitive, then it is an order relation, in such case, the operations $\sqcap$ and $\sqcup$ are associative. Further, if $\trianglelefteq$ is transitive, then the trellis $(X, \trianglelefteq, \sqcap, \sqcup)$ is a lattice.

Throughout this paper, X always denotes a trellis $(X, \trianglelefteq, \sqcap, \sqcup)$ and X^d denotes its pseudo-order dual trellis $(X^d, \trianglerighteq, \sqcup, \sqcap)$. Also, to avoid any confusion or miss reading of some equations, L denotes a bounded lattice, and the notations $(\leq, \wedge, \vee, 0, 1)$ refer respectively (the order, $\min$, $\max$, the least element, the greatest element) on it.

Example 2.6. Let $(X, \trianglelefteq)$ be the psoset given in Example 2.2 with $\sqcap$ and $\sqcup$ are given by the following table:

$\sqcap\sqcup$	a	b	c	d	e
a	a	b	c	d	e
b	a	b	c	e	e
c	a	b	c	d	e
d	a	a	c	d	e
e	a	b	c	d	e

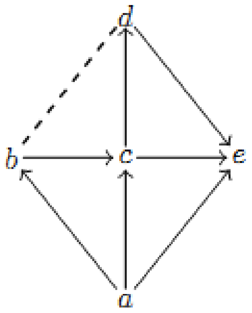

Figure 1: Hasse diagram of the trellis $(X = \{a, b, c, d, e\}, \trianglelefteq, \sqcap, \sqcup)$.

Then $(X, \trianglelefteq, \sqcap, \sqcup)$ is a trellis, and the following Hasse diagram depicts this. The dashed line between b and d means that b and d are incomparable elements.

Definition 2.7. [21] Let $(X, \trianglelefteq)$ be a psoset and $a \in X$. Then

(i) a is called left-transitive element on X if $x \trianglelefteq y \trianglelefteq a$ implies $x \trianglelefteq a$, for any $x, y \in X$;

(ii) a is called right-transitive element if $a \trianglelefteq x \trianglelefteq y$ implies $a \trianglelefteq y$, for any $x, y \in X$.

Next, we denote by $X^{\ell t}$ (resp. X^{rt}) the set of all left-transitive (resp. right-transitive) elements of X

Theorem 2.8. [21] *Let X be a trellis and $x, a \in X$. It holds that*

(i) If a is left-transitive element, then

  (a) $x \lesssim a$, implies $x \trianglelefteq a$;

  (b) $x \trianglelefteq a$, implies that $x \sqcup y \trianglelefteq a \sqcup y$, for any $y \in X$.

(ii) If a is right-transitive element, then

  (a) $a \lesssim x$, implies $a \trianglelefteq x$;

  (b) $a \trianglelefteq x$, implies that $a \sqcap y \trianglelefteq x \sqcap y$, for any $y \in X$.

2.2 L-fuzzy sets

The notion of fuzzy set was first introduced by Zadeh[27] as a generalization of the classical set, it is characterized by the fact of graduation of membership of elements in a set A on a universe X. He expressed the appartenance degrees in the real interval $[0, 1]$. Later, Goguen[13] extended this notion to L-fuzzy sets, where $(L, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice plays the role of the membership (or the truth) values set replacing the real interval $[0, 1]$.

Definition 2.9. [13] Let X be a nonempty set. An L-fuzzy set on X is an object $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ which is characterized by a membership function $\mu_A : X \to L$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy set A, for any $x \in X$.

Several operations are defined in the set of L-fuzzy sets, here we will present only those that are related to the present paper.

Let $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x)\rangle \mid x \in X\}$ be two L-fuzzy sets on X, then

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x),$ for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x),$ for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x)\rangle \mid x \in X\}$;

(v) $\overline{A} = \{\langle x, 1 - \mu_A(x)\rangle \mid x \in X\}$;

(vi) $Supp(A) = \{x \in X \mid \mu_A(x) > 0\}$;

(vii) $Ker(A) = \{x \in X \mid \mu_A(x) = 1\}$.

3 Fuzzy ideals and filters on a trellis

In this section, we introduce and study the notions of L-fuzzy ideal and L-fuzzy filter on a given trellis. Furthermore, we characterize these notions in terms of the trellis meet and join operations.

3.1 Definitions and examples

We start by recalling the notions of crisp ideal and its dual (filter, as often called) on a given trellis.

Definition 3.1. [21] Let X be a trellis, I and F be two subsets on X. Then

(i) I is called an ideal on X if the following two conditions hold

  (a) if $x \in I$ and $y \trianglelefteq x$, then $y \in I$;

  (b) if $x, y \in I$, then $x \sqcup y \in I$.

(ii) F is called a filter on X if the following two conditions hold

  (a) if $x \in F$ and $x \trianglelefteq y$, then $y \in F$;

  (b) if $x, y \in F$, then $x \sqcap y \in F$.

Example 3.2. Let X be the trellis given by the Hasse diagram in Figure 1. Then $I = \{a, b, c\}$ (resp. $F = \{c, d, e\}$) is an ideal (resp. a filter) on X.

Now, we introduce the notions of L-fuzzy trellis, L-fuzzy ideal and L-fuzzy filter.

Definition 3.3. Let $(X, \trianglelefteq, \sqcap, \sqcup)$ be a trellis and $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ be an L-fuzzy set on X. Then

(i) A is called a $\sqcap$-semi L-fuzzy trellis if $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$, for any $x, y \in X$;

(ii) A is called a $\sqcup$-semi L-fuzzy trellis if $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$, for any $x, y \in X$;

(iii) A is called an L-fuzzy trellis it is both $\sqcap$- and $\sqcup$-semi L-fuzzy trellis.

Example 3.4. Let X be the trellis given by the Hasse diagram in Figure 1, $L = [0, 1]$ and $A = \{< a, 0.5 >, < b, 0.4 >, < c, 0.4 >, < d, 0.7 >, < e, 0.5 >\}$ be an L-fuzzy set on X. Then A is an L-fuzzy trellis on X.

The following definitions of L-fuzzy ideals and L-fuzzy filters on a given trellis are natural generalization of the same notions on a lattice structure.

