



The Topology $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ in the Frame of Primal Topological Spaces

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Abstract

In this paper, we will use the family of regular \star -open subsets to present and examine two new operators $(\cdot)_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ and $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$. We demonstrate that, in contrast to the operator $(\cdot)_{\mathcal{P}\mathcal{R}\star}^{\otimes}$, the operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ is a Kuratowski closure operator. We show that each of these operators lies between two previously defined operators where for each subset $H \subseteq \mathcal{S}$, $H_{\mathcal{P}}^{\diamond} \subseteq H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \subseteq H_{\mathcal{P}\mathcal{R}}^{\diamond}$ and $H \subseteq Cl_{\mathcal{P}}^{\diamond}(H) \subseteq Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq Cl_{\mathcal{P}\mathcal{R}}^{\diamond}(H)$. Furthermore, we show that the topology, denoted by $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$, which is obtained by $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ is independent from \mathcal{T} and it is finer than $\mathcal{T}_{\eta\star}$, where $\mathcal{T}_{\eta\star}$ is the family of all union of regular \star -open subsets of $(\mathcal{S}, \mathcal{T})$. Then we demonstrate several fundamental results concerning this new structure and present many illustrative examples that relate to our conclusions. Finally, by using the operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ we introduce a new notion namely, \mathcal{P} -generalized closed sets, and study some of their basic properties.

Keywords: Kuratowski closure operator; primal; primal topological; The operator $(\cdot)_{\mathcal{P}\mathcal{R}\star}^{\otimes}$; the operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$.

1 Introduction

Many topologies with important applications in mathematics have been developed using new mathematical frameworks, such as filter,¹³ ideal¹² and partially ordered.²⁰ Around 1947, the concept of grill was introduced by Chpqut,⁹ which represents classic structures for general topology in the literature. It is an efficient tool for understanding topological concepts and has numerous applications in topological theory [see⁷⁻⁸]. Some structures based on filter and grill are defined and investigated [see¹⁸⁻¹⁹]. A grill's related topology has been defined and examined by Roy and Mukherjee.²⁴ They instructed important topological notions in terms of this new framework and looked over the fundamental theorems. Furthermore, some operators have been described and their properties examined to see if they meet Kuratowski's closure axioms [see²⁵].

Recently, Acharjee et al.,¹ introduced a new classical structure named primal. They utilize two new operators to construct the concept of primal topological space which is the dual structure of grill, and they explore numerous fundamental characteristics of both these two operators. Moreover, after the introduction of primal many new research has been conducted on primal topological spaces. In,⁵ the authors examined the relationships and stratification of single-valued neutrosophic primals (svn-primals) and grills (svn-grills). In addition, they introduced a novel structure known as an svn-primal in the context of single-valued neutrosophic topology.

Moreover, in,²¹ Özkoc et al. presented and investigated two new operators $(\cdot)_{\mathcal{P}\mathcal{R}}^{\diamond}$ and $Cl_{\mathcal{P}\mathcal{R}}^{\diamond}(\cdot)$ via primal, and they demonstrated and studied many properties of $\mathcal{T}_{\mathcal{P}\mathcal{R}}$, where $\mathcal{T}_{\mathcal{P}\mathcal{R}}$ is the topology obtained by the operator $Cl_{\mathcal{P}\mathcal{R}}^{\diamond}(\cdot)$. In general, after the introduction of primal many other new research have been developed regarding primal topological spaces [see²] and in the frameworks of soft setting and fuzzy setting, two new variants of primal topology have been provided [see,⁴⁶].

2 Preliminaries

Throughout this paper, (S, \mathcal{T}) represents a topological space. For any subset $H \subseteq S$, $Cl(H)$ and $Int(H)$ denote the closure and the interior of H respectively. The power set of a set S will be denoted by 2^S . The family of all closed subsets of (S, \mathcal{T}) will be denoted by $\mathcal{C}(S, \mathcal{T})$. Two subsets H and K are said to be separated in (S, \mathcal{T}) if $H \cap Cl(K) = Cl(H) \cap K = \varphi$. A subset H of (S, \mathcal{T}) is called regular open²⁶ if $H = Int(Cl(H))$ and the complement of a regular open subset is called regular closed.²⁶ A subset H is called δ -open²⁷ if $H = \delta - Int(H)$, where $\delta - Int(H)$ is the union of all regular open subsets contained in H . A subset H of (S, \mathcal{T}) is called pre-open if $H \subseteq Int(Cl(H))$.¹⁶ A subset H of (S, \mathcal{T}) is called generalized closed in S ¹⁵ if $Cl(H) \subseteq V$ whenever $H \subseteq V$ and $V \in \mathcal{T}$. The complement of a generalized closed subset in S ¹⁵ is called generalized open subset in S . The generalized closure (resp., generalized interior) of H , denoted by $Cl_{\star}(H)$ (resp., $Int_{\star}(H)$) is defined as the intersection of all generalized closed subsets in S containing H (resp., the union of all generalized open subsets in S contained in H).¹¹ The space (S, \mathcal{T}) is said to be locally indiscrete if $\mathcal{T} \subseteq \mathcal{C}(S, \mathcal{T})$ ¹⁰ and it is $\mathcal{T}_{\frac{1}{2}}$ -space if each generalized closed in S is closed.¹⁵

Now, we recall the following notions and results which will be required in the next sections.

Definition 2.1. Let (S, \mathcal{T}) be a topological space with $H \subseteq S$. Then H is called:

- (i) regular \star -open (resp., regular \star -closed) subset²² if $H = Int(Cl_{\star}(H))$ (resp., $H = Cl(Int_{\star}(H))$).
- (ii) η \star -open¹⁷ if it is a union of regular \star -open subsets.

The family of all regular open (resp., generalized closed, regular \star -open, regular \star -closed, η \star -open) subsets of (S, \mathcal{T}) will be denoted by $\mathcal{RO}(S, \mathcal{T})$ (resp., $\mathcal{GC}(S, \mathcal{T})$, $\mathcal{RO}^{\star}(S, \mathcal{T})$, $\mathcal{RC}^{\star}(S, \mathcal{T})$, $\mathcal{T}_{\eta\star}$). The family of all open (resp., regular open, regular \star -open) subsets of (S, \mathcal{T}) containing a point $x \in S$ will be denoted by $\mathcal{T}(x)$ (resp., $\mathcal{RO}(x, S, \mathcal{T})$, $\mathcal{RO}^{\star}(x, S, \mathcal{T})$). Also, $\mathcal{T}_{\eta\star}$ forms a topology on S ,¹⁷ but in general $\mathcal{RO}^{\star}(S, \mathcal{T})$ doesn't form a topology on S . Moreover, in,¹⁴ the authors re-introduced the concept of $\mathcal{RO}^{\star}(S, \mathcal{T})$, they denoted it by GR -open sets, as a generalization for regular open sets, and they presented additional properties to this family.

Theorem 2.2. Let (S, \mathcal{T}) be a topological space. Then:

- (i) $\mathcal{RO}(S, \mathcal{T}) \subseteq \mathcal{RO}^{\star}(S, \mathcal{T}) \subseteq \mathcal{T}$.²²
- (ii) $\mathcal{RO}^{\star}(S, \mathcal{T}) \subseteq \mathcal{T}_{\eta\star} \subseteq \mathcal{T}$.¹⁷

Theorem 2.3.²² Let (S, \mathcal{T}) be a topological space with $H, K \subseteq S$. If $H, K \in \mathcal{RO}^{\star}(S, \mathcal{T})$, then $H \cap K \in \mathcal{RO}^{\star}(S, \mathcal{T})$.

Definition 2.4.²³ Let (S, \mathcal{T}) be a topological space. Then (S, \mathcal{T}) is said to be \mathcal{R}^{\star} -regular if for each $H \in \mathcal{C}(S, \mathcal{T})$ and each point $x \notin H$, there are $V, U \in \mathcal{RO}^{\star}(S, \mathcal{T})$ with $x \in V$, $H \subseteq U$ and $V \cap U = \varphi$.

