An Efficient Symmetric Operational Matrix Method for Solving Tempered Fractional Differential Equations with Respect to Another Function

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Abstract

In this paper, we introduce a novel extension of the symmetry operational matrix method specifically designed to tackle tempered fractional differential equations (FDE) that incorporate an additional function. Our approach leverages the framework of shifted Legendre polynomials (SLP), which are well-suited for this context. While the operational matrix method has been widely recognized for its efficacy in addressing a range of problems within fractional calculus, its application to tempered fractional differential equations remains relatively uncharted territory. To bridge this gap, we begin by deriving the analytical expression for the tempered fractional derivative (TFD) of the term τ^p. This crucial step paves the way for the formulation of a new operational matrix that captures the behavior of fractional derivatives in conjunction with another function. We use a method that combines a limited number of terms from the shifted Legendre polynomial basis. This allows us to accurately solve tempered fractional differential equations that include an additional function. We show that our approach works well through several numerical examples, demonstrating how effective and accurate our results are in tackling these complex equations.

Keywords : Orthogonality; Tempered Fractional; Caputo Fractional Differential Equations; Shifted Legendre polynomial

1 Introduction

The history of fractional calculus began with a discussion between L'Hôpital and Leibniz dated in the 1695, marking its inception with a deeply rooted historical significance. Over centuries, it has benefited from the contributions of various mathematicians and physicists, enriching the domain of fractional calculus theories. It was not until the last century that significant literature, including works by, Podlubny from 1999, Oldham and Spanier from 1974, Marichev from 1993, and Samko, Kilbas began to emerge, highlighting its importance. Recently, the application of fractional calculus has been recognized in describing a broad spectrum of non-traditional phenomena across applied sciences and engineering, as evidenced by various theories and experiments.[1,2] Its strong mathematical foundation has made fractional calculus a crucial tool for modeling unusual kinetics in fields such as finance,[3] biology,[4] chemistry,[5,7] physics,[8] engineering,[9] and beyond.[10,11] In its practical use, different types of fractional calculus have been developed, which the works in Riemann-Liouville, and the Caputo tempered tempered fractional derivative in various types,[12,13] Riesz,[14] and Hilfer and fractional derivatives,[15,16] showcasing its versatility and wide application.[17,18]

The tempered fractional differential equations represent an advancement in fractional calculus[20] and are frequently utilized to model the shift from typical to anomalous diffusion, especially within finite time or limited physical space. This area of calculus is typically employed to describe the anomalous diffusion characteristics of living particles. The foundational definitions of fractional integration featuring exponential kernels and weak singular are attributed to Buschman's earlier research.[40]

This research focuses on finding reliable numerical methods to solve tempered fractional differential equations. These equations usually do not have exact solutions, and finding analytical solutions is often difficult. Limited research has been done on this topic so far. Some methods that have been explored include third-order semi-discretized scheme,[22] theory of fuzzy,[23] linearized technique,[24] the spectral Galerkin method,[25] and second order Exponential Time Differencing Finite Element Method (ETD-RDP-FEM).[26] However, many successful numerical methods used for fractional differential equations in the Caputo sense have not yet been applied to solve these tempered fractional differential equations focusing on tempered fractional differential equations with respect to another function. This includes the symmetric operational matrix method. This paper aims to develop a reliable numerical method that uses a symmetric operational matrix with shifted Legendre polynomials to solve the tempered fractional differential equations. The plan is to convert these equations into a set of algebraic equations while focusing on tempered fractional differential equations with respect to another function.

A square matrix that is identical to its transpose is referred to as a symmetric matrix. The symmetry operational matrix approach has not been effectively modified in recent times to tackle various issues related to fractional operators, including solving the Prabhakar fractional differential equation.[27] While this method enjoys widespread use in tackling various challenges in fractional calculus, its application to FDE based with respect to another functions remains an area for future exploration. So we extend this another function to solve the tempered fractional derivatives. Other literature used solely used single operator says, α, as a fractional operator.[28–30] Our innovative approach introduces another additional function (that is, with respect to another function), φ, thereby demonstrating its effectiveness. This call emphasizes the importance of fractional differential equations over their integer counterparts.

[31] in their paper explained how to obtain matrix elements $(J'K'M'|\lambda A_\alpha|JKM)$. Here, λA_α is the direction cosine between the A-axis (which can be X, Y, or Z) and the α-axis (which can be x, y, or z). These axes are part of a space-fixed reference frame (XYZ) and a body-fixed reference frame (xyz), used to describe the rotation of a rigid symmetrical rotator. The functions $\Psi(J, K, M)$ are the normalized wave functions for the symmetrical rotator, which did not include fractional-order.

