



New Class of Equivalence Classes of Neutrosophic Fuzzy Delta-Algebras

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Abstract

This work analyzes neutrosophic fuzzification in δ -algebra, applies novel classes of neutrosophic fuzzy (NF) to algebra, and explores the ideas of $NF\delta$ -ideal ($NF\delta - I$), $NF\delta$ -subalgebra ($IF\delta - SA$), δ -homomorphism, and $NF\bar{\delta}$ -ideal ($NF\bar{\delta} - I$), exploring some of their descriptions. We shall demonstrate a variety of applications, including the notations $\simeq_H, \simeq_G, \simeq_J$ and \bowtie_α on $NF\delta - SA(V) = \{M / M \text{ is } NF\delta - SA \text{ of } M\}$. We will also investigate their equivalence classes, evaluate our findings in light of the unique ideas offered in this work, and investigate related characteristics.

Keywords: Neutrosophic Fuzzy set; δ -algebra; Neutrosophic fuzzy δ -subalgebra; Equivalence Classes.

1. Introduction

In 1965, Zadeh [1] introduced the concept of a fuzzy set (FS), which is a class of objects with a continuous range of membership grades. Atanassov [2] presented the intuitionistic fuzzy set (IFS) in 1986. In addition to providing the opportunity to accurately describe the issue based on current knowledge and observations, the idea of non-membership, which seems to be more accurate at quantifying uncertainty, also makes it possible to use current information and perception to describe the problem. In 1999, Smarandache [3] proposed the notion of neutrosophic sets. Subsequently, numerous mathematicians in various fields [4–7] are studying the notion of neutrosophic fuzzy sets.

In 2021, the concept of δ -algebra [8] was shown. Next, the concept of intuitionistic fuzzy δ -algebra was described [9], and addressed its applications. After that, Abbas et al. [10] introduced new varieties of Neutrosophic. Atshan and Khalil [11] suggested and explored numerous varieties of ρ -algebra, including neutrosophic / full neutrosophic / complete neutrosophic ρ -ideal / subalgebra. Recently, the notion of algebra is discussed in neutrosophic [12] and other non-classical sets like fuzzy set [13-16], soft set [17,18], permutation set [19-21], and others.

In this study, we will examine notions such as ($NF\delta - I$), ($IF\delta - SA$), δ -homomorphism, and ($NF\bar{\delta} - I$). A number of applications, including the notations $\simeq_H, \simeq_G, \simeq_J$ and \bowtie_α on $NF\delta - SA(V)$ will also be demonstrated on the set $NF\delta - SA(V) = \{M / M \text{ is } NF\delta - SA \text{ of } M\}$. Additionally, we will examine and talk about their equivalency classes.

2. Preliminaries

Here, a few definitions that are necessary for our new results are reviewed in this section.

Definition 2.1[8]: A triple $(V, *, 0)$ is called a δ -algebra (δ -A) if $0 \in V$ and fulfilling the following assumptions:

- i. $\tau * \tau = 0$
- ii. $0 * \tau = 0$
- iii. If $\tau * \kappa = 0$ and $\kappa * \tau = 0 \rightarrow \tau = \kappa$, for all $\tau, \kappa \in V$,
- iv. For all $\tau \neq \kappa \in V - \{0\} \rightarrow \tau * \kappa = \kappa * \tau \neq 0$,
- v. For all $\tau \neq \kappa \in V - \{0\} \rightarrow (\tau * (\tau * \kappa)) * (\kappa * \tau) = 0$.

Definition 2.2 [8]: Let $V \neq \emptyset$. Then $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_I(\tau) \rangle \mid \tau \in V \}$ is a neutrosophic set, where $M_H(\tau): V \rightarrow [0,1]$, $M_G(\tau): V \rightarrow [0,1]$, and $M_I(\tau): V \rightarrow [0,1]$ are functions. Values $M_H(\tau)$, $M_G(\tau)$ and $M_I(\tau)$ indicate to the degree of membership (*Me.*), non-membership (*N - Me.*), and indeterminate (*In.*) of τ to M , respectively.

Definition 2.3 [9]: Let $\emptyset \neq K \subseteq V$, where $(V, *, 0)$ is a (δ -A), K is said to be δ -subalgebra (δ -SA) of V if $\tau * \kappa \in K$, for any $\tau, \kappa \in K$.

Definition 2.4[8]:

Let $(V, *, 0)$ be a (δ -A) and $\emptyset \neq K \subseteq V$. K is said to be a δ -ideal of V if

- i. $\tau, \kappa \in K \rightarrow \tau * \kappa \in K$,
- ii. $\tau * \kappa \in K$ and $\kappa \in K \rightarrow \tau \in K$, for all $\tau, \kappa \in V$.

Definition 2.5 [2]:

Let $\varepsilon_G(k) = 1 - \zeta_G(k) - \vartheta_G(k)$ be the IFS index or hesitation margin of k in G is the degree $k \in \mathcal{F}$ to the IFS G and $\varepsilon_G(k) \in [0,1]$, i.e., $\varepsilon_G(k): \mathcal{F} \rightarrow [0,1]$ and $0 \leq \varepsilon_G(k) \leq 1$, for all $k \in \mathcal{F}$.

Definition 2.6 [22]:

Let $H = (\zeta_H, \vartheta_H)$ be an IFS in \mathcal{F} with $s \in [0,1]$. Then $M(\zeta_H, s) = \{k \in \mathcal{F} \mid \zeta_H(k) \geq s\}$ is said to be a ζ -level s -cut of H .

Definition 2.7[22]:

Let $H = (\zeta_H, \vartheta_H)$ be an IFS in \mathcal{F} with $s \in [0,1]$. Then $N(\vartheta_H, s) = \{k \in \mathcal{F} \mid \vartheta_H(k) \leq s\}$ is said to be a ϑ -level s -cut of H .

3. A Neutrosophic Fuzzy δ -subalgebras in δ -algebras.

In this section, we look into and examine a number of new concepts, including δ -homomorphism, ($NF\delta$ -SA), ($NF\delta$ -I), and ($NF\bar{\delta}$ -I). Some basic properties are also given.

Definition 3.1: Assume that $(V, *, 0)$ is a δ -algebra (briefly, δ -A) and $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_I(\tau) \rangle \mid \tau \in V \}$ is an (NFS) of V . We say M is a neutrosophic fuzzy δ -subalgebra of V (briefly, $NF\delta$ -SA) if such that;

- (i) $M_H(\tau * \kappa) \geq \min\{M_H(\tau), M_H(\kappa)\}$,
- (ii) $M_G(\tau * \kappa) \leq \max\{M_G(\tau), M_G(\kappa)\}$,
- (iii) $M_I(\tau * \kappa) \geq \min\{M_I(\tau), M_I(\kappa)\}, \forall \tau, \kappa \in V$.