Theorem 2.5.¹⁴ A function $f : (S, \mathcal{T}) \rightarrow (S^*, \mathcal{T}^*)$ is GR -continuous iff $f^{-1}(H) \in \mathcal{RO}^{\star}(S, \mathcal{T})$ for each $H \in \mathcal{T}^*$.

Lemma 2.6.¹⁰ Let (S, \mathcal{T}) be a topological space. Then the following are equivalent:

- (i) (S, \mathcal{T}) is locally indiscrete.
- (ii) $\{x\}$ is pre-open for each $x \in S$.
- (iii) H is pre-open for each $H \subseteq S$.
- (iv) H is pre-open for each $H \in \mathcal{C}(S, \mathcal{T})$.

Theorem 2.7.¹¹ Let (S, \mathcal{T}) be a topological space. If $\mathcal{T}_{\blacktriangle}$ is a topology on S which is defined by the Kuratowski closure operator Cl_{\blacktriangle} , that is $\mathcal{T}_{\blacktriangle} = \{H \subseteq S : Cl_{\blacktriangle}(H^c) = H^c\}$, then $(S, \mathcal{T}_{\blacktriangle})$ is $\mathcal{T}_{\frac{1}{2}}$ -space.

Definition 2.8. ⁹ A family \mathfrak{G} of 2^S is said to be a grill on S if \mathfrak{G} satisfies the following:

- (i) $\varnothing \notin \mathfrak{G}$.
- (ii) if $U \cup V \in \mathfrak{G}$, then $U \in \mathfrak{G}$ or $V \in \mathfrak{G}$.
- (iii) if $U \in \mathfrak{G}$ and $U \subseteq V$, then $V \in \mathfrak{G}$.

Definition 2.9. ¹ A family \mathcal{P} of 2^S is said to be a primal on S if \mathcal{P} satisfies the following:

- (i) $S \notin \mathcal{P}$.
- (ii) if $U \cap V \in \mathcal{P}$, then $U \in \mathcal{P}$ or $V \in \mathcal{P}$.
- (iii) if $V \in \mathcal{P}$ and $U \subseteq V$, then $U \in \mathcal{P}$.

Theorem 2.10. ¹ If \mathcal{P} and \mathcal{P}^* are two primals on S , then $\mathcal{P} \cup \mathcal{P}^*$ is primal on S .

Definition 2.11. ¹ A topological space (S, \mathcal{T}) with a primal \mathcal{P} on S is called a primal topological space and denoted by $(S, \mathcal{T}, \mathcal{P})$.

Definition 2.12. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then a function:

- (i) $(\cdot)^\diamond : 2^S \rightarrow 2^S$ is defined by: $H^\diamond = \{x \in S : \forall V \in \mathcal{T}(x), H^c \cup V^c \in \mathcal{P}\}$ for each subset H of S .¹
- (ii) $(\cdot)_{\mathcal{R}}^\diamond : 2^S \rightarrow 2^S$ is defined by: $H_{\mathcal{R}}^\diamond = \{x \in S : \forall V \in \mathcal{RO}(x, S, \mathcal{T}), H^c \cup V^c \in \mathcal{P}\}$ for each subset H of S .²¹
- (iii) $Cl^\diamond : 2^S \rightarrow 2^S$ is defined by: $Cl^\diamond(H) = H^\diamond \cup H$ for each subset H of S .¹
- (iv) $Cl_{\mathcal{R}}^\diamond : 2^S \rightarrow 2^S$ is defined by: $Cl_{\mathcal{R}}^\diamond(H) = H_{\mathcal{R}}^\diamond \cup H$ for each subset H of S .²¹

For each subset H of S , we will use the notations $H_{\mathcal{P}}^\diamond, H_{\mathcal{P}\mathcal{R}}^\diamond, Cl_{\mathcal{P}}^\diamond(H)$ and $Cl_{\mathcal{P}\mathcal{R}}^\diamond(H)$ instead of $H^\diamond, H_{\mathcal{R}}^\diamond, Cl^\diamond(H)$ and $Cl_{\mathcal{R}}^\diamond(H)$ respectively. To read more properties of the operators $(\cdot)^\diamond, (\cdot)_{\mathcal{R}}^\diamond, Cl^\diamond$ and $Cl_{\mathcal{R}}^\diamond$ [see¹ and²¹].

Definition 2.13. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then:

- (i) $\mathcal{T}^\diamond = \{H \subseteq S : Cl_{\mathcal{P}}^\diamond(H^c) = H^c\}$ is a topology on S induced by \mathcal{T} and \mathcal{P} .¹
- (ii) $\mathcal{T}_{\mathcal{R}}^\diamond = \{H \subseteq S : Cl_{\mathcal{P}\mathcal{R}}^\diamond(H^c) = H^c\}$ is a topology on S induced by \mathcal{T} and \mathcal{P} .²¹

In^{1,21} the authors proved that $\mathcal{T} \subseteq \mathcal{T}^\diamond$ and $\mathcal{T}_\delta \subseteq \mathcal{T}_{\mathcal{R}}^\diamond$ respectively where \mathcal{T}_δ is the family of all δ -open subsets of (S, \mathcal{T}) . Moreover, we will use the notations $\mathcal{T}_{\mathcal{P}}^\diamond$ and $\mathcal{T}_{\mathcal{P}\mathcal{R}}^\diamond$ instead of \mathcal{T}^\diamond and $\mathcal{T}_{\mathcal{R}}^\diamond$ respectively.

Theorem 2.14. ³ Let (S, \mathcal{T}) and (S^*, \mathcal{T}^*) be topological spaces and $f : (S, \mathcal{T}) \rightarrow (S^*, \mathcal{T}^*)$ be a bijective function. If \mathcal{P} is a primal on S , then $f(\mathcal{P}) = \{f(H) : H \in \mathcal{P}\}$ is a primal on S^* .

3 The operator $(\cdot)_{\mathcal{P}\mathcal{R}\star}^\circledast$

In this section, we introduce a new operator in primal topological spaces and study some of its features. We not only set up aware essential features of this operator, but we also provide numerous examples.

Definition 3.1. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then define an operator $(\cdot)_{\mathcal{P}\mathcal{R}\star}^\circledast : 2^S \rightarrow 2^S$ by: $H_{\mathcal{P}\mathcal{R}\star}^\circledast = \{x \in S : \forall V \in \mathcal{RO}^\star(x, S, \mathcal{T}), H^c \cup V^c \in \mathcal{P}\}$ for each subset H of S .

Theorem 3.2. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then $H_{\mathcal{P}}^\diamond \subseteq H_{\mathcal{P}\mathcal{R}\star}^\circledast \subseteq H_{\mathcal{P}\mathcal{R}}^\diamond$ for each subset H of S .

Proof. The proof is a consequence of Theorem 2.2. □

The following example will show that there is a primal topological space $(S, \mathcal{T}, \mathcal{P})$ and $H \subseteq S$ such that $H_{\mathcal{P}}^{\diamond} \subsetneq H_{\mathcal{PR}^{\star}}^{\otimes} \subsetneq H_{\mathcal{PR}}^{\diamond}$. Also, this example will show that the inclusions of $H_{\mathcal{PR}^{\star}}^{\otimes} \subseteq H$ or $H \subseteq H_{\mathcal{PR}^{\star}}^{\otimes}$ is not true in general.

Example 3.3. Consider $S = \{x_1, x_2, x_3\}$ with $\mathcal{T} = \{\varphi, S, \{x_3\}, \{x_2, x_3\}\}$ and $\mathcal{P} = \{\varphi, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$. Since $\mathcal{RO}^{\star}(S, \mathcal{T}) = \{\varphi, S, \{x_3\}\}$, then:

(i) $\{x_1\}_{\mathcal{P}}^{\diamond} = \{x_1\}, \{x_1\}_{\mathcal{PR}^{\star}}^{\otimes} = \{x_1, x_2\}$ and $\{x_1\}_{\mathcal{PR}}^{\diamond} = S$.

(ii) $\{x_1, x_3\}_{\mathcal{PR}^{\star}}^{\otimes} = \{x_1, x_2\}$.