The paper in[32] explains how to solve the eigenvalue problem for symmetric matrices using a numerical method. This method simplifies the problem by turning it into matrix-matrix multiplications. For matrices of size n, the number of multiplications is roughly $\log_2 n$. In high-performance parallel computing, it is important to reduce memory references because moving data between memory and registers can greatly affect performance. Also,[33] presents a comprehensive matrix accompanied by various hints to aid in the analysis, including notable features like symmetry. Additionally, it outlines the specific operations that should be executed as part of the inspector routine, ensuring that the process is thorough and effective in achieving accurate results.[34]

Recently, there has been a growing interest in studying FDE. However, due to the involvement of a fractional-order derivative, developing and applying accurate numerical methods and conducting rigorous theoretical analysis for multi-dimensional cases is challenging. While some forms have been proposed for solving equations, many remain unsolved, especially for multi-dimensional cases. The complex nature of fractional operators is a significant challenge in this regard. To address these problems, numerical schemes like the symmetry operational matrix method can be developed. The operational matrix method allows for obtaining analytical expressions for fractional derivatives and reduces computational effort.

We aim to address this research gap with our approach. The symmetry operational matrix method offers several advantages over other existing techniques. Notably, it is straightforward to implement and can be easily programmed using any Computer Algebra System. Additionally, this method demonstrates efficiency in deriving numerical solutions for fractional differential equations, particularly when they involve multiple functions, variable coefficients, or higher-order derivatives.

In this present paper, we aim to build upon an Efficient Symmetry Operational Matrix Method by proposing a new algorithm that integrates an additional function, denoted as φ, into the Tempered Fractional Differential Equations.

This outline is designed as follows: In Sect. 2, we discuss ideas related to tempered fractional calculus, focusing on the Caputo form its derivative and provide definitions to support our points. Sect. 3 looks at the Symmetry Operational Matrix of SLP used for Tempered Fractional Derivatives with an Additional Function. In this section, we explain how the shifted Legendre functions can be represented as polynomials, as described in subsection 3.1. We also present the symmetry operational matrix linked to the SLP in subsection 3.2. Sect. 4 talks about the specific benefits of the additional function, φ, when using the Shifted Legendre Operational Matrix scheme. The numerical application of tempered fractional differential equations with another function is covered in Section 5 while Sect. 6 give the summary of the whole study in the form of conclusion.

2 Some Ideas Related to Tempered Fractional Calculus

Definition 2.1. [37]

For values of α, and for complex numbers $\alpha, \varphi \in \mathbb{C}$ with the conditions $\text{Re}(\alpha) > 0$ and $\text{Re}(\varphi) \geq 0$, the tempered fractional integral (TFI) of order (α, φ) for a function g that belongs to the space $L^1[a,b]$ is defined in the following manner.

$$\begin{aligned} {}_0^T I_\tau^{(\alpha,\varphi)} g(\tau) &= \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-\nu)^{\alpha-1} e^{-\varphi(\tau-\nu)} g(\nu)\, d\nu, \;\; \tau \in [a,b], \\ &= \frac{e^{-\varphi\tau}}{\Gamma(\alpha)} \int_0^\tau (\tau-\nu)^{\alpha-1} e^{\varphi\nu} g(\nu)\, d\nu. \end{aligned} \tag{1}$$

Definition 2.2. [37]

For complex numbers α and φ belonging to $\mathbb{C}$, given that $\text{Re}(\alpha) > 0$ and $\text{Re}(\varphi) \geq 0$, we can establish the Riemann-Liouville TFD of order (α, φ) for a function $g \in L^1[a,b]$ in the following way.

$$\begin{aligned} {}_0^{RT} D_\tau^{(\alpha,\varphi)} g(\tau) &= e^{-\varphi\tau} \left({}_0^T D_\tau^{(\alpha)} e^{\varphi\tau} g(\tau) \right) \\ &= \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \frac{d^n}{d\tau^n} \int_0^\tau (\tau-\nu)^{n-\alpha-1} e^{\varphi(\nu)} g(\nu)\, d\nu, \;\; \tau \in [a,b], \end{aligned} \tag{2}$$

where $n = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ is integer part of α.

In the context of the Caputo form of the tempered fractional derivative, we start by differentiating the function defined in the integral as shown in equation (2). This approach brings us to the formal definition described in Definition 2.3. This definition provides insights into the features and properties linked to this particular derivative, deepening our comprehension of its mathematical and practical uses across different domains.

Definition 2.3. [38]

For complex numbers α and φ in $\mathbb{C}$, with the condition $Re(\alpha) > 0$ and $Re(\varphi) \geq 0$, we define the Caputo time-fractional derivative (TFD) of order (α, φ) for a function f in $L^1[a,b]$. This definition generalizes classical differentiation to fractional orders while considering the integrability of f over the interval $[a,b]$.

$${}_0^{CT} D_\tau^{(\alpha,\varphi)} g(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau-\nu)^{n-\alpha-1} \frac{d^n e^{\varphi(\nu)} g(\nu)}{d\nu^n}\, d\nu, \;\; \tau \in [a,b], \tag{3}$$

To clarify, n is defined as $n = \lfloor \alpha \rfloor + 1$, where $\lfloor \alpha \rfloor$ represents the integer part of α. This definition establishes a clear relationship between n and α.