Definition 3.5. Let $(X, \trianglelefteq, \sqcap, \sqcup)$ be a trellis and $I = \{\langle x, \mu_I(x)\rangle \mid x \in X\}$ be an L-fuzzy set on X. Then I is called an L-fuzzy ideal on X if for any $x, y \in X$, the following conditions are satisfied:

(i) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$;

(ii) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$.

Definition 3.6. Let $(X, \trianglelefteq, \sqcap, \sqcup)$ be a trellis and $F = \{\langle x, \mu_F(x)\rangle \mid x \in L\}$ be an L-fuzzy set on X. Then F is called an L-fuzzy filter on X if for any $x, y \in X$, the following conditions are satisfied:

(i) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$;

(ii) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$.

Example 3.7. Let X be the trellis given by the Hasse diagram in Figure 1 and $L = [0, 1]$. Then

(i) the L-fuzzy set $I = \{< a, 0.5 >, < b, 0.3 >, < c, 0.2 >, < d, 0.1 >, < e, 0.1\}$ is an L-fuzzy ideal on X;

(ii) the L-fuzzy set $F = \{< a, 0.1 >, < b, 0.2 >, < c, 0.2 >, < d, 0.1 >, < e, 0.5 >\}$ is an L-fuzzy filter on X.

The following proposition is useful later to avoid writing dual proofs. Its proof is straightforward.

Proposition 3.8. *Let X be a trellis, X^d be its pseudo-order-dual trellis and A is an L-fuzzy set on X. Then it holds that A is an L-fuzzy ideal on X if and only if A is an L-fuzzy filter on X^d and conversely.*

3.2 Basic characterization of fuzzy ideals and filters on a trellis

In this subsection, we characterize the notion of L-fuzzy ideals and L-fuzzy filters on a trellis in terms of its meet and join operations.

We start with the following theorem which provides a characterization of L-fuzzy ideals on a trellis in terms of its join operation.

Theorem 3.9. *Let X be a trellis and I be an L-fuzzy set on X. Then it holds that I is an L-fuzzy ideal on X if and only if the following condition is satisfied:*

$$\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y), \textit{ for any } x, y \in X\,.$$

Proof. Suppose that I is an L-fuzzy ideal on X, then for any $x, y \in X$ it holds that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$. Since $x \trianglelefteq x \sqcup y$ and $y \trianglelefteq x \sqcup y$, it follows from Definition 3.5 (ii) that

$$\mu_I(x) = \mu_I(x \sqcap (x \sqcup y)) \geq \mu_I(x) \vee \mu_I(x \sqcup y) \geq \mu_I(x \sqcup y)\,.$$

In the same manner,

$$\mu_I(y) \geq \mu_I(x \sqcup y)\,.$$

Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$. Thus,

$$\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)\,.$$

Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ for any $x, y \in X$. Then it is easy to see that

$$\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$$

for any $x, y \in X$. Next, we will show that $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$ for any $x, y \in X$. Let $x, y \in X$, since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$ then it holds that $\mu_I(x \sqcup (x \sqcap y)) = \mu_I(x)$ and $\mu_I(y \sqcup (x \sqcap y)) = \mu_I(y)$. From hypothesis it follows that $\mu_I(x) \wedge \mu_I(x \sqcap y) = \mu_I(x)$ and $\mu_I(y) \wedge \mu_I(x \sqcap y) = \mu_I(y)$. Hence, $\mu_I(x \sqcap y) \geq \mu_I(x)$ and $\mu_I(x \sqcap y) \geq \mu_I(y)$. Thus, $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, for any $x, y \in X$. Therefore, I is an L-fuzzy ideal on X. □

In the same manner, the following theorem provides a characterization of L-fuzzy filters on a trellis in terms of its meet operation.

Theorem 3.10. *Let X be a trellis and F be an L-fuzzy set on X. Then it holds that F is an L-fuzzy filter on X if and only if the following condition is satisfied:*

$$\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y), \textit{ for any } x, y \in X\,.$$

Proof. The proof is a direct application of Proposition 3.8 and Theorem 3.9. □

As corollaries of the above theorems, we obtain the following interesting properties of L-fuzzy ideals and L-fuzzy filters on a trellis.

Corollary 3.11. *Let X be a trellis and I be an L-fuzzy ideal on X. Then for any $x, y \in X$ it holds that*

(i) If $x \trianglelefteq y$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the mapping $\mu_I : X \to L$ is antitone).

Corollary 3.12. *Let X be a trellis and F be an L-fuzzy filter on X. Then for any $x, y \in X$ it holds that*

(i) If $x \trianglelefteq y$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the mapping $\mu_F : X \to L$ is monotone).

4 Fuzzy down-sets and up-sets on a pseudo-ordered set

In this section, we introduce the notion of L-fuzzy down-set (resp. L-fuzzy up-set) on a pseudo-ordered set and study their properties.

4.1 Definitions and preliminary results

Let $(X, \trianglelefteq)$ be a psoset and S be an L-fuzzy set on X.

(i) S is called L-fuzzy down-set if $x \trianglelefteq y$, then $\mu_S(y) \leq \mu_S(x)$, for any $x, y \in S$;

(ii) Dually, S is called L-fuzzy up-set if $x \trianglelefteq y$, then $\mu_S(x) \leq \mu_S(y)$, for any $x, y \in S$.

For a given L-fuzzy set S on a psoset $(X, \trianglelefteq)$, we denote by:

(i) $\Downarrow S$: the L-fuzzy set associated to S defined as

$$\mu_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \mu_S(y)\,;$$

(ii) $\Uparrow S$: the L-fuzzy set associated to S defined as

$$\mu_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \mu_S(y)\,;$$

The absence of the property of transitivity for $\trianglelefteq$ leads to the following L-fuzzy sets associated to an L-fuzzy set S.

(iii) S: the L-fuzzy set associated to S defined as

$$\mu_S(x) = \sup_{y \in \upharpoonright x} \mu_S(y)\,;$$

(iv) S: the L-fuzzy set associated to S defined as

$$\mu_S(x) = \sup_{y \in \downharpoonleft x} \mu_S(y)\,.$$

The following proposition is an immediate result of the above definitions.