Example 3.2: Assume that $V = \{0, r, s, p\}$ is a set, and let $(*)$ be defined on V as indicated by the Table1:

Table 1: $(V, *, 0)$ is $(\delta - A)$.

*	0	r	s	p
0	0	0	0	0
r	r	0	r	r
s	s	r	0	r
p	p	r	r	0

Then $(V, *, 0)$ is a $(\delta - A)$. Let each one of $M_H = \begin{pmatrix} 0 & r & s & p \\ 0.5 & 0.3 & 0.3 & 0.5 \end{pmatrix}$, $M_G = \begin{pmatrix} 0 & r & s & p \\ 0.3 & 0.6 & 0.6 & 0.3 \end{pmatrix}$ and $M_J = \begin{pmatrix} 0 & r & s & p \\ 0.2 & 0.1 & 0.1 & 0.2 \end{pmatrix}$ be a (FS) for $(Me.)$, $(N - Me.)$, and $(In.)$ of M respectively. Then M is a $(NF\delta - SA)$ of V .

Definition 3.3: Let $(V, *, 0)$ be a $\delta - algebra$ (briefly, $\delta - A$) & $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ be a (NFS) of V . Then M is called a neutrosophic fuzzy $\delta - ideal$ of V (briefly, $NF\delta - I$) if such that

- i. $M_H(\tau * \kappa) \geq \min\{M_H(\tau), M_H(\kappa)\}$, $M_G(\tau * \kappa) \leq \max\{M_G(\tau), M_G(\kappa)\}$, and $M_J(\tau * \kappa) \geq \min\{M_J(\tau), M_J(\kappa)\}$.
- ii. $M_H(\tau) \geq \min\{M_H(\tau * \kappa), M_H(\kappa)\}$, $M_G(\tau) \leq \max\{M_G(\tau * \kappa), M_G(\kappa)\}$, $M_J(\tau) \geq \min\{M_J(\tau * \kappa), M_J(\kappa)\}$, $\forall \tau, \kappa \in V$.

Example 3.4:

By using Table1, we consider that $(V, *, 0)$ is a $(\delta - A)$. Let $M_H = \begin{pmatrix} 0 & r & s & p \\ 0.3 & 0.2 & 0.2 & 0.3 \end{pmatrix}$ be a (FS) for $(Me.)$ of M , $M_G = \begin{pmatrix} 0 & r & s & p \\ 0.4 & 0.6 & 0.6 & 0.4 \end{pmatrix}$ be a (FS) for $(N - Me.)$ of M , and $M_J = \begin{pmatrix} 0 & r & s & p \\ 0.3 & 0.2 & 0.2 & 0.3 \end{pmatrix}$ be a (FS) for $(In.)$ of M . Hence M is a $(NF\delta - I)$ of V .

Definition 3.5: Let $(V, *, 0)$ be a $\delta - algebra$ (briefly, $\delta - A$) and $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ is an (NFS) of V . We say M is a neutrosophic fuzzy pseudo $\bar{\delta} - ideal$ of V (briefly, $NF\bar{\delta} - I$) if such that

- i. $M_H(0) \geq M_H(\kappa)$, $M_G(0) \leq M_G(\kappa)$, and $M_J(0) \geq M_J(\kappa)$.
- ii. $M_H(\tau * \kappa) \geq \min\{M_H(\tau), M_H(\kappa)\}$, $M_G(\tau * \kappa) \leq \max\{M_G(\tau), M_G(\kappa)\}$ and $M_J(\tau * \kappa) \geq \min\{M_J(\tau), M_J(\kappa)\}$, $\forall \tau, \kappa \in V$.

Example 3.6: By using Table (1) in Example (3.2) and a neutrosophic fuzzy set, we get M is a neutrosophic fuzzy $\bar{\delta} - ideal$ of V .

Note 3.7: Based on the aforementioned definitions, we shall consider the following:

- i. $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ is $(NF\delta - SA)$ if it is $(NF\delta - I)$,
- ii. Let M be a $(NF\delta - SA)$ and such that [Definition (3.3)-(ii)], then M is $(NF\delta - I)$,
- iii. $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ is $(NF\delta - SA)$ if it is $(NF\bar{\delta} - I)$,
- iv. Let M be a $(NF\delta - SA)$ and such that [Definition (3.5)-(i)], then M is $(NF\bar{\delta} - I)$.

Lemma 3.8: Let $(V, *, 0)$ be a $(\delta - A)$ and $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ be a $(NF\delta - SA)$ of V , then $M_H(0) \geq M_H(\tau)$, $M_G(0) \leq M_G(\tau)$, and $M_J(0) \geq M_J(\tau)$, $\forall \tau \in V$.

Proof: Assume that $\tau \in V$. Hence $M_H(0) = M_H(\tau * \tau) \geq \min\{M_H(\tau), M_H(\tau)\} = M_H(\tau)$, $M_G(0) = M_G(\tau * \tau) \leq \max\{M_G(\tau), M_G(\tau)\} = M_G(\tau)$, also $M_J(0) = M_J(\tau * \tau) \geq \min\{M_J(\tau), M_J(\tau)\} = M_J(\tau)$. ■

Proposition 3.9: Assume that $(V, *, 0)$ is a $(\delta - A)$ and $M_i = \{ \langle \tau, M_{H_i}(\tau), M_{G_i}(\tau), M_{J_i}(\tau) \rangle \mid \tau \in V, i \in I \}$ is a collection of $(NF\delta - SA)$ of V , then $\bigcap_{i \in I} M_i$ is a $(NF\bar{\delta} - I)$ of V , such that $\bigcap_{i \in I} M_i = \{ \langle \tau, \min\{M_{H_i}(\tau)\}, \max\{M_{G_i}(\tau)\}, \min\{M_{J_i}(\tau)\} \rangle \mid \tau \in V \}$.

Proof: Let $\tau, \kappa \in V$. Then, we have $\min\{M_{H_i}(\tau * \kappa)\} \geq \min\{\min\{M_{H_i}(\tau), M_{H_i}(\kappa)\}\} = \min\{\min\{M_{H_i}(\tau)\}, \min\{M_{H_i}(\kappa)\}\}$ and $\max\{M_{G_i}(\tau * \kappa)\} \leq \max\{\max\{M_{G_i}(\tau), M_{G_i}(\kappa)\}\} = \max\{\max\{M_{G_i}(\tau)\}, \max\{M_{G_i}(\kappa)\}\}$. Furthermore, $\min\{M_{J_i}(\tau * \kappa)\} \geq \min\{\min\{M_{J_i}(\tau), M_{J_i}(\kappa)\}\} = \min\{\min\{M_{J_i}(\tau)\}, \min\{M_{J_i}(\kappa)\}\}$.