We now turn our attention to the tempered fractional derivative. By definition, the tempered fractional derivative ${}_0^T D_\tau^{(\alpha,\varphi)} g(\tau)$ is obtained by repeatedly applying the proportional derivative operator $\left(\frac{d}{d\tau}+\varphi\right)^n$ times to the tempered fractional integral $\left({}_0^T I_\tau^{(n-\alpha,\varphi)} g(\tau)\right)$, where n is a natural number. We will proceed by induction on n, beginning with the following observations:

Therefore, the relationship between the tempered fractional integral and derivative is established as follows:

$$ {}_0^T D_\tau^{(\alpha,\varphi)} g(\tau) = \left(\frac{d}{d\tau}+\varphi\right)^n \left({}_0^T I_\tau^{(n-\alpha,\varphi)} g(\tau)\right), \tag{4} $$

In this context, let n be defined as $n = \lfloor Re(\alpha) \rfloor + 1$.

The tempered fractional derivative of the function τ^p, with p being a positive integer, is represented in the following manner:

$$ \begin{aligned} {}_0^T D_\tau^{(\alpha,\varphi)} \tau^p &= \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \int_0^\tau (\tau-\nu)^{n-\alpha-1} \frac{d^n}{d\nu^n}\left(e^{\varphi\nu}\nu^p\right) d\nu, \\ &= \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \int_0^\tau (\tau-\nu)^{n-\alpha-1} \frac{d^n}{d\nu^n} \sum_{k=0}^{\infty} \frac{\varphi^k \nu^{k+p}}{k!} d\nu \\ &= \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \int_0^\tau (\tau-\nu)^{n-\alpha-1} \sum_{k=0}^{\infty} \frac{\varphi^k}{k!} \frac{\Gamma(k+p+1)\nu^{k+p-n}}{\Gamma(k+p+1-n)} d\nu \\ &= \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{\varphi^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1-n)} \int_0^\tau (\tau-\nu)^{n-\alpha-1} \nu^{k+p-n} d\nu. \end{aligned} \tag{5} $$

By applying the integration formula

$$ \int_0^\tau (\tau-t)^{a-1} t^{b-1}\, dt = B(a,b)\tau^{a+b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\tau^{a+b-1}, $$

where $B(a,b)$ denotes the beta function and $a, b > 0$, we derive the following result.

So that,

$$ {}_0^T D_\tau^{(\alpha,\varphi)} \tau^p = \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{\varphi^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1-n)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tau^{a+b-1}. \tag{6} $$

We simply to get

$$ \begin{aligned} &{}_0^T D_\tau^{(\alpha,\varphi)} \tau^p \\ &= \frac{e^{-\varphi\tau}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{\varphi^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p-n+1)} \frac{\Gamma(n-\alpha)\Gamma(k+p-n+1)}{\Gamma(n-\alpha+k+p-n+1)} \tau^{n-\alpha+k+p-n} \\ &= e^{-\varphi\tau} \sum_{k=0}^{\infty} \frac{\varphi^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p-\alpha+1)} \tau^{k+p-\alpha}. \end{aligned} \tag{7} $$

For the function $g(\tau) = (\tau-a)^p$ with $\mathrm{Re}(p) > -1$, it is established that,[39]

$$ {}_a^T D_\tau^{(\alpha,\varphi)} (\tau-a)^k = (\tau-a)^{k-\alpha} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\, {}_1F_1(-\alpha; k-\alpha+1; -\varphi(\tau-a)). \tag{8} $$

We treat k as a positive integer. When a is equal to 0, the results from both expressions in (7) and (8) are the same.

In the process of obtaining the symmetry operational matrix from (8), we can confidently depend on the subsequent results pertaining to the hypergeometric function, ${}_1F_1, {}_2F_2$.

$$\begin{aligned}
{}_1F_1(a;c;\tau) &= \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)} \frac{\Gamma(c)}{\Gamma(c+k)} \frac{\tau^k}{k!}, \\
{}_2F_2(a,b;c,d;\tau) &= \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)} \frac{\Gamma(b+k)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+k)} \frac{\Gamma(d)}{\Gamma(d+k)} \frac{\tau^k}{k!}, \\
\int_0^1 {}_1F_1(a;c;\tau)d\tau &= \frac{c-1}{a-1}\, {}_1F_1(a-1;c-1;\tau), \\
\int_0^1 \tau^l\, {}_1F_1(a;c;m\tau)d\tau &= \frac{1}{l+1}\, {}_2F_2(a,l+1;c,l+2;m).
\end{aligned} \tag{9}$$