Theorem 3.4. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then:

(i) $H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond}$ for each subset H of S whenever (S, \mathcal{T}) is $T_{\frac{1}{2}}$ -space.

(ii) $H_{\mathcal{P}}^{\diamond} = H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond}$ for each subset H of S whenever (S, \mathcal{T}) is locally indiscrete.

Proof. (i) Assume that $H \subseteq S$. Since $\mathcal{GC}(S, \mathcal{T}) \subseteq \mathcal{C}(S, \mathcal{T})$, then $Cl(H) = Cl_{\star}(H)$ and hence $\mathcal{RO}(S, \mathcal{T}) = \mathcal{RO}^{\star}(S, \mathcal{T})$. Therefore $H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond}$.

(ii) Assume that $H \subseteq S$. If (S, \mathcal{T}) is locally indiscrete, then $\mathcal{RO}(S, \mathcal{T}) = \mathcal{T}$ and hence $H_{\mathcal{P}}^{\diamond} = H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond}$. □

Note that, if $\mathcal{P} = \{\varphi\}$, then $H_{\mathcal{P}}^{\diamond} = H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond} = \varphi$, but we did not consider it in Theorem 3.4 since $\mathcal{P} = \{\varphi\}$ is a primal on S iff S is a singleton.

Example 3.5. Consider $S = \{x_1, x_2, x_3\}$ with $\mathcal{T} = \{\varphi, S, \{x_1\}\}$ and $\mathcal{P} = \{\varphi, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$. Then $H_{\mathcal{P}}^{\diamond} = H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond}$ for each subset H of S while (S, \mathcal{T}) is neither locally indiscrete nor $T_{\frac{1}{2}}$ -space. Therefore, the converse of Theorem 3.4 is not true in general.

Corollary 3.6. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then $H_{\mathcal{P}}^{\diamond} = H_{\mathcal{PR}^{\star}}^{\otimes} = H_{\mathcal{PR}}^{\diamond}$ for each subset H of S if one of the following holds:

(i) each subset of S is preopen.

(ii) each singleton in S is preopen.

(iii) each closed subset of S is preopen.

Proof. The proof is a consequence of Lemma 2.6 and Theorem 3.4. □

Proposition 3.7. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq S$. If

$$\mathcal{T}_{\eta}^{\star} = \{\varphi, S\}, \text{ then } H_{\mathcal{PR}^{\star}}^{\otimes} = \begin{cases} \varphi & : H^c \notin \mathcal{P} \\ S & : H^c \in \mathcal{P} \end{cases} .$$

Proof. Assume that $x \in S$ and $H \subseteq S$. If $H^c \notin \mathcal{P}$, then $H^c \cup S^c \notin \mathcal{P}$ and hence $x \notin H_{\mathcal{PR}^{\star}}^{\otimes}$. Therefore, $H_{\mathcal{PR}^{\star}}^{\otimes} = \varphi$. Now, since $\mathcal{RO}^{\star}(x, S, \mathcal{T}) = \{S\}$ for each $x \in S$, then $H^c \cup S^c \in \mathcal{P}$ whenever $H^c \in \mathcal{P}$ and thus $H_{\mathcal{PR}^{\star}}^{\otimes} = S$. □

The converse of Proposition 3.7 may be not true as shown in Example 3.5.

Theorem 3.8. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H, K \subseteq S$. Then the following statements hold:

- (i) if $H^c \in \mathcal{T}_{\eta^*}$, then $H_{\mathcal{PR}^*}^{\otimes} \subseteq H$ and hence $\varphi_{\mathcal{PR}^*}^{\otimes} = \varphi$.
- (ii) $Cl(H_{\mathcal{PR}^*}^{\otimes}) = Cl_{\eta^*}(H_{\mathcal{PR}^*}^{\otimes}) = H_{\mathcal{PR}^*}^{\otimes}$ where Cl_{η^*} is the closure operator with respect to \mathcal{T}_{η^*} .
- (iii) if $H \subseteq K$, then $H_{\mathcal{PR}^*}^{\otimes} \subseteq K_{\mathcal{PR}^*}^{\otimes}$.
- (iv) $(H_{\mathcal{PR}^*}^{\otimes})_{\mathcal{PR}^*}^{\otimes} \subseteq H_{\mathcal{PR}^*}^{\otimes}$.
- (v) $H_{\mathcal{PR}^*}^{\otimes} \cup K_{\mathcal{PR}^*}^{\otimes} = (H \cup K)_{\mathcal{PR}^*}^{\otimes}$.
- (vi) $(H \cap K)_{\mathcal{PR}^*}^{\otimes} \subseteq H_{\mathcal{PR}^*}^{\otimes} \cap K_{\mathcal{PR}^*}^{\otimes}$.

Proof. We need to prove part(ii) only since all other implications can be proved directly from Definition 3.1. Let $x \in Cl_{\eta^*}(H_{\mathcal{PR}^*}^{\otimes})$ and $V \in \mathcal{RO}^*(x, S, \mathcal{T})$. As there is $t \in H_{\mathcal{PR}^*}^{\otimes} \cap V$, then $V \in \mathcal{RO}^*(t, S, \mathcal{T})$ and hence $H^c \cup V^c \in \mathcal{P}$ for each $V \in \mathcal{RO}^*(x, S, \mathcal{T})$. Thus, $Cl_{\eta^*}(H_{\mathcal{PR}^*}^{\otimes}) \subseteq H_{\mathcal{PR}^*}^{\otimes}$. Since $H_{\mathcal{PR}^*}^{\otimes} \subseteq Cl_{\eta^*}(H_{\mathcal{PR}^*}^{\otimes})$ is always true, then $Cl_{\eta^*}(H_{\mathcal{PR}^*}^{\otimes}) = H_{\mathcal{PR}^*}^{\otimes}$. Now, by Theorem 2.2, $H_{\mathcal{PR}^*}^{\otimes} \subseteq Cl(H_{\mathcal{PR}^*}^{\otimes}) \subseteq Cl_{\eta^*}(H_{\mathcal{PR}^*}^{\otimes}) = H_{\mathcal{PR}^*}^{\otimes}$ and therefore $Cl(H_{\mathcal{PR}^*}^{\otimes}) = H_{\mathcal{PR}^*}^{\otimes}$. \square

Remark 3.9. Referring to Theorem 3.8 and note the following:

(i) $\varphi = \{x_3\}_{\mathcal{PR}^*}^{\otimes} \subseteq \{x_3\}$ in Example 3.5, while $\{x_1, x_2\} \notin \mathcal{T}_{\eta^*}$ and hence the converse of part(i) may not be true.

(ii) consider the set of real numbers \mathbb{R} with topology $\mathcal{T} = \{(-\infty, x) : x \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Then $\mathcal{RO}^*(S, \mathcal{T}) = \{\mathbb{R}, \emptyset\}$. Let \mathcal{P}_f be the primal of all subsets of the real line which complement is not finite. Let $H = \{0, 1\}$. Then $H_{\mathcal{PR}^*}^{\otimes} = \{x \in \mathbb{R} : \forall V \in \mathcal{RO}^*(x, S, \mathcal{T}), H^c \cup V^c = H^c \in \mathcal{P}_f\} = \emptyset$. Let $K = (0, 1)$. Then $K_{\mathcal{PR}^*}^{\otimes} = \{x \in \mathbb{R} : \forall V \in \mathcal{RO}^*(x, S, \mathcal{T}), K^c \cup V^c = K^c \in \mathcal{P}_f\} = \mathbb{R}$. Hence the converse of part(iii) may not be true.

(iii) consider $S = \{x_1, x_2, x_3\}$ and $\mathcal{T} = \{\varphi, S, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$ with $\mathcal{P} = \{\varphi, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}\}$. Then $\{x_1, x_2\}_{\mathcal{PR}^*}^{\otimes} = \{x_2, x_3\}$ and $\{x_3\}_{\mathcal{PR}^*}^{\otimes} = \{x_3\}$ and hence $\{x_1, x_2\}_{\mathcal{PR}^*}^{\otimes} \cap \{x_3\}_{\mathcal{PR}^*}^{\otimes} = \{x_3\}$, while $(\{x_1, x_2\} \cap \{x_3\})_{\mathcal{PR}^*}^{\otimes} = \varphi_{\mathcal{PR}^*}^{\otimes} = \varphi$. Therefore, the reverse inclusion of part(vi) may not be true.