Proposition 4.1. *Let X be a trellis, X^d be its pseudo-order-dual trellis and S be an L-fuzzy set on X. The following statements hold:*

(i) S is an L-fuzzy down-set (resp. up-set) on X if and only if S is an L-fuzzy up-set (resp. down-set) on X^d;

(ii) $\Downarrow S$ (resp. S) on X coincides with $\Uparrow S$ (resp. S) on X^d;

(iii) $\Uparrow S$ (resp. S) on X coincides with $\Downarrow S$ (resp. S) on X^d.

Remark 4.2. (i) In general, $S \subseteq \Downarrow S \subseteq \ S$ and $S \subseteq \Uparrow S \subseteq \ S$, but one can observe that if $(X, \trianglelefteq)$ is an ordered set, then $\Downarrow S = S$ and $\Uparrow S = S$, for any L-fuzzy set S on X.

(ii) In contrary to the case of ordered sets, one easily verifies that the elimination of the property of transitivity of $\trianglelefteq$ makes that $\Downarrow S$ (resp. $\Uparrow S$) would not necessarily L-fuzzy down-set (resp. L-fuzzy up-set) on $(X, \trianglelefteq)$, for any L-fuzzy set S on X.

For a given psoset $(X, \trianglelefteq)$, the following proposition shows that the associated set S (resp. S) to a given L-fuzzy set S on X is an L-fuzzy down-set (resp. an L-fuzzy up-set).

Proposition 4.3. *Let $(X, \trianglelefteq)$ be a psoset and S be an L-fuzzy set on X. It holds that*

(i) S is an L-fuzzy down-set;

(ii) S is an L-fuzzy up-set.

Proof. (i) Let $x, y \in X$ such that $x \trianglelefteq y$. We show that $\mu_S(y) \geq \mu_S(x)$.

$$\begin{aligned} \mu_S(y) &= \sup_{y \in \upharpoonleft t} \mu_S(t) = \sup_{y \lesssim t} \mu_S(t) \\ &\leq \sup_{x \trianglelefteq y \lesssim t} \mu_S(t) = \sup_{x \lesssim t} \mu_S(t) \\ &= \mu_S(x). \end{aligned}$$

Hence, S is an L-fuzzy down-set.

(ii) Follows from Proposition 4.1 and (i).

□

Next, we introduce the notion of L-fuzzy singleton on a given psoset.

Definition 4.4. An L-fuzzy singleton on a psoset $(X, \trianglelefteq)$ is an L-fuzzy set on X given by $\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t)\rangle \mid t \in X\}$, where

$$\mu_{\tilde{x}}(t) = \begin{cases} 1\,, \ if\ x = t \\ \alpha(t)\,, \ \ otherwise \end{cases}$$

such that $\alpha : X \to L - \{1\}$ is an arbitrary mapping.

For a given L-fuzzy singleton $\tilde{x}$ on a psoset $(X, \trianglelefteq)$, the associated sets $\Downarrow \tilde{x}$ and $\Uparrow \tilde{x}$ are defined as:

$$\Downarrow \tilde{x} = \{\langle t, \mu_{\Downarrow \tilde{x}}(t)\rangle \mid t \in X\}$$

and

$$\Uparrow \tilde{x} = \{\langle t, \mu_{\Uparrow \tilde{x}}(t)\rangle \mid t \in X\},$$

where

$$\mu_{\Downarrow \tilde{x}}(t) = \sup_{y \in \uparrow t} \mu_{\tilde{x}}(y) \text{ and } \mu_{\Uparrow \tilde{x}}(t) = \sup_{y \in \downarrow t} \mu_{\tilde{x}}(y).$$

One easily verifies that

$$\mu_{\Downarrow \tilde{x}}(t) = \begin{cases} 1, \ if \ t \trianglelefteq x \\ \sup\limits_{y \in \uparrow t} \alpha(y), \ \ otherwise \end{cases}$$

and

$$\mu_{\Uparrow \tilde{x}}(t) = \begin{cases} 1, \ if \ x \trianglelefteq t \\ \sup\limits_{y \in \downarrow t} \alpha(y), \ \ otherwise \end{cases} .$$

In the same line, the L-fuzzy sets $\tilde{x}$ and $\tilde{x}$ associated set to a given L-fuzzy singleton $\tilde{x}$ on a psoset $(X, \trianglelefteq)$ are defined as:

$$\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t)\rangle \mid t \in X\}$$

and

$$\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t)\rangle \mid t \in X\},$$

where

$$\mu_{\tilde{x}}(t) = \sup_{y \in \upharpoonright t} \mu_{\tilde{x}}(y) \text{ and } \mu_{\tilde{x}}(t) = \sup_{y \in \downharpoonleft t} \mu_{\tilde{x}}(y).$$

One easily verifies that

$$\mu_{\tilde{x}}(t) = \begin{cases} 1, \ if \ t \lesssim x \\ \sup\limits_{y \in \upharpoonright t} \alpha(y), \ \ otherwise \end{cases}$$

and

$$\mu_{\tilde{x}}(t) = \begin{cases} 1, \ if \ x \lesssim t \\ \sup\limits_{y \in \downharpoonleft t} \alpha(y), \ \ otherwise \end{cases} .$$

Note that for a given crisp set S on a given psoset $(X, \trianglelefteq)$, it holds that

(i) $\Downarrow S = \downarrow S = \{x \in X \ : \ x \trianglelefteq y,$ for some $y \in S\}$;

(ii) $\Uparrow S = \uparrow S = \{x \in X \ : \ y \trianglelefteq x,$ for some $y \in S\}$;

(iii) $S = \downharpoonleft S = \{x \in X \ : \ x \lesssim y,$ for some $y \in S\}$;

(iv) $S = \upharpoonright S = \{x \in X \ : \ y \lesssim x,$ for some $y \in S\}$;

(v) $\Downarrow x = \downarrow x$ and $\Uparrow x = \uparrow x$;

(vi) $x = \downharpoonleft x$ and $x = \upharpoonright x$.

The following proposition provides a necessary and sufficient condition that $\Downarrow \tilde{x}$ (resp. $\Uparrow \tilde{x}$) would be coincides with $\tilde{x}$ (resp. $\tilde{x}$).