Hence $\bigcap_{i \in I} M_i = \{ \langle \tau, \min\{M_{H_i}(\tau)\}, \max\{M_{G_i}(\tau)\}, \min\{M_{J_i}(\tau)\} \rangle \mid \tau \in V \}$. Hence that the condition (ii) in Definition (3.5) is held. Now, let $\tau \in V$. Thus, we have:

$\min\{M_{H_i}(0)\} = \min\{M_{H_i}(\tau * \tau)\} \geq \min\{M_{H_i}(\tau), M_{H_i}(\tau)\} = \min\{M_{H_i}(\tau)\}$ and $\max\{M_{G_i}(0)\} = \max\{M_{G_i}(\tau * \tau)\} \leq \max\{M_{G_i}(\tau), M_{G_i}(\tau)\} = \max\{M_{G_i}(\tau)\}$. Also

$\min\{M_{J_i}(0)\} = \min\{M_{J_i}(\tau * \tau)\} \geq \min\{M_{J_i}(\tau), M_{J_i}(\tau)\} = \min\{M_{J_i}(\tau)\}$. Thus (i) in Definition [3.5] is verified. So, $\bigcap_{i \in I} M_i$ is a $(NF\bar{\delta} - I)$ of V . ■

Proposition 3.10: Assume that $(V, *, 0)$ is a $(\delta - A)$ and $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ be a $(NF\delta - SA)$ of V , then both of the sets $T_H = \{ \tau \in V \mid M_H(\tau) = M_H(0) \}$, $T_G = \{ \tau \in V \mid M_G(\tau) = M_G(0) \}$ and $T_J = \{ \tau \in V \mid M_J(\tau) = M_J(0) \}$ are $\delta -$ subalgebras of V .

Proof: Let $\tau, \kappa \in T_H$. Then $M_H(\tau) = M_H(0) = M_H(\kappa)$, also $M_H(\tau * \kappa) \geq \min\{M_H(\tau), M_H(\kappa)\} = M_H(0)$. From Lemma (3.7), we get $M_H(\tau * \kappa) = M_H(0)$ or correspondingly $\tau * \kappa \in T_H$. Also, let $\tau, \kappa \in T_G$. Therefore $M_G(\tau * \kappa) \leq \max\{M_G(\tau), M_G(\kappa)\} = M_G(0)$. So, by Lemma (3.7), we obtain $M_G(\tau * \kappa) = M_G(0)$. Hence $\tau * \kappa \in T_G$. Next let $\tau, \kappa \in T_J$. Then $M_J(\tau * \kappa) \geq \min\{M_J(\tau), M_J(\kappa)\} = M_J(0)$. By Lemma (3.7), we get $M_J(\tau * \kappa) = M_J(0)$. Hence $\tau * \kappa \in T_J$. ■

Definition 3.11: Let $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ be a $(NF\delta - SA)$ of V . Then M has finite image, if any image of M_H, M_G and M_J with finite cardinality (i.e $\text{Im}(M_H) = \{M_H(\tau) \mid \tau \in V\}$, $\text{Im}(M_G) = \{M_G(\tau) \mid \tau \in V\}$ and $\text{Im}(M_J) = \{M_J(\tau) \mid \tau \in V\}$ such that $|\text{Im}(M_H)| < \infty$, $|\text{Im}(M_G)| < \infty$ and $|\text{Im}(M_J)| < \infty$.

Definition 3.12: Let that $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ is a $(NF\delta - SA)$ of V and $\alpha \in [0, 1]$. The sets $Q(M_H, \alpha) = \{ \tau \in V \mid M_H(\tau) \geq \alpha \}$ (resp., $W(M_G, \alpha) = \{ \tau \in V \mid (M_G(\tau) \leq \alpha) \}$ and $R(M_J, \alpha) = \{ \tau \in V \mid M_J(\tau) \geq \alpha \}$ are said to be $H -$ level $\alpha -$ cut (resp., $G -$ level $\alpha -$ cut and $J -$ level $\alpha -$ cut) of M .

Proposition 3.13: Assume that $(V, *, 0)$ is a $(\delta - A)$ and $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ be a $(NF\delta - SA)$ of V , then each one of the $H -$ level $\alpha -$ cut, $G -$ level $\alpha -$ cut and $J -$ level $\alpha -$ cut of M is $(\delta - SA)$ of V . Any $\alpha \in [0, 1]$ with $\alpha \in \text{Im}(M_H) \cap \text{Im}(M_G) \cap \text{Im}(M_J)$ are said to be $H -$ level $\delta -$ subalgebra $(HL\delta - SA)$, $G -$ level $\delta -$ subalgebra $(GL\delta - SA)$, and $J -$ level $\delta -$ subalgebra $(JL\delta - SA)$ if and only if H is an $(NF\delta - SA)$ of V .

Proof: Assume $\tau, \kappa \in Q(M_H, \alpha)$. Therefore $M_H(\tau) \geq \alpha$ and $M_H(\kappa) \geq \alpha$. So, we consider that $M_H(\tau * \kappa) \geq \min\{M_H(\tau), M_H(\kappa)\} \geq \alpha$ so that $\tau * \kappa \in Q(M_H, \alpha)$. Hence $Q(M_H, \alpha)$ is a $(\delta - SA)$ of V . Moreover, let $\tau, \kappa \in W(M_G, \alpha)$. We get $M_G(\tau * \kappa) \leq \max\{M_G(\tau), M_G(\kappa)\} \leq \alpha$ and $\tau * \kappa \in W(M_G, \alpha)$. Then $W(M_G, \alpha)$ is a $(\delta - SA)$ of V . Also $\tau, \kappa \in R(M_J, \alpha)$, then $M_J(\tau * \kappa) \geq \min\{M_J(\tau), M_J(\kappa)\} \geq \alpha$ so that $\tau * \kappa \in R(M_J, \alpha)$. Hence $R(M_J, \alpha)$ is a $(\delta - SA)$ of V .

Conversely, assume τ and κ be two elements in V such that $M_H(\tau * \kappa) < \min\{M_H(\tau), M_H(\kappa)\}$. Let $a = \frac{1}{2} [M_H(\tau * \kappa) + \min\{M_H(\tau), M_H(\kappa)\}]$. Then $M_H(\tau * \kappa) < a < \min\{M_H(\tau), M_H(\kappa)\}$, thus $\tau * \kappa \notin Q(M_H, \alpha)$. However $\tau, \kappa \in Q(M_H, \alpha)$. But that is in conflict with. Hence $M_H(\tau * \kappa) \geq \min\{M_H(\tau), M_H(\kappa)\}, \forall \tau, \kappa \in V$. In other side, if $M_G(\tau * \kappa) > \max\{M_G(\tau), M_G(\kappa)\}$ for some $\tau, \kappa \in V$. Put $b = \frac{1}{2} [M_G(\tau * \kappa) + \max\{M_G(\tau), M_G(\kappa)\}]$, then we get that $M_G(\tau * \kappa) > b > \max\{M_G(\tau), M_G(\kappa)\}$ and hence $\tau, \kappa \in W(M_G, \alpha)$ and $\tau * \kappa \notin W(M_G, \alpha)$. This is in conflict with. Thus, we have $M_G(\tau * \kappa) \leq \max\{M_G(\tau), M_G(\kappa)\}, \forall \tau, \kappa \in V$. Also, assume $M_J(\tau * \kappa) < \min\{M_J(\tau), M_J(\kappa)\}$. Let $c = \frac{1}{2} [M_J(\tau * \kappa) + \min\{M_J(\tau), M_J(\kappa)\}]$. Then $M_J(\tau * \kappa) < c < \min\{M_J(\tau), M_J(\kappa)\}$, thus $\tau * \kappa \notin R(M_J, \alpha)$. However