Theorem 3.10. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H, K \subseteq S$. If $H, H^c \in \mathcal{T}_{\eta^*}$, then $(H \cap K)_{\mathcal{PR}^*}^{\otimes} = H \cap K_{\mathcal{PR}^*}^{\otimes} = H_{\mathcal{PR}^*}^{\otimes} \cap K_{\mathcal{PR}^*}^{\otimes}$.

Proof. Assume that $H, K \subseteq S$. Since $H^c \in \mathcal{T}_{\eta^*}$, then by Theorem 3.8, $(H \cap K)_{\mathcal{PR}^*}^{\otimes} \subseteq H_{\mathcal{PR}^*}^{\otimes} \cap K_{\mathcal{PR}^*}^{\otimes} \subseteq H \cap K_{\mathcal{PR}^*}^{\otimes}$. Now, let $x \in H \cap K_{\mathcal{PR}^*}^{\otimes}$ and $V \in \mathcal{RO}^*(x, S, \mathcal{T})$. Then $H = \cup_{\gamma \in \Delta} V_{\gamma}$ with $V_{\gamma} \in \mathcal{RO}^*(S, \mathcal{T})$ for each $\gamma \in \Delta$ and so there is $V_{\gamma_0} \in \mathcal{RO}^*(x, S, \mathcal{T})$ for some $\gamma_0 \in \Delta$. By Theorem 2.3, $V_{\gamma_0} \cap V \in \mathcal{RO}^*(x, S, \mathcal{T})$ and hence $K^c \cup (\cup_{\gamma \in \Delta} V_{\gamma} \cap V)^c \subseteq K^c \cup (V_{\gamma_0} \cap V)^c \in \mathcal{P}$, which leads to $(H \cap K)^c \cup V^c = K^c \cup (H \cap V)^c \in \mathcal{P}$. Thus, $x \in (H \cap K)_{\mathcal{PR}^*}^{\otimes}$ and therefore $H_{\mathcal{PR}^*}^{\otimes} \cap K_{\mathcal{PR}^*}^{\otimes} \subseteq H \cap K_{\mathcal{PR}^*}^{\otimes} \subseteq (H \cap K)_{\mathcal{PR}^*}^{\otimes} \subseteq H_{\mathcal{PR}^*}^{\otimes} \cap K_{\mathcal{PR}^*}^{\otimes}$ and hence $H_{\mathcal{PR}^*}^{\otimes} \cap K_{\mathcal{PR}^*}^{\otimes} = H \cap K_{\mathcal{PR}^*}^{\otimes} = (H \cap K)_{\mathcal{PR}^*}^{\otimes}$. \square

Theorem 3.11. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then the following are equivalent:

- (i) $\mathcal{RC}^*(S, \mathcal{T}) - \{S\} \subseteq \mathcal{P}$.
- (ii) if $H^c \notin \mathcal{P}$, then $Int^*(H) = \emptyset$, where $Int^*(H) = \cup\{V \in \mathcal{RO}^*(S, \mathcal{T}) : V \subseteq H\}$.
- (iii) for each $H \in \mathcal{RO}^*(S, \mathcal{T})$, we have $H \subseteq H_{\mathcal{PR}^*}^{\otimes}$.
- (iv) $S = S_{\mathcal{PR}^*}^{\otimes}$.

Proof. (i \Rightarrow ii) Suppose that $H^c \notin \mathcal{P}$ and $\mathcal{RC}^\star(\mathcal{S}, \mathcal{T}) - \{\mathcal{S}\} \subseteq \mathcal{P}$ with $x \in \text{Int}^\star(H)$. Thus, we can find $V \in \mathcal{RO}^\star(\mathcal{S}, \mathcal{T})$ such that $x \in V \subseteq H$. Since $H^c \subseteq V^c$ and $H^c \notin \mathcal{P}$, then $V^c \notin \mathcal{P}$ which contradicts that $\mathcal{RC}^\star(\mathcal{S}, \mathcal{T}) - \{\mathcal{S}\} \subseteq \mathcal{P}$. Hence, $\text{Int}^\star(H) = \emptyset$.

(ii \Rightarrow iii) In case $H = \emptyset$, we obviously have $H_{\mathcal{PR}^\star}^\otimes = \emptyset = H$. Let $x \in H$ and suppose $x \notin H_{\mathcal{PR}^\star}^\otimes$. Then there is $V \in \mathcal{RO}^\star(x, \mathcal{S}, \mathcal{T})$ with $H^c \cup V^c \notin \mathcal{P}$ which implies that $(H \cap V)^c \notin \mathcal{P}$. Since $H \in \mathcal{RO}^\star(\mathcal{S}, \mathcal{T})$, then by part(ii) we have $H \cap V = \text{Int}^\star(H \cap V) = \emptyset$ which is a contradiction since $x \in H \cap V$. Therefore, $x \in H_{\mathcal{PR}^\star}^\otimes$ and hence $H \subseteq H_{\mathcal{PR}^\star}^\otimes$.

(iii \Rightarrow iv) It is clear since $\mathcal{S} \in \mathcal{RO}^\star(\mathcal{S}, \mathcal{T})$ and hence $\mathcal{S} = \mathcal{S}_{\mathcal{PR}^\star}^\otimes$.

(iv \Rightarrow i) Since $\mathcal{S} = \mathcal{S}_{\mathcal{PR}^\star}^\otimes = \{x \in \mathcal{S} : V^c \cup \mathcal{S}^c = V^c \in \mathcal{P} \text{ for each } x \in V \in \mathcal{RO}^\star(\mathcal{S}, \mathcal{T})\}$, then $\mathcal{RC}^\star(\mathcal{S}, \mathcal{T}) - \{\mathcal{S}\} \subseteq \mathcal{P}$. □

Theorem 3.12. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ and $(\mathcal{S}, \mathcal{T}, \mathcal{P}^*)$ be two primal topological spaces with $H \subseteq \mathcal{S}$. Then:

(i) $H_{\mathcal{PR}^\star}^\otimes \subseteq H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes$ whenever $\mathcal{P} \subseteq \mathcal{P}^*$.

(ii) $H_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^\star}^\otimes = H_{\mathcal{PR}^\star}^\otimes \cup H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes$.

Proof. (i) Let $x \in H_{\mathcal{PR}^\star}^\otimes$ and $V \in \mathcal{RO}^\star(x, \mathcal{S}, \mathcal{T})$. Then $H^c \cup V^c \in \mathcal{P} \subseteq \mathcal{P}^*$ and hence $x \in H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes$.

(ii) From part(i), we have $H_{\mathcal{PR}^\star}^\otimes \cup H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes \subseteq H_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^\star}^\otimes$. Now assume that $x \notin H_{\mathcal{PR}^\star}^\otimes \cup H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes$, then there are $V, U \in \mathcal{RO}^\star(x, \mathcal{S}, \mathcal{T})$ with $H^c \cup V^c \notin \mathcal{P}$ and $H^c \cup U^c \notin \mathcal{P}^*$. Set $K = V \cap U$, then $H^c \cup K^c \notin \mathcal{P} \cup \mathcal{P}^*$ with $K \in \mathcal{RO}^\star(x, \mathcal{S}, \mathcal{T})$ and hence $x \notin H_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^\star}^\otimes$. Therefore, $H_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^\star}^\otimes = H_{\mathcal{PR}^\star}^\otimes \cup H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes$. □

Theorem 3.13. Let $f : (\mathcal{S}, \mathcal{T}, \mathcal{P}) \rightarrow (\mathcal{S}^*, \mathcal{T}^*, f(\mathcal{P}))$ be bijective GR-continuous function, where $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ and $(\mathcal{S}^*, \mathcal{T}^*, f(\mathcal{P}))$ are primal topological spaces. Then:

(i) $(f^{-1}(H))_{\mathcal{PR}^\star}^\otimes \subseteq f^{-1}(H_{f(\mathcal{P})\mathcal{R}^\star}^\otimes)$ for each $H \subseteq \mathcal{S}^*$.