Proposition 4.5. *Let $(X, \trianglelefteq)$ be a psoset and $x \in X$. The following equivalences hold:*

(i) $\Downarrow \tilde{x} = \tilde{x}$ *if and only if* x *is a left-transitive element;*

(ii) $\Uparrow \tilde{x} = \tilde{x}$ *if and only if* x *is a right-transitive element.*

Proof. (i) Let $(X, \trianglelefteq)$ be a psoset and $x \in X$. Then

$$\mu_{\tilde{x}}(t) = \begin{cases} 1\ ,\ if\ t \lesssim x \\ \sup\limits_{y \in \upharpoonleft t} \alpha(y)\ ,\ \ otherwise \end{cases}$$

Since x is a left-transitive element, it follows from Theorem 2.8 that $t \lesssim x$ if and only if $t \trianglelefteq x$. Hence,

$$\begin{aligned} \mu_{\tilde{x}}(t) &= \begin{cases} 1\ ,\ if\ t \trianglelefteq x \\ \sup\limits_{y \in \uparrow t} \alpha(y)\ ,\ \ otherwise \end{cases} \\ &= \mu_{\Downarrow \tilde{x}}(t). \end{aligned}$$

(ii) Follows from Proposition 4.1 and (i).

□

The following corollary provides a necessary and sufficient condition that $\Downarrow \tilde{x}$ (resp. $\Uparrow \tilde{x}$) would be an L-fuzzy down-set (resp. an L-fuzzy up-set).

Corollary 4.6. *Let* $(X, \trianglelefteq)$ *be a psoset and* $x \in X$*. The following equivalences hold:*

(i) $\Downarrow \tilde{x}$ *is an L-fuzzy down-set if and only if* x *is a left-transitive element;*

(ii) $\Uparrow \tilde{x}$ *is an L-fuzzy up-set if and only if* x *is a right-transitive element.*

Remark 4.7. If we restrict X by X^{ltr} (resp. X^{rtr}), then $\Downarrow S$ (resp. $\Uparrow S$) is an L-fuzzy down-set on X^{ltr} (resp. L-fuzzy up-set on X^{rtr}).

4.2 Properties of fuzzy down-sets and up-sets on a trellis

In this subsection, we show some interesting properties of L-fuzzy down-sets and up-sets on a trellis.

The following proposition is immediate.

Proposition 4.8. *Let* $(X, \trianglelefteq)$ *be a psoset and* S *be an L-fuzzy set on* X*. The following implications hold:*

(i) if S *is an L-fuzzy down-set, then* $S = \Downarrow S$*;*

(ii) if S *is an L-fuzzy up-set, then* $S = \Uparrow S$*.*

Remark 4.9. The converse of the above implications does not necessarily hold. Indeed, let X be the trellis given by the Hasse diagram in Figure 1 and S be the L-fuzzy set on X given by $S = \{< a, 0.8 >, < b, 0.6 >, < c, 0.2 >, < d, 0.7 >, < e, 0.1 >\}$. It is easy to verify that $\Downarrow S = \{< a, 0.8 >, < b, 0.6 >, < c, 0.2 >, < d, 0.7 >, < e, 0.1 >\} = S$, but S is not an L-fuzzy down-set.

In the same line, the following result holds. The proof is straightforward.

Proposition 4.10. *Let* $(X, \trianglelefteq)$ *be a psoset and* S *be an L-fuzzy set on* X*. The following equivalences hold:*

(i) S *is an L-fuzzy down-set if and only if* $S = \Downarrow S = S$*;*

(ii) S *is an L-fuzzy up-set if and only if* $S = \Uparrow S = S$*.*

The following proposition shows that L-fuzzy down-sets (resp. L-fuzzy up-sets) on a trellis are closed under the union and intersection of L-fuzzy sets.

Proposition 4.11. *Let X be a trellis and R, S be two L-fuzzy sets on X. It holds that*

(i) if R and S are two L-fuzzy down-sets, then $R \cup S$ and $R \cap S$ are L-fuzzy down-sets;

(ii) if R and S are two L-fuzzy up-sets, then $R \cup S$ and $R \cap S$ are L-fuzzy up-sets.

Proof. (i) We only show that $R \cup S$ is an L-fuzzy down-set, as $R \cap S$ can be proved analogously. Let $x, y \in X$ such that $x \trianglelefteq y$. Since R and S are L-fuzzy down-sets, and $\mu_{R \cup S}(x) = \mu_R(x) \vee \mu_S(x)$, it follows that $\mu_{R \cup S}(x) \geq \mu_R(y) \vee \mu_S(y) = \mu_{R \cup S}(y)$. Thus, $R \cup S$ is an L-fuzzy down-set.

(ii) Follows from Proposition 4.1 and (i).

□

The following propositions list some properties of L-fuzzy down-sets and L-fuzzy up-sets.

Proposition 4.12. *Let X be a trellis and R, S be two L-fuzzy sets on X. The following statements hold*

(i) if $S \subseteq R$, then $S \subseteq \ R$ and $\Downarrow S \subseteq \Downarrow R$;

(ii) $(S \cup R) = S \cup R$ and $\Downarrow (S \cup R) \subseteq \Downarrow S \cup \Downarrow R$;

(iii) $(S \cap R) \subseteq S \cap R$ and $\Downarrow (S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

Proof. We only give the proof for as for $\Downarrow$ can be proved analogously.

(i) Suppose that $S \subseteq R$, then $\mu_S(x) = \sup\limits_{y \in \upharpoonright x} \mu_S(y) \leq \sup\limits_{y \in \upharpoonright x} \mu_R(y) = \mu_R(x)$. Hence, $S \subseteq \ R$.

(ii) On the one hand, we easily verify from (i) that $S \cup R \subseteq (S \cup R)$. On the other hand, from Propositions 4.3 and 4.11 it holds that $S \cup R$ is an L-fuzzy down-set. Now the fact that $S \cup R \subseteq S \cup R$ implies that $(S \cup R) = S \cup R$.

(iii) The proof is similar to that of (ii).

□

In the same direction, a dual version of Proposition 4.12 can also be obtained for L-fuzzy up-sets. Its proof follows from Propositions 4.1 and 4.12.