$\tau, \kappa \in R(M_J, \alpha)$. But that is in conflict with. Hence $M_J(\tau * \kappa) \geq \min\{M_J(\tau), M_J(\kappa)\}, \forall \tau, \kappa \in V$. Hence M is an $(NF\delta - SA)$ of V . ■

Proposition 3.14: Assume φ be a $(\delta - SA)$ of V , then φ can be realized as both $(HL\delta - SA)$, $(GL\delta - SA)$ and $(JL\delta - SA)$ of some a $(NF\delta - SA)$ of V .

Proof: Assume $HL\delta - SA$ be a $(\delta - SA)$ of V , and take M_H, M_G and, M_J be fuzzy sets in V as specified by $M_H(\tau) = \begin{cases} r, & \text{if } \tau \in \varphi \\ 0, & \text{Otherwise.} \end{cases}, M_G(\tau) = \begin{cases} s, & \text{if } \tau \in \varphi \\ 1, & \text{Otherwise.} \end{cases}$ and, $M_J(\tau) = \begin{cases} p, & \text{if } \tau \in \varphi \\ 0, & \text{Otherwise.} \end{cases}, \forall \tau \in V$, where $r, s, p \in (0,1)$ are fixed real numbers with $r + s + p < 1$. Let $\tau, \kappa \in V$. Therefore $\tau * \kappa \in \varphi$ whenever $\tau, \kappa \in \varphi$. So, we get $M_H(\tau * \kappa) = \min\{M_H(\tau), M_H(\kappa)\}, M_G(\tau * \kappa) = \max\{M_G(\tau), M_G(\kappa)\}$ and $M_J(\tau * \kappa) = \min\{M_J(\tau), M_J(\kappa)\}$. If either $\tau \notin \varphi$ or $\kappa \notin \varphi$, then either $M_H(\tau) = 0$ or $M_H(\kappa) = 0$ and $M_J(\tau) = 0$ or $M_J(\kappa) = 0$. Moreover, we have either $M_G(\tau) = 1$ or $M_G(\kappa) = 1$. It follows that $M_H(\tau * \kappa) \geq 0 = \min\{M_H(\tau), M_H(\kappa)\}, M_G(\tau * \kappa) \leq 1 = \max\{M_G(\tau), M_G(\kappa)\}$ and, $M_J(\tau * \kappa) \geq 0 = \min\{M_J(\tau), M_J(\kappa)\}$. Then $M = \{\langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V\}$ is an $(NF\delta - SA)$ of V . Of course, $Q(M_H, \alpha) = \varphi = W(M_G, \alpha)$ and $R(M_J, \alpha)$. ■

Definition 3.15: Assume $Y: (V, *, 0) \rightarrow (K, \diamond, \zeta)$ is a mapping of $(\delta - A)$. We say Y is δ -homomorphism $(\delta - H)$ if $Y(\tau * \kappa) = Y(\tau) \diamond Y(\kappa), \forall \tau, \kappa \in V$. Also, $\mathfrak{S}^{-1}(E) = \{\langle \tau, Y^{-1}E_H(\tau), Y^{-1}E_G(\tau), Y^{-1}E_J(\tau) \rangle \mid \tau \in V\}$ is a (NFS) in $(\delta - A) V$ for any $(NFS) E = \{\langle \kappa, E_H(\kappa), E_G(\kappa), E_J(\kappa) \rangle \mid \kappa \in K\}$ of $(\delta - A) K$. Moreover, if $M = \{\langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V\}$ is an (NFS) in $(\delta - A) V$, then $Y(M)$ is (NFS) in K and specified by: $Y(M) = \{\langle \kappa, Y_{sup}E_H(\kappa), Y_{inf}E_G(\kappa), Y_{sup}E_J(\kappa) \rangle \mid \kappa \in K\}$, where

$$Y_{sup}E_H(\kappa) = \begin{cases} \sup\{E_H(\tau) \mid \tau \in Y^{-1}(\kappa)\}, & \text{if } Y^{-1}(\kappa) \neq \emptyset, \\ 0, & \text{Otherwise,} \end{cases}$$

$$Y_{inf}E_G(\kappa) = \begin{cases} \inf\{E_G(\tau) \mid \tau \in Y^{-1}(\kappa)\}, & \text{if } Y^{-1}(\kappa) \neq \emptyset, \\ 1, & \text{Otherwise,} \end{cases}, \quad \text{and}$$

$$Y_{sup}E_J(\kappa) = \begin{cases} \sup\{E_J(\tau) \mid \tau \in Y^{-1}(\kappa)\}, & \text{if } Y^{-1}(\kappa) \neq \emptyset, \\ 0, & \text{Otherwise,} \end{cases}, \forall \kappa \in K$$

Proposition 3.16: Assume $Y: (V, *, 0) \rightarrow (K, \diamond, \zeta)$ be a δ -epimorphism of δ -algebras. If $E = \{\langle \kappa, E_H(\kappa), E_G(\kappa), E_J(\kappa) \rangle \mid \kappa \in K\}$ a $(NF\delta - SA)$ of K , then $\mathfrak{S}^{-1}(E)$ is a $(NF\delta - SA)$ of V .

Proof: Let E under Y is a $(NF\delta - SA)$, for any $\tau, \kappa \in V$, we have

$$Y^{-1}E_H(\tau * \kappa) = E_H(Y(\tau * \kappa)) = E_H(Y(\tau) \diamond Y(\kappa)) \geq \min\{E_H(Y(\tau)), E_H(Y(\kappa))\} \\ = \min\{Y^{-1}E_H(\tau), Y^{-1}E_H(\kappa)\},$$

$$Y^{-1}E_G(\tau * \kappa) = E_G(Y(\tau * \kappa)) = E_G(Y(\tau) \diamond Y(\kappa)) \leq \max\{E_G(Y(\tau)), E_G(Y(\kappa))\} \\ = \max\{Y^{-1}E_G(\tau), Y^{-1}E_G(\kappa)\}.$$

Also, $Y^{-1}E_J(\tau * \kappa) = E_J(Y(\tau * \kappa)) = E_J(Y(\tau) \diamond Y(\kappa)) \geq \min\{E_J(Y(\tau)), E_J(Y(\kappa))\} = \min\{Y^{-1}E_J(\tau), Y^{-1}E_J(\kappa)\}$. Hence $Y^{-1}(E)$ is a $(NF\delta - SA)$ of V . ■

Proposition 3.17: Assume $Y: (V, *, 0) \rightarrow (K, \diamond, \zeta)$ be a δ -epimorphism of δ -algebras. If $E = \{\langle \tau, E_H(\tau), E_G(\tau), E_J(\tau) \rangle \mid \tau \in V\}$ a $(NF\delta - SA)$ of V , then $Y(E)$ is a $(NF\delta - SA)$ of K .