(ii) $f(H_{\mathcal{PR}^\star}^\otimes) \subseteq (f(H))_{f(\mathcal{P})\mathcal{R}^\star}^\otimes$ for each $H \subseteq \mathcal{S}$.

Proof. (i) Assume that $x \notin f^{-1}(H_{f(\mathcal{P})\mathcal{R}^\star}^\otimes)$. Then there is $V \in \mathcal{RO}^\star(f(x), \mathcal{S}^*, \mathcal{T}^*)$ with $H^c \cup V^c \notin f(\mathcal{P})$ and hence $(f^{-1}(H))^c \cup (f^{-1}(V))^c = f^{-1}(H^c \cup V^c) \notin \mathcal{P}$. Since $f^{-1}(V) \in \mathcal{RO}^\star(x, \mathcal{S}, \mathcal{T})$, then $x \notin (f^{-1}(H))_{\mathcal{PR}^\star}^\otimes$.

(ii) Assume that $f(x) \in f(H_{\mathcal{PR}^\star}^\otimes) - (f(H))_{f(\mathcal{P})\mathcal{R}^\star}^\otimes$. Then there is $V \in \mathcal{RO}^\star(f(x), \mathcal{S}^*, \mathcal{T}^*)$ with $(f(H))^c \cup V^c \notin f(\mathcal{P})$ and hence $(f^{-1}f(H))^c \cup (f^{-1}(V))^c \notin \mathcal{P}$. Since $(f^{-1}f(H))^c \cup (f^{-1}(V))^c = H^c \cup (f^{-1}(V))^c$, then $H^c \cup (f^{-1}(V))^c \notin \mathcal{P}$ with $f^{-1}(V) \in \mathcal{RO}^\star(x, \mathcal{S}, \mathcal{T})$, therefore $x \notin H_{\mathcal{PR}^\star}^\otimes$ which is a contradiction. □

Example 3.14. (i) Define $f : (\mathcal{S}, \mathcal{T}, \mathcal{P}) \rightarrow (\mathcal{S}^*, \mathcal{T}^*, f(\mathcal{P}))$ by: $f(x_1) = 2, f(x_2) = 1$ and $f(x_3) = 3$ where $\mathcal{S} = \{x_1, x_2, x_3\}, \mathcal{T} = \{\varphi, \mathcal{S}, \{x_1\}\}, \mathcal{P} = \{\varphi, \{x_1\}, \{x_3\}, \{x_1, x_3\}\}, \mathcal{S}^* = \{1, 2, 3\}$ and $\mathcal{T}^* = \{\varphi, \mathcal{S}, \{1\}\}$. Take $H = \{1\}$, then $(f^{-1}\{1\})_{\mathcal{PR}^\star}^\otimes = \{x_2\}_{\mathcal{PR}^\star}^\otimes = \{x_2, x_3\}$ while $f^{-1}(H_{f(\mathcal{P})\mathcal{R}^\star}^\otimes) = f^{-1}(\{1\}_{f(\mathcal{P})\mathcal{R}^\star}^\otimes) = f^{-1}\{\mathcal{S}^*\} = \mathcal{S}$. Moreover, if $H = \{x_2, x_3\}$, then $f(H_{f(\mathcal{P})\mathcal{R}^\star}^\otimes) = f(\{x_2, x_3\}) = \{1, 3\}$ while $(f(H))_{f(\mathcal{P})\mathcal{R}^\star}^\otimes = \{1, 3\}_{f(\mathcal{P})\mathcal{R}^\star}^\otimes = \mathcal{S}^*$. Therefore, the reverse inclusion of Theorem 3.13 [Parts(i) and (ii)] are not true in general. Also, note that although f is a bijective GR-continuous function, the equality in the Theorem 3.13 is not achieved.

(ii) Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ be defined as in Example 3.5. Assume that $\mathcal{S}^* = \{1, 2, 3\}$ with $\mathcal{T}^* = \{\varphi, \mathcal{S}, \{1\}\}$ and $\mathcal{P}^* = \{\varphi, \{1\}, \{3\}, \{1, 3\}\}$. Define $f : (\mathcal{S}, \mathcal{T}, \mathcal{P}) \rightarrow (\mathcal{S}^*, \mathcal{T}^*, \mathcal{P}^*)$ by: $f(x_1) = 1, f(x_2) = 2$ and $f(x_3) = 3$. Take $H = \{1\}$, then $(f^{-1}\{1\})_{\mathcal{PR}^\star}^\otimes = \{x_1\}_{\mathcal{PR}^\star}^\otimes = \mathcal{S}$ while $f^{-1}(H_{\mathcal{P}^*\mathcal{R}^\star}^\otimes) = f^{-1}(\{1\}_{\mathcal{P}^*\mathcal{R}^\star}^\otimes) = f^{-1}\{\varphi\} = \varphi$. Moreover, if $H = \{x_1\}$, then $f(\{x_1\}_{\mathcal{PR}^\star}^\otimes) = \mathcal{S}^*$ while $(f(H))_{\mathcal{P}^*\mathcal{R}^\star}^\otimes = \{1\}_{\mathcal{P}^*\mathcal{R}^\star}^\otimes = \varphi$. Therefore, Theorem 3.13 is not necessarily true if $(\mathcal{S}^*, \mathcal{T}^*, f(\mathcal{P}))$ was different from $f(\mathcal{P})$.

Corollary 3.15. Let $f : (S, \mathcal{T}, \mathcal{P}) \rightarrow (S^*, \mathcal{T}^*, f(\mathcal{P}))$ be bijective function with $(S, \mathcal{T}, \mathcal{P})$ and $(S^*, \mathcal{T}^*, f(\mathcal{P}))$ are primal topological spaces. If f and f^{-1} are GR-continuous functions, then:

(i) $(f^{-1}(H))_{\mathcal{P}\mathcal{R}\star}^{\otimes} = f^{-1}(H_{f(\mathcal{P})\mathcal{R}\star}^{\otimes})$ for each $H \subseteq S^*$.

(ii) $f(H_{\mathcal{P}\mathcal{R}\star}^{\otimes}) = (f(H))_{f(\mathcal{P})\mathcal{R}\star}^{\otimes}$ for each $H \subseteq S$.

Proof. The proof is a consequence of Theorem 3.13. □

4 The operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ with the topology $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$

In this section, we use the operator $(\cdot)_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ to define a new operator, denoted by $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$, and investigate some of its properties. We obtain a new topology, denoted by $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$, by using a kuratowski closure operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}$ and study the relationships between the topologies $\mathcal{T}_{\eta\star}, \mathcal{T}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}, \mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond}$ and $\mathcal{T}_{\mathcal{P}}^{\diamond}$.

Definition 4.1. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then define an operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes} : 2^S \rightarrow 2^S$ by: $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cup H$ for each $H \subseteq S$.

Theorem 4.2. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq S$. If $K, K^c \in \mathcal{T}_{\eta\star}$, then $K \cap Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K \cap H)$.

Proof. By using Theorem 3.10 we have $K \cap Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = K \cap (H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cup H) = (K \cap H_{\mathcal{P}\mathcal{R}\star}^{\otimes}) \cup (K \cap H) = (H \cap K)_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cup (K \cap H) = Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K \cap H)$. □

Theorem 4.3. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H, K \subseteq S$. Then the following statements hold:

- (i) $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(\varphi) = \varphi$ and $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(S) = S$.
- (ii) $H \subseteq Cl(H)_{\mathcal{P}}^{\diamond} \subseteq Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq Cl_{\mathcal{P}\mathcal{R}}^{\diamond}(H)$.
- (iii) if $H \subseteq K$, then $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K)$.
- (iv) $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H \cup K) = Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \cup Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K)$.
- (v) $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H \cap K) \subseteq Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \cap Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K)$.
- (vi) $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H))$.