Proposition 4.13. *Let X be a trellis and R, S be two L-fuzzy sets on X. The following statements hold*

(i) if $S \subseteq R$, then $S \subseteq R$ and $\Uparrow S \subseteq \Uparrow R$;

(ii) $(S \cup R) = S \cup R$ and $\Uparrow S \cup \Uparrow R \subseteq \Uparrow (S \cup R)$;

(iii) $(S \cap R) \subseteq S \cap R$ and $\Uparrow (S \cap R) \subseteq \Uparrow S \cap \Uparrow R$.

Combining Propositions 4.10, 4.12 and 4.13 leads to the following corollaries.

Corollary 4.14. *Let X be a trellis and S be an L-fuzzy set on X. Then it holds that*

(i) $(S) = \Downarrow (S) = (\Downarrow S) = S$;

(ii) $(S) = \Uparrow (S) = (\Uparrow S) = S$.

Corollary 4.15. *Let X be a trellis and S be an L-fuzzy set on X. Then it holds that*

(i) *S is the smallest L-fuzzy down-set containing S;*

(ii) *S is the smallest L-fuzzy up-set containing S.*

Remark 4.16. Note that if S be an L-fuzzy set on a trellis X, then $\Downarrow (\Downarrow S) \neq \Downarrow S$ $(resp.\ \Uparrow (\Uparrow S) \neq \Uparrow S)$. Indeed, let X be the trellis given by the Hasse diagram in Figure 1 and S be the L-fuzzy set on X given by:

$S = \{< a, 0.8 >, < b, 0.6 >, < c, 0.2 >, < d, 0.7 >, < e, 0.1 >\}$. It holds that

$\mu_{\Downarrow S}(a) = \sup\limits_{y \in \uparrow a} \mu_S(y) = 0.8$ such that $\uparrow a = \{a, b, c, d, e\}$,

$\mu_{\Downarrow S}(b) = \sup\limits_{y \in \uparrow b} \mu_S(y) = 0.6$ such that $\uparrow b = \{b, c, e\}$,

$\mu_{\Downarrow S}(c) = \sup\limits_{y \in \uparrow c} \mu_S(y) = 0.7$ such that $\uparrow c = \{c, d, e\}$,

$\mu_{\Downarrow S}(d) = \sup\limits_{y \in \uparrow d} \mu_S(y) = 0.7$ such that $\uparrow d = \{d, e\}$.

$\mu_{\Downarrow S}(e) = \sup\limits_{y \in \uparrow e} \mu_S(y) = 0.1$ such that $\uparrow e = \{e\}$.

Therefore, $\Downarrow S = \{< a, 0.8 >, < b, 0.6 >, < c, 0.7 >, < d, 0.7 >, < e, 0.1 >\}$. In the same way, we obtain that $\Downarrow (\Downarrow S) = \{< a, 0.8 >, < b, 0.7 >, < c, 0.7 >, < d, 0.7 >, < e, 0.1 >\}$. Thus, $\Downarrow (\Downarrow S) \neq \Downarrow S$.

The following proposition shows the interaction of the support and the kernel with the (resp. $\Downarrow$) and (resp. $\Uparrow$) of an L-fuzzy set on a given trellis.

Proposition 4.17. *Let X be a trellis and S be an L-fuzzy set on X. The following equalities hold:*

(i) *$Supp(S) = \downharpoonleft Supp(S)$ and $Supp(\Downarrow S) = \downarrow Supp(S)$;*

(ii) *$Supp(S) = \upharpoonright Supp(S)$ and $Supp(\Uparrow S) = \uparrow Supp(S)$;*

(iii) *$Ker(S) = \downharpoonleft Ker(S)$ and $Ker(\Downarrow S) = \downarrow Ker(S)$;*

(iv) *$Ker(S) = \upharpoonright Ker(S)$ and $Ker(\Uparrow S) = \uparrow Ker(S)$.*

Proof. We only give the proof of (i) for , as the others can be proved analogously.

Let $x \in Supp(S)$, then it holds that $\mu_S(x) > 0$. Hence, $\sup\limits_{y \in \upharpoonright x} \mu_S(y) > 0$. This implies that there exists $t \in \upharpoonright x$ such that $\mu_S(t) > 0$. Hence, $t \in Supp(S)$. Since $t \in \upharpoonright x$, it follows that $x \in \downharpoonleft Supp(S)$. Thus, $Supp(S) \subseteq \downharpoonleft Supp(S)$.

On the other hand, let $x \in \downharpoonleft Supp(S)$. Then it holds that $y \in \upharpoonright x$ such that $\mu_S(y) > 0$, then $\sup\limits_{y \in \upharpoonright x} \mu_S(y) = \mu_S(x) > 0$. Hence, $x \in Supp(S)$. Therefore, $Supp(S) = \downharpoonleft Supp(S)$

□

In the following result, we show that any L-fuzzy ideal (resp. L-fuzzy filter) on a trellis X is an L-fuzzy down-set (resp. L-fuzzy up-set) on X.

Theorem 4.18. *Let X be a trellis and S be an L-fuzzy set on X. The following implications hold* *(i) if S is an L-fuzzy ideal, then S is an L-fuzzy down-set;*

(ii) if S is an L-fuzzy filter, then S is an L-fuzzy up-set.

Proof. (i) Let $x, y \in X$ such that $x \trianglelefteq y$. Since S is an L-fuzzy ideal, it follows that $\mu_S(x) = \mu_S(x \sqcap y) \geq \mu_S(x) \vee \mu_S(y)$. Hence, $\mu_S(x) \geq \mu_S(y)$. Thus, S is an L-fuzzy down-set.

(ii) Follows from Proposition 3.8, (i) and Proposition 4.1.

□

In view of Theorem 4.18 and Proposition 4.10, we obtain the following corollary.

Corollary 4.19. *Let X be a trellis and S be an L-fuzzy set on X. The following implications hold*

(i) if S is an L-fuzzy ideal, then $S = S = \Downarrow S$;

(ii) if S is an L-fuzzy filter, then $S = S = \Uparrow S$.

Remark 4.20. One can verify that the converse of the above implications of Theorem 4.18 and Corollary 4.19 does not necessarily hold.

5 Principal (crisp) L-fuzzy ideals and L-fuzzy filters on a trellis

In this section, we provide some interesting properties of principal ideal (resp. principal filter) on a given trellis. In the fuzzy setting, we introduce and study the notion of principal L-fuzzy ideal (resp. principal L-fuzzy filter) on a trellis. Moreover, we pay particular attention to their characterizations in the absence of the (associativity) transitivity property.