Proof: Assume E be a $(NF\delta - SA)$ of V and let $\kappa, \sigma \in K$. Observing that $\{\tau * \alpha \mid \tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\} \subseteq \{\tau \in V \mid \tau \in \mathfrak{S}^{-1}(\kappa \diamond \sigma)\}$, we have $Y_{sup}(E_H)(\kappa \diamond \sigma) = \sup\{E_H(\tau) \mid \tau \in Y^{-1}(\kappa \diamond \sigma)\}$

$$\geq \sup\{E_H(\tau * \alpha) \mid \tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\}$$

$$\geq \sup\{\min\{E_H(\tau), E_H(\alpha)\} \mid \tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\}$$

$$\begin{aligned}
&= \min\{\sup\{E_H(\tau)|\tau \in Y^{-1}(\kappa)\}, \sup\{E_H(\alpha)|\alpha \in Y^{-1}(\sigma)\}\} \\
&= \min\{Y_{\sup}(E_H)(\kappa), Y_{\sup}(E_H)(\sigma)\}, \\
Y_{\inf}(E_G)(\kappa \diamond \sigma) &= \inf\{E_G(\tau)|\tau \in Y^{-1}(\kappa \diamond \sigma)\} \\
&\leq \inf\{E_G(\tau * \alpha)|\tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\} \\
&\leq \inf\{\max\{E_G(\tau), E_G(\alpha)\}|\tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\} \\
&= \max\{\inf\{E_G(\tau)|\tau \in Y^{-1}(\kappa)\}, \inf\{E_G(\alpha)|\alpha \in Y^{-1}(\sigma)\}\} \\
&= \max\{Y_{\sup}(E_G)(\kappa), Y_{\sup}(E_G)(\sigma)\}.
\end{aligned}$$

$$\begin{aligned}
\text{And, } Y_{\sup}(E_j)(\kappa \diamond \sigma) &= \sup\{E_j(\tau)|\tau \in Y^{-1}(\kappa \diamond \sigma)\} \\
&\geq \sup\{E_j(\tau * \alpha)|\tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\} \\
&\geq \sup\{\min\{E_j(\tau), E_j(\alpha)\}|\tau \in Y^{-1}(\kappa) \text{ and } \alpha \in Y^{-1}(\sigma)\} \\
&= \min\{\sup\{E_j(\tau)|\tau \in Y^{-1}(\kappa)\}, \sup\{E_j(\alpha)|\alpha \in Y^{-1}(\sigma)\}\} \\
&= \min\{Y_{\sup}(E_j)(\kappa), Y_{\sup}(E_j)(\sigma)\}.
\end{aligned}$$

Then $Y(E) = \{\langle \kappa, Y_{\sup}E_H(\kappa), Y_{\inf}E_G(\kappa), Y_{\sup}E_j(\kappa) \rangle | \kappa \in K\}$ is a $(NF\delta - SA)$ of K . ■

Proposition 3.18: Assume $Y: (V, *, 0) \rightarrow (K, \diamond, \zeta)$ be a δ -epimorphism of δ -algebras. If $M = \{\langle \tau, M_H(\tau), M_G(\tau), M_j(\tau) \rangle | \tau \in V\}$ a $(NF\delta - I)$ of V , then $Y(M)$ is a $(NF\delta - I)$ of K .

Proof: Given that M is a $(NF\delta - I)$ of V . Hence from Proposition [3.17] and Note[3.7] we have $Y(M)$ is a $(NF\delta - I)$ of K . Consequently, condition (1) in Definition [3.3] is maintained. Since Y is surjective, thus for any $\kappa, \sigma \in K$, $\exists \tau, \alpha \in V$ such that $\tau \in Y^{-1}Y(\tau) = Y^{-1}(\kappa)$ and $\alpha \in Y^{-1}Y(\alpha) = Y^{-1}(\sigma)$. Also, $\tau * \alpha \in Y^{-1}(\kappa) * Y^{-1}(\sigma) = Y^{-1}(\kappa \diamond \sigma)$. Furthermore, noticing that $M_H(\tau) \geq \min\{M_H(\tau * \alpha), M_H(\alpha)\}$, $M_G(\tau) \leq \max\{M_G(\tau * \alpha), M_G(\alpha)\}$ and $M_j(\tau) \geq \min\{M_j(\tau * \alpha), M_j(\alpha)\}$. For any $\kappa, \sigma \in K$, we have

$$\begin{aligned}
Y_{\sup}(M_H)(\kappa) &= \sup\{M_H(\tau)|\tau \in Y^{-1}(\kappa)\} \\
&\geq \sup\{\min\{M_H(\tau * \alpha), M_H(\alpha)|\tau * \alpha \in Y^{-1}(\kappa \diamond \sigma) \text{ and } \alpha \in Y^{-1}(\sigma)\}\} \\
&= \min\{\sup\{M_H(\tau * \alpha)|\tau * \alpha \in Y^{-1}(\kappa \diamond \sigma)\}, \sup\{M_H(\alpha)|\alpha \in Y^{-1}(\sigma)\}\} \\
&= \min\{Y_{\sup}(M_H)(\kappa \diamond \sigma), Y_{\sup}(M_H)(\sigma)\}.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } Y_{\inf}(M_G)(\kappa) &= \inf\{M_G(\tau)|\tau \in Y^{-1}(\kappa)\} \\
&\leq \inf\{\max\{M_G(\tau * \alpha), M_G(\alpha)|\tau * \alpha \in Y^{-1}(\kappa \diamond \sigma) \text{ and } \alpha \in Y^{-1}(\sigma)\}\} \\
&= \max\{\inf\{M_G(\tau * \alpha)|\tau * \alpha \in Y^{-1}(\kappa \diamond \sigma)\}, \inf\{M_G(\alpha)|\alpha \in Y^{-1}(\sigma)\}\} \\
&= \max\{Y_{\inf}(M_G)(\kappa \diamond \sigma), Y_{\inf}(M_G)(\sigma)\}.
\end{aligned}$$

$$\begin{aligned}
\text{Next, } Y_{\sup}(M_j)(\kappa) &= \sup\{M_j(\tau)|\tau \in Y^{-1}(\kappa)\} \\
&\geq \sup\{\min\{M_j(\tau * \alpha), M_j(\alpha)|\tau * \alpha \in Y^{-1}(\kappa \diamond \sigma) \text{ and } \alpha \in Y^{-1}(\sigma)\}\} \\
&= \min\{\sup\{M_j(\tau * \alpha)|\tau * \alpha \in Y^{-1}(\kappa \diamond \sigma)\}, \sup\{M_j(\alpha)|\alpha \in Y^{-1}(\sigma)\}\} \\
&= \min\{Y_{\sup}(M_j)(\kappa \diamond \sigma), Y_{\sup}(M_j)(\sigma)\}.
\end{aligned}$$