Proof. The proof is immediately from Theorems 3.2 and 3.8. □

Corollary 4.4. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then, the operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes} : 2^S \rightarrow 2^S$ is a Kuratowski closure operator.

Definition 4.5. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq S$. Then the family $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes} = \{H \subseteq S : Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H^c) = H^c\}$ is a topology on S , denoted by $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$, induced by \mathcal{T} and \mathcal{P} . Furthermore, $Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = \cup\{V \in \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes} : V \subseteq H\}$.

Theorem 4.6. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq S$. Then:

- (i) $\mathcal{T}_{\eta\star} \subseteq \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$.
- (ii) $\mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond} \subseteq \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes} \subseteq \mathcal{T}_{\mathcal{P}}^{\diamond}$.

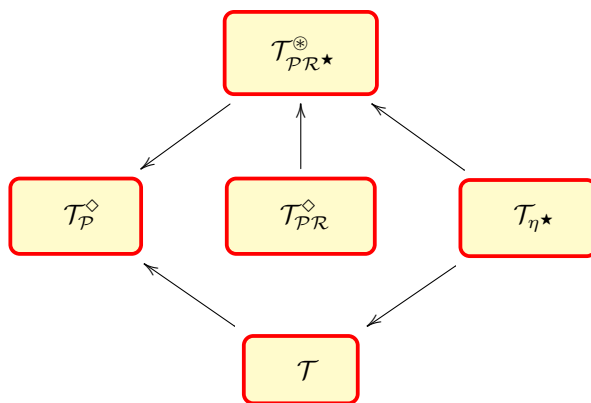
Proof. (i) Assume that $H \in \mathcal{T}_{\eta^*}$, then by Theorem 3.8, $(H^c)_{\mathcal{P}\mathcal{R}^*}^{\otimes} \subseteq H^c$ and hence $Cl_{\mathcal{P}\mathcal{R}^*}^{\otimes}(H^c) = H^c$. Therefore, $H \in \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes}$.

(ii) Assume that $H \in \mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond}$. Then $(H^c)_{\mathcal{P}\mathcal{R}}^{\diamond} \subseteq H^c$ and hence by Theorem 3.2 we have $(H^c)_{\mathcal{P}\mathcal{R}^*}^{\otimes} \subseteq H^c$. Therefore $Cl_{\mathcal{P}\mathcal{R}^*}^{\otimes}(H^c) = H^c$ and thus $H \in \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes}$. By the same way $\mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} \subseteq \mathcal{T}_{\mathcal{P}}^{\diamond}$ can be proved. \square

Referring to Example 3.3. Simple calculations show that the following:

- (i) $\mathcal{T}_{\eta^*} = \{\varphi, \mathcal{S}, \{x_3\}\}$ and $\mathcal{T}_{\mathcal{P}}^{\diamond} = 2^{\mathcal{S}}$,
- (ii) $\mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} = \{\varphi, \mathcal{S}, \{x_1\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}\}$ and $\mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond} = \{\varphi, \mathcal{S}, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\}$.

Therefore, we generate the following diagram where none of these implications being reversible.



Theorem 4.7. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then:

- (i) $\mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond} = \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes}$ whenever $(\mathcal{S}, \mathcal{T})$ is $\mathcal{T}_{\frac{1}{2}}$ -space.
- (ii) $\mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond} = \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} = \mathcal{T}_{\mathcal{P}}^{\diamond}$ whenever $(\mathcal{S}, \mathcal{T})$ is locally indiscrete.

Proof. (i) By using Theorem 3.4 [Part(i)] we have $H_{\mathcal{P}\mathcal{R}^*}^{\otimes} = H_{\mathcal{P}\mathcal{R}}^{\diamond}$ for each subset H of \mathcal{S} and hence $\mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond} = \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes}$.

(ii) By using Theorem 3.4 [Part(ii)] we have $H_{\mathcal{P}}^{\diamond} = H_{\mathcal{P}\mathcal{R}^*}^{\otimes} = H_{\mathcal{P}\mathcal{R}}^{\diamond}$ for each subset H of \mathcal{S} and hence $\mathcal{T}_{\mathcal{P}\mathcal{R}}^{\diamond} = \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} = \mathcal{T}_{\mathcal{P}}^{\diamond}$. \square

Referring to Example 3.5, to see that the converse of Theorem 4.7 is not true in general.

Corollary 4.8. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ be a primal topological space. Then $(\mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes})_{\mathcal{P}\mathcal{R}^*}^{\otimes} = (\mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes})_{\mathcal{P}\mathcal{R}}^{\diamond}$.

Proof. From Theorems 2.7, $(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes})$ is $\mathcal{T}_{\frac{1}{2}}$ -space and hence by Theorems 4.7 (i) $(\mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes})_{\mathcal{P}\mathcal{R}^*}^{\otimes} = (\mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes})_{\mathcal{P}\mathcal{R}}^{\diamond}$. \square

Theorem 4.9. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ and $(\mathcal{S}, \mathcal{T}, \mathcal{P}^*)$ be two primal topological spaces with $H \subseteq \mathcal{S}$. Then:

- (i) $\mathcal{T}_{\mathcal{P}^*\mathcal{R}^*}^{\otimes} \subseteq \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes}$ whenever $\mathcal{P} \subseteq \mathcal{P}^*$.
- (ii) $\mathcal{T}_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^*}^{\otimes} = \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} \cap \mathcal{T}_{\mathcal{P}^*\mathcal{R}^*}^{\otimes}$.

Proof. (i) Assume that $H \in \mathcal{T}_{\mathcal{P}^*\mathcal{R}^*}^{\otimes}$. By using Theorem 3.12, $H^c \cup (H^c)_{\mathcal{P}\mathcal{R}^*}^{\otimes} \subseteq H^c \cup (H^c)_{\mathcal{P}^*\mathcal{R}^*}^{\otimes} = H^c$ and hence $H \in \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes}$.

(ii) By part(i) we have $\mathcal{T}_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^*}^{\otimes} \subseteq \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} \cap \mathcal{T}_{\mathcal{P}^*\mathcal{R}^*}^{\otimes}$. Now assume that $H \in \mathcal{T}_{\mathcal{P}\mathcal{R}^*}^{\otimes} \cap \mathcal{T}_{\mathcal{P}^*\mathcal{R}^*}^{\otimes}$. Then $Cl_{\mathcal{P}^*\mathcal{R}^*}^{\otimes}(H^c) = Cl_{\mathcal{P}\mathcal{R}^*}^{\otimes}(H^c) = H^c$ and hence by using Theorem 3.12 we have $H^c = H^c \cup ((H^c)_{\mathcal{P}\mathcal{R}^*}^{\otimes} \cup (H^c)_{\mathcal{P}^*\mathcal{R}^*}^{\otimes}) = H^c \cup (H^c)_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^*}^{\otimes}$. Therefore, $H \in \mathcal{T}_{(\mathcal{P} \cup \mathcal{P}^*)\mathcal{R}^*}^{\otimes}$. \square

5 $\mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}$ – Regular via \mathcal{P} –generalized closed subsets in $(\mathcal{S}, \mathcal{T}, \mathcal{P})$

In this section, we define \mathcal{P} –generalized closed and \mathcal{P} –generalized open sets. Then we study some of their basic features and use \mathcal{P} –generalized open sets to introduce the notion of $\mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}$ – regular spaces.

Definition 5.1. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq \mathcal{S}$. Then H is called \mathcal{P} –generalized closed subset in $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ if $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq V$ whenever $H \subseteq V$ and $V \in \mathcal{T}$.

The complement of \mathcal{P} –generalized closed subset in $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ is called \mathcal{P} –generalized open subset in $(\mathcal{S}, \mathcal{T}, \mathcal{P})$. The family of all \mathcal{P} –generalized closed (resp., \mathcal{P} –generalized open) subsets in $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ will be denoted by $\mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ (resp., $\mathcal{P}\mathcal{G}\mathcal{O}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$).