5.1 Principal ideals and filters on a trellis

In this subsection, we provide some interesting properties and characterizations of principal ideals (resp. filters) on a given trellis. First, we need to recall the following definitions of an ideal (resp. a filter) generated by a non-empty subset of a trellis.

Definition 5.1. Let X be a trellis and S be a non-empty subset of X. The ideal generated by S, denoted by $\langle S$, is the intersection of all ideals of X containing S. The filter generated by S, denoted by $S\rangle$, is the intersection of all filters of X containing S. If S is a singleton $\{x\}$, then we write $\langle x$ the ideal generated by x and $x\rangle$ the filter generated by x.

Definition 5.2. Let X be a trellis. I and F be two subsets on X. Then

(i) I is called a principal ideal on X, if there exists an element $x \in X$ such that $I = \langle x$;

(ii) F is called a principal filter on X, if there exists an element $x \in X$ such that $I = x\rangle$.

In the case of a lattice L, for any element $x \in L$, the set $\downarrow x$ (resp. $\uparrow x$) is an ideal (resp. a filer) on L, and the principal ideals (resp. principal filters) of L are characterized by these sets. In contrary, in the case of a trellis X the set $\downarrow x$ (resp. $\uparrow x$) is not necessarily an ideal (resp. a filer) on X. In the following proposition, we show a necessary and sufficient condition that $\downarrow x$ (resp. $\uparrow x$) is an ideal (resp. a filer) on X.

Proposition 5.3. *Let X be a trellis and x be an element on X. Then it holds that*

(i) $\downarrow x$ is an ideal on X if and only if x is a left-transitive element;

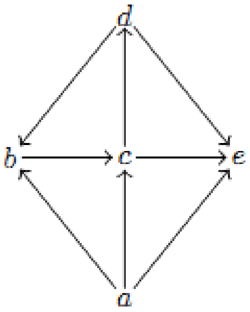

Figure 2: The trellis $X = \{a, b, c, d, e\}$ of Remark 5.5.

(ii) $\uparrow x$ *is a filter on* X *if and only if* x *is a right-transitive element.*

Proof. It is easy to check that if $\downarrow x$ is an ideal (resp. $\uparrow x$ is a filter) on X, then x is a left-transitive (resp. x is right-transitive) element. Conversely,

(i) Suppose that x is a left-transitive element on X and we show that $\downarrow x$ is an ideal.

  (a) Let $y \in \downarrow x$ and $z \trianglelefteq y$. Since x is left-transitive, it follows that $z \trianglelefteq x$. Hence, $z \in \downarrow x$.

  (b) Let $y, z \in \downarrow x$, then it holds that $y \trianglelefteq x$ and $z \trianglelefteq x$. This implies that $y \sqcup z \trianglelefteq x$. Hence, $y \sqcup z \in \downarrow x$. Thus, $\downarrow x$ is an ideal.

(ii) Follows from Proposition 3.8 and (i).

□

Proposition 5.4. *Let* X *be a trellis and* x *be an element on* X*. The following implications hold:*

(i) if x *is a left-transitive element, then* $\downharpoonleft x$ *is an ideal on* X*;*

(ii) if x *is a right-transitive element, then* $\upharpoonright x$ *is a filter on* X*.*

Proof. (i) Suppose that x is a left-transitive element on X and we show that $\downharpoonleft x$ is an ideal.

  (a) Let $y \in \downharpoonleft x$ and $z \trianglelefteq y$. Then it holds that $z \trianglelefteq y \lesssim x$. Hence, $z \lesssim x$. Thus, $z \in \downharpoonleft x$.

  (b) Let $y, z \in \downharpoonleft x$, then it holds that $y \lesssim x$ and $z \lesssim x$. From Theorem 2.8 it follows that $y \sqcup z \trianglelefteq x$. Hence, $y \sqcup z \in \downharpoonleft x$. Thus, $\downharpoonleft x$ is an ideal.

(ii) Follows from (i) and Propositions 3.8, 4.1.

□

Remark 5.5. The converse of the above implications does not necessarily hold. Indeed, let X be the trellis given by the Hasse diagram in the following figure. Notice that $\downharpoonleft d = \{a, b, c, d\}$ is ideal on X. But, the fact $b \trianglelefteq c$, $c \trianglelefteq d$ and $b \ntrianglelefteq d$ implies that d is not a left-transitive element on X.

In general, the following result characterizes principal ideals (resp. principal filters) on a given trellis.

Theorem 5.6. *Let* X *be a trellis,* x *an element on* X*. The following equivalences hold:*

(i) $\downharpoonleft x = \langle x$ *if and only if* $\downharpoonleft x$ *is a* $\sqcup$*-semi trellis of* X*;*

(ii) $\upharpoonright x = x\rangle$ *if and only if* $\upharpoonright x$ *is an* $\sqcap$*-semi trellis of* X*.*

Proof. We only give the proof of (i), as (ii) can be proved analogously. Let $x \in X$ such that $\downharpoonleft x$ is a $\sqcup$-semi trellis of X, and we show that $\downharpoonleft x = \langle x$.

(a) Let $y \in \downharpoonleft x$ and $z \in X$ such that $z \trianglelefteq y$. This implies that $z \lesssim y \lesssim x$. Since $\lesssim$ is transitive, it follows that $z \lesssim x$. Hence, $z \in \downharpoonleft x$.

(b) Let $y, z \in \downharpoonleft x$, then $y \lesssim x$ and $z \lesssim x$. Since $\downharpoonleft x$ is a $\sqcup$-semi trellis of X (i.e.; $\{a \sqcup b\} \in \downharpoonleft x$ for any $a, b \in \downharpoonleft x$), it follows that $y \sqcup z \in \downharpoonleft x$. Hence, $\downharpoonleft x$ is an ideal on X.