Hence $Y(M) = \{\langle \kappa, Y_{\sup}M_H(\kappa), Y_{\inf}M_G(\kappa), Y_{\sup}M_j(\kappa) \rangle | \kappa \in K\}$ is a $(NF\delta - I)$ of K . ■

Proposition 3.19: Assume $Y: (V, *, 0) \rightarrow (K, \diamond, \zeta)$ be a δ -epimorphism of δ -algebras. If $M = \{\langle \tau, M_H(\tau), M_G(\tau), M_j(\tau) \rangle | \tau \in V\}$ a $(NF\delta - I)$ of V , then $Y(M)$ is a $(NF\delta - I)$ of K .

Proof: Given that $M = \{\langle \tau, M_H(\tau), M_G(\tau), M_j(\tau) \rangle | \tau \in V\}$ is a $(NF\delta - I)$ of V . Thus from Proposition (3.17) and Note (3.7) we have $Y(M) = \{\langle \kappa, Y_{\sup}M_H(\kappa), Y_{\inf}M_G(\kappa), Y_{\sup}M_j(\kappa) \rangle | \kappa \in K\}$ is a $(NF\delta - SA)$ of K . Then M such

that [Definition (3.5)-(2)]. Because $M = \{ \langle \tau, (M_H(\tau), M_G(\tau), M_J(\tau)) \rangle \mid \tau \in V \}$ is a $(NF\bar{\delta} - I)$ of V , hence $M_H(0) \geq M_H(\tau)$, $M_G(0) \leq M_G(\tau)$, and $M_J(0) \geq M_J(\tau)$ for any $\tau \in V$. Since Y is $(\delta - H)$ of δ -algebras, then $Y(0) = \zeta$. Also, $0 \in Y^{-1}(\zeta)$ and $\{ \tau \mid \tau \in Y^{-1}(\kappa) \} \subseteq \{ \tau \mid \tau \in V \}$, $\forall \kappa \in K$. Hence we get, $Y_{\sup}(M_H)(\zeta) = \sup\{M_H(\tau) \mid \tau \in Y^{-1}(\zeta)\}$

$$\begin{aligned} &= M_H(0) \geq \sup\{M_H(\tau) \mid \tau \in V\} \geq \sup\{M_H(\tau) \mid \tau \in Y^{-1}(\kappa)\} \\ &= Y_{\sup}(M_H)(\kappa). \end{aligned}$$

$$\begin{aligned} \text{Also, } Y_{\inf}(M_G)(\zeta) &= \inf\{M_G(\tau) \mid \tau \in Y^{-1}(\zeta)\} = M_G(0) \leq \inf\{M_G(\tau) \mid \tau \in V\} \\ &\leq \inf\{M_G(\tau) \mid \tau \in Y^{-1}(\kappa)\} = Y_{\sup}(M_G)(\kappa) \end{aligned}$$

$$\begin{aligned} \text{On the opposite end, } Y_{\sup}(M_J)(\zeta) &= \sup\{M_J(\tau) \mid \tau \in Y^{-1}(\zeta)\} \\ &= M_J(0) \geq \sup\{M_J(\tau) \mid \tau \in V\} \geq \sup\{M_J(\tau) \mid \tau \in Y^{-1}(\kappa)\} \\ &= Y_{\sup}(M_J)(\kappa). \end{aligned}$$

Then $Y(M) = \{ \langle \kappa, Y_{\sup}M_H(\kappa), Y_{\inf}M_G(\kappa), Y_{\sup}M_J(\kappa) \rangle \mid \kappa \in K \}$ is a $(NF\bar{\delta} - I)$ of K . ■

4. The Equivalence Classes of Neutrosophic Fuzzy Delta-Algebras

This section provides examples of some $NF\delta - SA(V)$ applications on $NF\delta - SA(V)$, including $\simeq_H, \simeq_G, \simeq_J$ and \bowtie_α . Additionally, this section contains some of their crucial elements.

4.1. The Equivalence Classes of $NF\delta - SA(V)$ Modulo $(\simeq_H, \simeq_G, \simeq_J)$:

Signify to the set of all $(NF\delta - SA)$ of V by $NF\delta - SA(V)$. Also, assume $\simeq_H, \simeq_G, \simeq_J$ be three binary relations on $NF\delta - SA(V)$, they are identified with:

$M \simeq_H K \Leftrightarrow Q(M_H, \alpha) = Q(K_H, \alpha)$, $M \simeq_G K \Leftrightarrow W(M_G, \alpha) = W(K_G, \alpha)$ and $M \simeq_J K \Leftrightarrow R(M_J, \alpha) = R(K_J, \alpha)$ respectively, for some, $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ and $K = \{ \langle \tau, K_H(\tau), K_G(\tau), K_J(\tau) \rangle \mid \tau \in V \}$ in $NF\delta - SA(V)$. Furthermore, we consider \simeq_H, \simeq_G and \simeq_J are equivalence relations on $NF\delta - SA(V)$. Assume $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \} \in NF\delta - SA(V)$. The equivalence class of $M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \}$ modulo \simeq_H, \simeq_G , (resp. \simeq_J) is symbolized by $[M]_H, [M]_G$ (resp. $[M]_J$). Also, these symbols $(NF\delta - SA(V)/\simeq_H)$, $(NF\delta - SA(V)/\simeq_G)$ (resp. $NF\delta - SA(V)/\simeq_J$) refer to the set of all equivalence classes of M modulo \simeq_H, \simeq_G , (resp. \simeq_J). That means $NF\delta - SA(V)/\simeq_H = \{ [M]_H \mid M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \} \in NF\delta - SA(V) \}$, $NF\delta - SA(V)/\simeq_G = \{ [M]_G \mid M = \{ \langle \tau, (M_H(\tau), M_G(\tau), M_J(\tau)) \rangle \mid \tau \in V \} \in NF\delta - SA(V) \}$ (resp. $NF\delta - SA(V)/\simeq_J = \{ [M]_J \mid M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \} \in NF\delta - SA(V) \}$). Also, this symbol $\delta_l(V)$ refer to the set of all δ -ideals of V . Let $\Psi_\alpha, \psi_\alpha, X_\alpha: NF\delta - SA(V) \rightarrow \delta_l(V) \cup \{\emptyset\}$ be maps are defined by $\Psi_\alpha(M) = Q(M_H, \alpha)$, $\psi_\alpha(M) = W(M_G, \alpha)$ and $X_\alpha = R(M_J, \alpha)$, where $\alpha \in [0, 1]$, $\forall M = \{ \langle \tau, M_H(\tau), M_G(\tau), M_J(\tau) \rangle \mid \tau \in V \} \in NF\delta - SA(V)$. Also, Ψ_α, ψ_α , and X_α are well-defined.