Example 5.2. Consider $\mathcal{S} = \{x_1, x_2, x_3\}$ with $\mathcal{T} = \{\varphi, \mathcal{S}, \{x_3\}, \{x_2, x_3\}, \{x_3, x_1\}\}$ and $\mathcal{P} = \{\varphi, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$. Then we can note the following:

(i) $\mathcal{C}(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}) \subseteq \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ where $\mathcal{C}(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes})$ is the family of all closed subsets of $(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes})$. This holds since $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = H$ for each $H \in \mathcal{C}(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes})$ while the reverse inclusion may not be true since $\{x_1, x_2\} \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P}) - \mathcal{C}(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes})$.

(ii) $\mathcal{G}\mathcal{C}(\mathcal{S}, \mathcal{T})$ and $\mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ are independent since $\{x_1\} \in \mathcal{G}\mathcal{C}(\mathcal{S}, \mathcal{T}) - \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ and $\{x_2, x_3\} \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P}) - \mathcal{G}\mathcal{C}(\mathcal{S}, \mathcal{T})$.

Theorem 5.3. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq \mathcal{S}$. Then $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ if and only if $U \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$, whenever $U \subseteq H$ and $U \in \mathcal{C}(\mathcal{S}, \mathcal{T})$.

Proof. Let $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ with $U \subseteq H$ and $U \in \mathcal{C}(\mathcal{S}, \mathcal{T})$. Then $\mathcal{S} - H \subseteq \mathcal{S} - U$ with $\mathcal{S} - H \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ and hence $\mathcal{S} - Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) = Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(\mathcal{S} - H) \subseteq \mathcal{S} - U$. Thus $U \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$. Conversely, let $U \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$, whenever $U \subseteq H$ and $U \in \mathcal{C}(\mathcal{S}, \mathcal{T})$. Suppose $\mathcal{S} - H \subseteq V$ and $V \in \mathcal{T}$, then $\mathcal{S} - V \subseteq H$ and $\mathcal{S} - V \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$. Hence, $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(\mathcal{S} - H) = \mathcal{S} - Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq V$ and therefore, $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$. \square

Theorem 5.4. Let $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq \mathcal{S}$. Then the following are equivalent:

(i) $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$.

(ii) $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \subseteq V$ whenever $H \subseteq V$ and $V \in \mathcal{T}$.

(iii) $Cl(\{x\}) \cap H \neq \varphi$ for each $x \in Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$.

(iv) $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} - H$ dose not contain non-empty closed set.

(v) $H = G \cap K^c$ where $G \in \mathcal{C}(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes})$ and K dose not contain non-empty closed set.

Proof. (i \Rightarrow ii) Follows immediately from Definition 5.1.

(ii \Rightarrow iii) Assume that $Cl(\{x\}) \cap H = \varphi$ for some $x \in Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$. Then $H \subseteq \mathcal{S} - Cl(\{x\})$ and hence $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \subseteq \mathcal{S} - Cl(\{x\})$ which is a contradiction.

(iii \Rightarrow iv) Assume that $K \in \mathcal{C}(\mathcal{S}, \mathcal{T})$ with $x \in K \subseteq H_{\mathcal{P}\mathcal{R}\star}^{\otimes} - H$. Since $Cl(\{x\}) \subseteq K \subseteq \mathcal{S} - H$, then $Cl(\{x\}) \cap H = \varphi$ which is a contradiction.

(iv \Rightarrow v) Take $G = Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$ and $K = H_{\mathcal{P}\mathcal{R}\star}^{\otimes} - H$. Then $H = G \cap K^c$ with $G \in \mathcal{C}(\mathcal{S}, \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes})$ and K , by part (iv), dose not contain non-empty closed set.

(v \Rightarrow i) Let $H \subseteq V$ with $V \in \mathcal{T}$. Then $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cap V^c \subseteq G_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cap V^c \subseteq G \cap V^c \subseteq K$. By Theorem 3.8, $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cap V^c \in \mathcal{C}(\mathcal{S}, \mathcal{T})$ and hence $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \cap V^c = \varphi$. Therefore $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq V$. \square

Theorem 5.5. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H, K \subseteq S$. If $H \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$, then:

- (i) $H \cup K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever $K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$.
- (ii) $H \cap K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever $K^c \in \mathcal{T}_{\eta\star}$.
- (iii) $K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever $H \subseteq K \subseteq Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$.

Proof. Follows from Definition 5.1 and Theorems 3.8 and 4.3. □

In Example 3.5, $\{x_1, x_2\}, \{x_1, x_3\} \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ while $\{x_1\} \notin \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ and hence it is not true in general if $H, K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$, then $H \cap K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$. Also, in Example 5.2, $Cl_{\mathcal{P}\mathcal{R}\star}^{\otimes}\{x_1\} = \{x_1, x_2, x_3\}$ and hence $S \cap \{x_1\} \notin \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$. Therefore, Theorem 5.5 [Part (ii)] is no longer true if $K \in \mathcal{C}(S, \mathcal{T})$.

Corollary 5.6. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H \subseteq S$. Then the following statements hold:

- (i) $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever $H \in \mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\otimes}$.
- (ii) $H \cap K \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever $H, K \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$.
- (iii) $K \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever $Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \subseteq K \subseteq H$ and $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$.

Proof. Follows immediately from the definition of $\mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ and Theorem 5.5. □

Theorem 5.7. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H, K \subseteq S$. If H and K are separated in (S, \mathcal{T}) and $H, K \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$, then $H \cup K \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$.

Proof. Assume that $V \subseteq H \cup K$ with $V \in \mathcal{C}(S, \mathcal{T})$. Then $V \cap Cl(H) \subseteq H$ and hence by Theorem 5.3, $V \cap Cl(H) \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H)$. Also, we can get $V \cap Cl(K) \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K)$. Therefore, $V = V \cap (Cl(H) \cup Cl(K)) \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H) \cup Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(K) \subseteq Int_{\mathcal{P}\mathcal{R}\star}^{\otimes}(H \cup K)$ and thus $H \cup K \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$. □

Corollary 5.8. Let $(S, \mathcal{T}, \mathcal{P})$ be a primal topological space with $H, K \subseteq S$. If $H, K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ and H^c, K^c are separated in (S, \mathcal{T}) , then $H \cap K \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$.

Proof. Follows immediately from Theorem 5.7. □

Theorem 5.9. Let $f : (S, \mathcal{T}, \mathcal{P}) \rightarrow (S^*, \mathcal{T}^*, f(\mathcal{P}))$ be bijective GR–continuous function, where $(S, \mathcal{T}, \mathcal{P})$ and $(S^*, \mathcal{T}^*, f(\mathcal{P}))$ are primal topological spaces. Then:

- (i) $f^{-1}(H) \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ for each $H \in \mathcal{P}\mathcal{G}\mathcal{C}_{f(\mathcal{P})\mathcal{R}\star}(S^*, \mathcal{T}^*, f(\mathcal{P}))$ whenever f^{-1} is continuous.
- (ii) $f(H) \in \mathcal{P}\mathcal{G}\mathcal{C}_{f(\mathcal{P})\mathcal{R}\star}(S^*, \mathcal{T}^*, f(\mathcal{P}))$ for each $H \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever f^{-1} is GR–continuous.

Proof. (i) Assume that $f^{-1}(H) \subseteq V$ and $V \in \mathcal{T}$. Since $H \subseteq f(V)$ and $H \in \mathcal{P}\mathcal{G}\mathcal{C}_{f(\mathcal{P})\mathcal{R}\star}(S^*, \mathcal{T}^*, f(\mathcal{P}))$, then $H_{f(\mathcal{P})\mathcal{R}\star}^{\otimes} \subseteq f(V)$ and hence $f^{-1}(H_{f(\mathcal{P})\mathcal{R}\star}^{\otimes}) \subseteq V$. Therefore by Theorem 3.13, $(f^{-1}(H))_{\mathcal{P}\mathcal{R}\star}^{\otimes} \subseteq V$.