Now, we show that $\downharpoonleft x$ is the smallest ideal containing x. Suppose that there exists another ideal J containing x such that $\downharpoonleft x \nsubseteq J$. Then there exists $z \in X$ such that $z \in \downharpoonleft x$ and $z \notin J$. The fact that $z \in \downharpoonleft x$ implies that $z \lesssim x$, i.e.; there exist a finite sequence of elements $(a_1, ..., a_n)$ of X such that $z \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq a_n \trianglelefteq x$. Since J is an ideal containing x, it follows that $z \in J$ is a contradiction. Hence, $\downharpoonleft x \subseteq J$. Thus, $\downharpoonleft x = \langle x$.

□

Combining Proposition 5.3 and Theorem 5.6 leads to the following corollary.

Corollary 5.7. *Let X be a trellis and $x \in X$. The following equivalences hold:*

(i) $\downarrow x = \downharpoonleft x = \langle x$ if and only if x is a left-transitive element;

(ii) $\uparrow x = \upharpoonleft x = x\rangle$ if and only if x is a right-transitive element.

5.2 Principal L-fuzzy ideals and L-fuzzy filters on a trellis

In this section, we introduce the notion of principal L-fuzzy ideal (resp. principal L-fuzzy filter) on a given trellis. Similarly to the crisp case, we characterize these notions in terms of down- and up-sets generated by L-fuzzy singletons.

Definition 5.8. Let X be a trellis and $\tilde{x}$ be an L-fuzzy singleton on X, Then

(i) the principal L-fuzzy ideal generated by an L-fuzzy singleton $\tilde{x}$ is the smallest L-fuzzy ideal contains $\tilde{x}$, denoted by $\langle\langle \tilde{x}$;

(ii) the principal L-fuzzy filter generated by an L-fuzzy singleton $\tilde{x}$ is the smallest L-fuzzy filter contains $\tilde{x}$, denoted by $\tilde{x}\rangle\rangle$.

Next, we characterize principal L-fuzzy ideals (resp. filters) on a trellis in terms of the down-sets (resp. up-sets) generated by L-fuzzy singletons on that trellis. We start with the following key results.

Proposition 5.9. *Let X be a trellis and $\tilde{x}$ be an L-fuzzy singleton on X. The following equivalences hold:*

(i) $\Downarrow \tilde{x}$ is an L-fuzzy ideal on X if and only if x is a left-transitive element;

(ii) $\Uparrow \tilde{x}$ is an L-fuzzy filter on X if and only if x is a right-transitive element.

Proof. (i) Similarly to the crisp case, one can easily prove that if $\Downarrow \tilde{x}$ is an L-fuzzy ideal, then x is a left-transitive element. Conversely, let $x \in X$, then from Theorem 3.9, it suffices to show that

$$\mu_{\Downarrow \tilde{x}}(a \sqcup b) = \mu_{\Downarrow \tilde{x}}(a) \wedge \mu_{\Downarrow \tilde{x}}(b), \text{ for any } a, b \in X\,.$$

The fact that x is a left-transitive element implies that ($a \sqcup b \trianglelefteq x$ if and only if $a \trianglelefteq x$ and $b \trianglelefteq x$). Hence,

$$\mu_{\Downarrow \tilde{x}}(a \sqcup b) = \sup_{a \sqcup b \trianglelefteq t} \mu_{\tilde{x}}(t) = \sup_{a \trianglelefteq t} \mu_{\tilde{x}}(t) \,\wedge\, \sup_{b \trianglelefteq t} \mu_{\tilde{x}}(t)\,.$$

Thus,

$$\mu_{\Downarrow \tilde{x}}(a \sqcup b) = \mu_{\Downarrow \tilde{x}}(a) \wedge \mu_{\Downarrow \tilde{x}}(b), \text{ for any } a, b \in X\,.$$

Therefore, $\Downarrow \tilde{x}$ is an L-fuzzy ideal on X.

(ii) Follows dually by using (i) and Propositions 3.8, 4.1.

□

In the following result, we show a characterization of a principal L-fuzzy ideal (resp. L-fuzzy filter) in terms of a down-set (resp. up-set) generated by an L-fuzzy singleton.

Theorem 5.10. *Let X be a trellis, x an element on X. The following equivalences hold:*

(i) $\tilde{x} = \langle\langle \tilde{x}$ if and only if $\tilde{x}$ is a $\sqcup$-semi L-fuzzy trellis of X;

(ii) $\tilde{x} = \tilde{x}\rangle\rangle$ if and only if $\tilde{x}$ is a $\sqcap$-semi L-fuzzy trellis of X.

Proof. (i) Let $a, b \in X$. First, we show that $\tilde{x}$ is an L-fuzzy ideal on X.

$$\begin{aligned} \mu_{\tilde{x}}(a \sqcup b) &= \sup_{t \in \upharpoonright(a \sqcup b)} \mu_{\tilde{x}}(t) \\ &= \sup_{a \sqcup b \lesssim t} \mu_{\tilde{x}}(t). \end{aligned}$$

Since $\tilde{x}$ is a $\sqcup$-semi L-fuzzy trellis then, it holds that $a \sqcup b \lesssim t$ if and only if $a \lesssim t$ and $b \lesssim t$. Hence,

$$\begin{aligned} \mu_{\tilde{x}}(a \sqcup b) &= \sup_{t \in \upharpoonright a} \mu_{\tilde{x}}(t) \wedge \sup_{t \in \upharpoonright b} \mu_{\tilde{x}}(t) \\ &= \mu_{\tilde{x}}(a) \wedge \mu_{\tilde{x}}(b). \end{aligned}$$

Thus, $\mu_{\tilde{x}}$ is an L-fuzzy ideal containing x.
Now, we show that $\mu_{\tilde{x}}$ is the smallest L-fuzzy ideal containing $\tilde{x}$. Let J be another L-fuzzy ideal containing $\tilde{x}$ i.e., $\mu_{\tilde{x}}(t) \leq \mu_J(t)$, for any $t \in X$.

$$\begin{aligned} \mu_{\tilde{x}}(t) &= \sup_{y \in \upharpoonright t} \mu_{\tilde{x}}(y) \leq \sup_{y \in \upharpoonright t} \mu_J(y) \\ &= \mu_J(t) = \mu_J(t). \end{aligned}$$

Thus, $\tilde{x} \subseteq J$.

(ii) Follows dually by using (i) and Propositions 3.8, 4.1.

□

In the following proposition, we show that the kernel of a principal L-fuzzy ideal (resp. L-fuzzy filter) is a crisp principal ideal (resp. filter).