Proposition 4.2: Assume $\Psi_\alpha, \psi_\alpha, X_\alpha: NF\delta - SA(V) \rightarrow \delta_l(V) \cup \{\emptyset\}$ be the maps. Then $\Psi_\alpha, \psi_\alpha, X_\alpha$ are onto, $\forall \alpha \in (0, 1)$.

Proof:

Let $\alpha \in (0, 1)$. Then $\tilde{0} = \langle \tau, \bar{0}, \bar{1}, \bar{0} \rangle$ is in $NF\delta - SA(V)$, where $\bar{0}, \bar{1}$ and $\bar{0}$ are defined by $\bar{0}(\tau) = 0$, $\bar{1}(\tau) = 1$ and, $\bar{0}(\tau) = 0 \forall \tau \in V$. Also, $\Psi_\alpha(\tilde{0}) = Q(\bar{0}, \alpha) = \emptyset = W(\bar{1}, \alpha) = \psi_\alpha(\tilde{0}) =$

$R(\bar{0}, \alpha) = X_\alpha(\tilde{0})$. Suppose that $\emptyset \neq \varphi \in \delta_l(V)$. Let $\varphi_H(\tau) = \begin{cases} 1, & \text{if } \tau \in \varphi \\ 0, & \text{if } \tau \notin \varphi \end{cases}$, $\varphi_G(\tau) = 1 - \varphi_G(\tau)$ and $\varphi_J(\tau) = 1 - \varphi_H(\tau) - \varphi_G(\tau)$, $\forall \tau \in V$, thus $\Psi_\alpha(\tilde{\varphi}) = Q(\varphi_H, \alpha) = \varphi = W(\varphi_G, \alpha) = \psi_\alpha(\tilde{\varphi}) = R(\varphi_J, \alpha) = X_\alpha(\tilde{\varphi})$. Now, we want to show that $\tilde{\varphi} = \langle \tau, \varphi_H, \varphi_G, \varphi_J \rangle \in NF\delta - SA(V)$. From [Definition 2.3] we get $\tilde{\varphi}$ is $(\delta - SA)$ of V [since $\varphi \in \delta_l(V)$]. So, $Q(\varphi_H, \alpha)$, $W(\varphi_G, \alpha)$ and $R(\varphi_J, \alpha)$ are δ -subalgebras of V . Also, $\tilde{\varphi} = \langle \tau, \varphi_H, \varphi_G, \varphi_J \rangle \in NF\delta - SA(V)$ [From Proposition [3.13]]. Therefore, $\forall \varphi \in \delta_l(V)$ we consider $\Psi_\alpha(\tilde{\varphi}) = \varphi$, $\psi_\alpha(\tilde{\varphi}) = \varphi$ and, $X_\alpha(\tilde{\varphi}) = \varphi$. for some $\tilde{\varphi} \in NF\delta - SA(V)$. This completes the proof. ■

Proposition 4.3: Let $NF\delta - SA(V)/\simeq_H, NF\delta - SA(V)/\simeq_G$ and $NF\delta - SA(V)/\simeq_J$ are quotient sets. Then for any $\alpha \in (0,1)$ they are equivalent $\delta_I(V) \cup \{\emptyset\}$.

Proof:

Consider that $\alpha \in (0,1)$ and assume $\overline{\Psi}_\alpha: NF\delta - SA(V)/\simeq_H \rightarrow \delta_I(V) \cup \{\emptyset\}, \overline{\Psi}_\alpha: NF\delta - SA(V)/\simeq_G \rightarrow \delta_I(V) \cup \{\emptyset\}$ and $\overline{X}_\alpha: NF\delta - SA(V)/\simeq_J \rightarrow \delta_I(V) \cup \{\emptyset\}$ are maps, where they are defined as $\overline{\Psi}_\alpha([M]_H) = \Psi_\alpha(M), \overline{\Psi}_\alpha([M]_G) = \psi_\alpha(M)$ (resp. $\overline{X}_\alpha([M]_J) = X_\alpha(M)$), for all $M = \langle \tau, M_H, M_G, M_J \rangle \in NF\delta - SA(V)$. Hence, $M \simeq_H K$ and $M \simeq_G K, M \simeq_J K$, for all $M = \langle \tau, M_H, M_G, M_J \rangle$ and $K = \langle \tau, K_H, K_G, K_J \rangle$ in $NF\delta - SA(V)$, if $Q(M_H, \alpha) = Q(K_H, \alpha), W(M_G, \alpha) = W(K_G, \alpha)$ and $R(M_J, \alpha) = R(K_J, \alpha)$. Then $[M]_H = [K]_H, [M]_G = [K]_G$ and $[M]_J = [K]_J$. Then $\overline{\Psi}_\alpha, \overline{\psi}_\alpha$ and \overline{X}_α are injective. Furthermore, let $\emptyset \neq \varphi \in \delta_I(V)$ and $\forall \tau \in V$, let $\varphi_H(\tau) = \begin{cases} 1, & \text{if } \tau \in \varphi \\ 0, & \text{if } \tau \notin \varphi \end{cases}, \varphi_G(\tau) = 1 - \varphi_H(\tau)$, and $\varphi_J(\tau) = 1 - \varphi_H(\tau) - \varphi_G(\tau)$ thus $\tilde{\varphi} = \langle \tau, \varphi_H, \varphi_G, \varphi_J \rangle \in NF\delta - SA(V)$. We consider that $\overline{\Psi}_\alpha([\tilde{\varphi}]_H) = \Psi_\alpha(\tilde{\varphi}) = Q(\varphi_H, \alpha) = \varphi, \overline{\psi}_\alpha([\tilde{\varphi}]_G) = \psi_\alpha(\tilde{\varphi}) = W(\varphi_G, \alpha) = \varphi$ and $\overline{X}_\alpha([\tilde{\varphi}]_J) = X_\alpha(\tilde{\varphi}) = R(\varphi_J, \alpha) = \varphi$. Next, for $\tilde{0} = \langle \tau, \tilde{0}, \tilde{1}, \tilde{0} \rangle \in NF\delta - SA(V)$ we have $\overline{\Psi}_\alpha([\tilde{0}]_H) = \Psi_\alpha(\tilde{0}) = Q(\tilde{0}, s) = \emptyset$. Also, $\overline{\psi}_\alpha([\tilde{0}]_G) = \psi_\alpha(\tilde{0}) = W(\tilde{1}, s) = \emptyset$ and $\overline{X}_\alpha([\tilde{0}]_J) = X_\alpha(\tilde{0}) = R(\tilde{0}, s) = \emptyset$. Thus $\overline{\Psi}_\alpha, \overline{\psi}_\alpha$ and \overline{X}_α are surjective and hence $NF\delta - SA(V)/\simeq_H, NF\delta - SA(V)/\simeq_G$ and $NF\delta - SA(V)/\simeq_J$ are equivalent to $\delta_I(V) \cup \{\emptyset\}$. ■

5. The equivalence classes of $NF\delta - SA(V)$ modulo \bowtie_α .

In this section, the relation \bowtie_α on $NF\delta - SA(V)$ is investigated and discussed.