(ii) Assume that $f(H) \subseteq V$ and $V \in \mathcal{T}$. Since $H \subseteq f^{-1}(V)$ and $H \in \mathcal{P}\mathcal{G}\mathcal{C}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$, then $H_{\mathcal{P}\mathcal{R}\star}^{\otimes} \subseteq f^{-1}(V)$ and hence $f(H_{\mathcal{P}\mathcal{R}\star}^{\otimes}) \subseteq V$. Therefore by Corollary 3.15, $(f(H))_{f(\mathcal{P})\mathcal{R}\star}^{\otimes} \subseteq V$. □

Corollary 5.10. Let $f : (S, \mathcal{T}, \mathcal{P}) \rightarrow (S^*, \mathcal{T}^*, f(\mathcal{P}))$ be bijective GR–continuous function, where $(S, \mathcal{T}, \mathcal{P})$ and $(S^*, \mathcal{T}^*, f(\mathcal{P}))$ are primal topological spaces. Then:

- (i) $f^{-1}(H) \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ for each $H \in \mathcal{P}\mathcal{G}_{f(\mathcal{P})\mathcal{R}\star}(S^*, \mathcal{T}^*, f(\mathcal{P}))$ whenever f^{-1} is continuous.
- (ii) $f(H) \in \mathcal{P}\mathcal{G}_{f(\mathcal{P})\mathcal{R}\star}(S^*, \mathcal{T}^*, f(\mathcal{P}))$ for each $H \in \mathcal{P}\mathcal{G}_{\mathcal{P}\mathcal{R}\star}(S, \mathcal{T}, \mathcal{P})$ whenever f^{-1} is GR–continuous.

Proof. Follows immediately from Theorem 5.9. □

Definition 5.11. Let (S, T, P) be a primal topological space. Then (S, T, P) is said to be $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular if for each $H \in \mathcal{C}(S, T)$ and each point $x \notin H$, there are $V, U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $x \in V, H \subseteq U$ and $V \cap U = \varphi$.

Each \mathcal{R}^\star -regular space is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular while the converse is not true in general as shown in Example 3.3.

Theorem 5.12. Let (S, T, P) be a primal topological space. Then the following are equivalent:

- i) (S, T, P) is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular.
- ii) for each $V \in \mathcal{T}(x)$, there is $U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $x \in U \subseteq Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(U) \subseteq V$.
- iii) for each subset H and $K \in \mathcal{T}$ with $H \cap K \neq \varphi$, there is $V \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ such that $H \cap V \neq \varphi$ and $Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V) \subseteq K$.
- iv) for each subset $H \neq \varphi$ and $K \in \mathcal{C}(S, T)$ with $H \cap K = \varphi$, there are $V, U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ such that $V \cap U = \varphi, H \cap V \neq \varphi$ and $K \subseteq U$.

Proof. $i \Rightarrow ii$) Assume that $V \in \mathcal{T}(x)$. Then there are $U, K \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ such that $x \in U$ and $V^c \subseteq K$ with $U \subseteq K^c$. By Theorem 5.3, $V^c \subseteq (Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(K^c))^c$ and hence $Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(K^c) \subseteq V$. Therefore, $x \in U \subseteq Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(U) \subseteq Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(K^c) \subseteq V$.

$ii \Rightarrow iii$) Let $H \subseteq S$ and $K \in \mathcal{T}$ with $H \cap K \neq \varphi$. Then there is $x \in H \cap K$ and hence by part(ii) there is $V \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $x \in V \subseteq Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V) \subseteq K$.

$iii \Rightarrow iv$) Assume that $\varphi \neq H \subseteq S$ and $K \in \mathcal{C}(S, T)$ with $H \cap K = \varphi$. Then $H \cap K^c \neq \varphi$ and hence by part(iii) there is $V \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $H \cap V \neq \varphi$ and $Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V) \subseteq K^c$. Set $U = S - Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V)$, then $U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $K \subseteq U$.

$iv \Rightarrow i$) Assume that $K \in \mathcal{C}(S, T)$ and $x \notin K$. Set $H = \{x\}$ in part(iv), then the result follows. □

Proposition 5.13. Let (S, T, P) be a primal topological space. If (S, T, P) is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular, then for each $H \in \mathcal{C}(S, T)$ and each point $x \notin H$, there are $V, U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $x \in V, V^c \cup U^c \notin \mathcal{P}$ and $(H - U)^c \notin \mathcal{P}$.

Proof. Let $H \in \mathcal{C}(S, T)$ with $x \notin H$. Then there are $V, U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $x \in V, H \subseteq U$ and $V \cap U = \varphi$. Since $H \subseteq U$, then $H - U = \varphi$ and hence $(H - U)^c \notin \mathcal{P}$ with $V^c \cup U^c \notin \mathcal{P}$. □

Theorem 5.14. Let (S, T, P) be a primal topological space. If (S, T, P) is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular, then:

- (i) each open subset of (S, T) can be written as union of \mathcal{P} -generalized closed subset in (S, T, P) .
- (ii) each closed subset of (S, T) can be written as intersection of \mathcal{P} -generalized open subset in (S, T, P) .

Proof. (i) Let $x \in H \in \mathcal{T}$. Then there are $V_x, U \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ with $x \in V_x, H^c \subseteq U$ and $V_x \cap U = \varphi$. Hence $Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V_x) \subseteq Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(U^c) \subseteq H$. Now since $Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V_x) \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}(S, T, P)$ and $H = \cup\{Cl_{\mathcal{P}\mathcal{R}^\star}^\otimes(V_x) : x \in H\}$, then the result follows.

(ii) The proof is a consequence of part(i). □

Theorem 5.15. Let $f : (S, T, P) \rightarrow (S^*, T^*, f(P))$ be bijective GR-continuous function, where (S, T, P) and $(S^*, T^*, f(P))$ are primal topological spaces. Then:

- (i) if $(S^*, T^*, f(P))$ is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular, then so is (S, T, P) whenever f^{-1} is continuous.
- (ii) if (S, T, P) is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}^\star}$ -regular, then so is $(S^*, T^*, f(P))$ whenever f^{-1} is GR-continuous.

Proof. (i) Let $H \in \mathcal{C}(\mathcal{S}, \mathcal{T})$ with $x \notin H$. Then there are $V, U \in \mathcal{P}\mathfrak{G}_{f(\mathcal{P})\mathcal{R}\star}(\mathcal{S}^*, \mathcal{T}^*, f(\mathcal{P}))$ with $f(x) \in V$, $f(H) \subseteq U$ and $V \cap U = \varnothing$. By Corollary 5.10, $f^{-1}(V), f^{-1}(U) \in \mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}\star}(\mathcal{S}, \mathcal{T}, \mathcal{P})$ with $x \in f^{-1}(V)$, $H \subseteq f^{-1}(U)$ and $f^{-1}(V) \cap f^{-1}(U) = \varnothing$. Therefore, $(\mathcal{S}, \mathcal{T}, \mathcal{P})$ is $\mathcal{P}\mathfrak{G}_{\mathcal{P}\mathcal{R}\star}$ -regular.

(ii) Similar technique of part (i). □

6 Conclusion

In this manuscript, we introduced and examined the characteristics of two new operators, denoted by $(\cdot)_{\mathcal{P}\mathcal{R}\star}^{\circledast}$ and $Cl_{\mathcal{P}\mathcal{R}\star}^{\circledast}$, via the concept of primal. It turns out that the second one is a Kuratowski closure operator, but the first one is not. As a result, we obtained a new topology, denoted by $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\circledast}$, and proved some relationships between this new topology and the other topologies that existed in the literature. On the other hand, we have shown the notions of $\mathcal{T}_{\mathcal{P}\mathcal{R}\star}^{\circledast}$ and \mathcal{T} are independent. Moreover, we use the operator $Cl_{\mathcal{P}\mathcal{R}\star}^{\circledast}$ to introduce the notion of \mathcal{P} -generalized closed sets and study some of their basic properties. We hope that this paper will inspire additional investigation into primal and rough sets as ideals. In further research, we will look into rough approximations derived from various topologies using primals. New topologies will also be created using various types of primals and neighborhoods.

Conflict of interest

The author declares have no conflict of interest.

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