Proposition 5.11. *Let X be a trellis and x be an element on X. Then it holds that*

(i) $Ker(\tilde{x}) = \downharpoonright x$ and $Ker(\Downarrow \tilde{x}) = \downarrow x$;

(ii) $Ker(\tilde{x}) = \upharpoonright x$ and $Ker(\Uparrow \tilde{x}) = \uparrow x$.

Proof. (i) We only give the proof of $Ker(\tilde{x}) = \downharpoonright x$, as $Ker(\Downarrow \tilde{x}) = \downarrow x$ can be proved analogously. From Proposition 4.17, it holds that $Ker(\tilde{x}) = \downharpoonright Ker(\tilde{x})$. This means that

$$Ker(\tilde{x}) = \downharpoonright \{t \in X \mid \mu_{\tilde{x}}(t) = 1\}.$$

By the definition of L-fuzzy singleton, we know that $\mu_{\tilde{x}}(t) = 1$ if and only if $t = x$. Hence, $Ker(\tilde{x}) = \downharpoonright \{x\} = \downharpoonright x$.

(ii) Follows from Proposition 4.1 and (i).

□

6 Conclusion

In this paper, we have studied the notions of principal ideal and principal filter on a trellis. Their characterizations are shown in terms of left-transitive (resp. right-transitive) elements on a trellis. We have also introduced an L-fuzzy version of these notions on a trellis and provided their various proprieties and characterizations. In addition, we have introduced and studied the notions of L-fuzzy down-sets (resp. up-sets) on a trellis to facilitate the study of these notions of principal L-fuzzy ideals and L-fuzzy filters. We have paid close attention to what happens in the absence of the (associativity) transitivity property on trellises. Future work will be directed to study (intuitionistic) fuzzy ideals and filters on fuzzy trellises.

References

[1] S.P. Bhatta, H. Shashirekha, Some characterizations of completeness for trellises in terms of joins of cycles, Czechoslovak Mathematical Journal, vol. 54, pp. 267–272, 2004.

[2] H. A. Abu-Donia and S. S. Hassan, “Fixed points in complete fuzzy metric spaces using common property (E.A.),” Mathematics, vol. 9, no. 8, p. 867, 2021.

[3] S.P. Bhatta, S. George, Some fixed point theorems for pseudo ordered sets, Algebra and Discrete Mathematics, vol. 11, no. 1, pp. 17–22, 2011.

[4] Y. Bo, W. Wangming, Fuzzy ideals on a distributive lattice, Fuzzy Sets and Systems, vol. 35, no. 2, pp. 231–240, 1990.

[5] S. Boudaoud, L. Zedam, S. Milles, Principal intuitionistic fuzzy ideals and filters on a lattice, Discussiones Mathematicae General Algebra and Applications, vol. 40, no. 1, pp. 75–88, 2020.

[6] N. Bourbaki, Topologie générale, Springer-Verlag, Berlin Heidelberg, 2007.

[7] B.A. Davey, H.A. Priestley, Introduction to trelliss and Order. Second ed, Cambridge University Press, Cambridge, 2002.

[8] B. Davvaz, O. Kazanci, A new kind of fuzzy Sublattice (Ideal, Filter) of a lattice, Int. J. Fuzzy. Syst., vol. 13, no. 1, pp. 55–63,2011.

[9] T. Yang, L. Liu, and C. Xu, “On fuzzy closure operators in lattice theory,” Symmetry, vol. 12, no. 11, p. 1875, 2020.

[10] E. Fried, Tournaments and non-associative lattices, Ann. Univ. Sci. Budapest, Sect. Math, vol. 13, pp. 151–164, 1970.

[11] E. Fried, G. Grätzer, Some examples of weakly associative lattices, Colloquium Mathematicum, vol. 27, no. 2, pp. 215–221, 1973.

[12] K. Gladstien, Characterization of completeness for trellises of finite length, Algebra Universalis, vol. 3, pp. 341–344, 1973.

[13] J. A. Goguen, L-Fuzzy Sets, Journal of Mathematical Analysis and Applications, vol. 18, no. 1, pp. 145–174, 1967.

[14] G. Grätzer, General Lattice Theory, Academic Press, New York, 1978.

[15] Y. Liu, K. Qin, Y. Xu, Fuzzy prime filters of lattice implication algebras, Fuzzy Information and Engineering, vol. 3, no. 3, pp. 235–246, 2011.

[16] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, Types of fuzzy ideals in fuzzy lattices, Journal of Intelligent and Fuzzy Systems, vol. 28, no. 2, pp. 929–945, 2015.

[17] S. Milles, L. Zedam, E. Rak, Characterizations of intuitionistic fuzzy ideals and filters based on lattice operations, Journal of Fuzzy Set Valued Analysis, vol. 2, pp. 143–159, 2017.

[18] H. Prodinger, Congruences defined by languages and filters, Information and Control, vol. 44, no. 1, pp. 36–46, 1980.

[19] B.S. Schröder, Ordered Sets, Birkhauser, Boston, 2002.

[20] H. Skala, Trellis theory, Algebra Universalis, vol. 1, pp. 218–233, 1971.

[21] H. Skala, Trellis theory, American Mathematical Society, Providence, 1972.

[22] M. H. Stone, The theory of representation of Boolean algebras, Trans. Amer. Math. Soc., vol. 40, no. 1, pp. 37–111, 1936.

[23] M. A. Rodríguez, J. R. Castro, and F. Moreno, “New approaches for fuzzy ideals and lattice structures in decision-making problems,” Applied Soft Computing, vol. 118, p. 108466, 2022.

[24] F. Včelař and Z. Pátíková, On fuzzification of Tarski’s fixed point theorem without transitivity, Fuzzy Sets and Systems, vol. 320, pp. 93–113, 2017.

[25] B. Van Gasse, G. Deschrijver, C. Cornelis, E. Kerre, Filters of residuated lattices and triangle algebras, Inform. Sci., vol. 480, no. 16, pp. 3006–3020, 2010.

[26] S. Willard, General Topology, Addison-Wesley Publishing Company, Massachusetts, 1970.

[27] L.A. Zadeh, Fuzzy sets, Information and Control, vol. 8, pp. 331–352, 1965.