Definition 5.1: We define the relation \bowtie_α on $NF\delta - SA(V)$ by $(M, K) \in \bowtie_\alpha \Leftrightarrow Q(M_H, \alpha) \cap W(M_G, \alpha) \cap R(M_J, \alpha) = Q(K_H, \alpha) \cap W(K_G, \alpha) \cap R(K_J, \alpha), \alpha \in [0,1], \forall M = \langle \tau, M_H, M_G, M_J \rangle$ and $K = \langle \tau, K_H, K_G, K_J \rangle \in NF\delta - SA(V)$. The equivalence class of $M = \langle \tau, M_H, M_G, M_J \rangle$ modulo $\bowtie_\alpha \forall M = \langle \tau, M_H, M_G, M_J \rangle \in NF\delta - SA(V)$ is symbolized as $[M]_{\bowtie_\alpha}$.

Proposition 5.2:

A map $\Pi_\alpha: NF\delta - SA(V) \rightarrow \delta_I(V) \cup \{\emptyset\}$ is surjective if $\Pi_\alpha(M) = \Psi_\alpha(M) \cap \psi_\alpha(M) \cap X_\alpha(M), \forall M = \langle \tau, M_H, M_G, M_J \rangle \in NF\delta - SA(V)$ and $\alpha \in (0,1)$.

Proof:

Assume that $\alpha \in (0,1)$ and let $\varphi \in \delta_I(V) \cup \{\emptyset\}$, if $\varphi = \emptyset$, then there is $\tilde{0} = \langle \tau, \tilde{0}, \tilde{1}, \tilde{0} \rangle \in NF\delta - SA(V)$ such that $\Pi_\alpha(\tilde{0}) = \Psi_\alpha(\tilde{0}) \cap \psi_\alpha(\tilde{0}) \cap X_\alpha(\tilde{0}) = Q(0, \alpha) \cap W(1, \alpha) \cap R(0, \alpha) = \emptyset = \varphi$. If $\varphi \neq \emptyset$, then $\exists \tilde{\varphi} = \langle \tau, \varphi_H, \varphi_G, \varphi_J \rangle \in NF\delta - SA(V)$, where $\varphi_H(\tau) = \begin{cases} 1, & \text{if } \tau \in \varphi \\ 0, & \text{if } \tau \notin \varphi \end{cases}, \varphi_G(\tau) = 1 - \varphi_H(\tau)$, and $\varphi_J(\tau) = 1 - \varphi_H(\tau) - \varphi_G(\tau)$ such that $\Pi_\alpha(\tilde{\varphi}) = \Psi_\alpha(\tilde{\varphi}) \cap \psi_\alpha(\tilde{\varphi}) \cap X_\alpha(\tilde{\varphi}) = Q(\varphi_H, \alpha) \cap W(\varphi_G, \alpha) \cap R(\varphi_J, \alpha) = \varphi$. Then Π_α is surjective.

Proposition 5.3: The quotient set $NF\delta - SA(V)/\bowtie_\alpha$ is equivalent to $\delta_I(V) \cup \{\emptyset\}$, for all $\alpha \in (0,1)$.

Proof:

Suppose that $\alpha \in (0,1)$ and $\overline{\Pi}_\alpha: NF\delta - SA(V)/\bowtie_\alpha \rightarrow \delta_I(V) \cup \{\emptyset\}$ is a map with

$$\overline{\Pi}_\alpha([M]_{\bowtie_\alpha}) = \Pi_\alpha(M), \forall [M]_{\bowtie_\alpha} \in NF\delta - SA(V)/\bowtie_\alpha. \text{ Assume that}$$

$$\overline{\Pi}_\alpha([M]_{\bowtie_\alpha}) = \overline{\Pi}_\alpha([K]_{\bowtie_\alpha}), \forall [M]_{\bowtie_\alpha}, [K]_{\bowtie_\alpha} \in NF\delta - SA(V)/\bowtie_\alpha.$$

We obtain $\Psi_\alpha(M) \cap \psi_\alpha(M) \cap X_\alpha(M) = \Psi_\alpha(K) \cap \psi_\alpha(K) \cap X_\alpha(K)$. That means

$Q(M_H, \alpha) \cap W(M_G, \alpha) \cap R(M_J, \alpha) = Q(K_H, \alpha) \cap W(K_G, \alpha) \cap R(K_J, \alpha)$. Therefore $(M, K) \in \bowtie_\alpha$, and hence $[M]_{\bowtie_\alpha} = [K]_{\bowtie_\alpha}$. So, $\overline{\Pi}_\alpha$ is injective. Also, for $\tilde{0} = \langle \tau, \tilde{0}, \tilde{1}, \tilde{0} \rangle \in NF\delta - SA(V)$ we get $\overline{\Pi}_\alpha([\tilde{0}]_{\bowtie_\alpha}) = \Pi_\alpha(\tilde{0}) = \Psi_\alpha(\tilde{0}) \cap \psi_\alpha(\tilde{0}) \cap X_\alpha(\tilde{0}) = Q(\tilde{0}, s) \cap W(\tilde{1}, s) \cap R(\tilde{0}, \alpha) = \emptyset$. Take $\tilde{\varphi} = \langle \tau, \varphi_H, \varphi_G, \varphi_J \rangle \in NF\delta - SA(V), \forall \varphi \in NF\delta - SA(V)$, have been the same $(NF\delta - SA)$ as in the proving of the Theorem [4.1.15]. Hence $\overline{\Pi}_\alpha([\tilde{\varphi}]_{\bowtie_\alpha}) = \Pi_\alpha(\tilde{\varphi}) = \Psi_\alpha(\tilde{\varphi}) \cap \psi_\alpha(\tilde{\varphi}) \cap X_\alpha(\tilde{\varphi}) = Q(\varphi_H, \alpha) \cap W(\varphi_G, \alpha) \cap R(\varphi_J, \alpha) = \varphi$. Hence $\overline{\Pi}_\alpha$ is surjective. Then $NF\delta - SA(V)/\bowtie_\alpha$ is equivalent to $\delta_I(V) \cup \{\emptyset\}$. ■

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