In another aspect, we can construct a solution for the single tempered fractional differential equation given by ${}^{T}D_x^{\alpha,\varphi} y(x) = \kappa y(x) + h(x)$. Here, α is in the range $(0,1]$ and $\varphi > 0$, alongside the initial condition $y(0) = y_0$. The solution can be effectively articulated as follows.[40]

$$y(\tau) = y_0 e^{-\varphi\tau} E_{\alpha,1}(\kappa\tau^{\alpha}) + \int_0^{\tau} h(\tau-s) e^{-\varphi s} s^{\alpha-1} E_{\alpha,\alpha}(\kappa s^{\alpha}) ds. \tag{10}$$

The expression $E_{a,b}(\tau)$ is defined as follows:

$$E_{a,b}(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(a\tau+b)}. \tag{11}$$

However, this approach may prove inadequate in the context of multi-order tempered fractional differential equations. To address this challenge, we propose employing a collocation scheme that utilizes a shifted Legendre operational matrix for solving these equations. Furthermore, alternative polynomial types can also be considered to derive a new operational matrix, which may effectively handle the complexities associated with tempered fractional derivatives.

3 symmetry Operational Matrix of Shifted Legendre for Tempered Fractional Derivatives with an Additional Function.

In this section, we examine the polynomial representation of the shifted Legendre functions as outlined in subsection 3.1, as well as the symmetry operational matrix associated with the SLP presented in subsection 3.2.

3.1 Shifted Legendre Polynomials (SLP)

Definition 3.1. The SLP definition is given below:[43]

$$\widetilde{L_b}\ (t) = L_b(2t-1). \tag{12}$$

The orthogonality of the polynomials $\widetilde{L_b}\ (t)$ on the interval $[0,1]$ can be inferred from the observation that the transformation $t \mapsto 2t-1$ is an affine mapping, which bijectively transforms the interval $[0,1]$ into the interval $[-1,1]$.

Then,

$$\int_0^1 \widetilde{L_b}\ (t)\widetilde{L_b}\ (t)dt = \frac{1}{2b+1}\partial ab. \tag{13}$$

where where ∂ab denotes the Kronecker delta, equal to 1 if $a = b$ and to 0 otherwise

Definition 3.2. The following is a detailed expression for the SLP, as referenced in source:[43]

$$\widetilde{L_b}\ (t) = (-1)^b \sum_{n=0}^{b} \binom{k}{n}\binom{k+1}{n}(-t)^n. \tag{14}$$

Legendre polynomials (LP) are special types of polynomials that are defined on the interval from $[-1, 1]$. One way to derive these polynomials is by using a recurrence relation, which is explained below:

$$L_{i+1}(t) = \frac{2i+1}{i+1} t L_i(t) - \frac{i}{i+1} L_{i-1}(t), \ \ i = 1, 2, \cdots \tag{15}$$

The LP are established with $L_1(t) = t$ and $L_0(t) = 1$. To adapt the given polynomials from the interval $[-1, 1]$ to the interval $[0, 1]$, a transformation utilizing the substitution $t = 2\tau - 1$ is applied. This process yields the SLPs, represented as $\tilde{L}_i(\tau)$.

$$\tilde{L}_{i+1}(\tau) = \frac{(2i+1)(2\tau-1)}{i+1} \tilde{L}_i(\tau) - \frac{i}{i+1} \tilde{L}_{i-1}(\tau), \ \ i \geq 0, \tag{16}$$

The SLPs are defined as follows: $\tilde{L}_1(\tau) = 2\tau - 1$ and $\tilde{L}_0(\tau) = 1$. You can find the ith degree polynomials $\tilde{L}_i(\tau)$ using a specific formula.

$$\tilde{L}_i(\tau) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!\tau^k}{(i-k)!(k!)^2}, \qquad i = 0, 1, 2, \cdots, \tag{17}$$

The following conditions apply: $\tilde{L}_i(0) = (-1)^i$ and $\tilde{L}_i(1) = 1$. The condition of orthogonality is established as follows.

$$\int_0^1 \tilde{L}_i(\tau)\tilde{L}_j(\tau)d\tau = \begin{cases} \frac{1}{2i+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{18}$$

The SLPs have a special feature which is useful for approximating functions. For a function $f(\tau)$ that can be squared and integrated over the interval from 0 to 1, (that is, $f(\tau) \in L^2[0, 1]$) we can express it systematically using SLPs.

$$g(\tau) = \sum_{j=0}^{\infty} c_j \tilde{L}_j(\tau), \tag{19}$$

The coefficients c_j are defined as follows:

$$c_j = (2j+1) \int_0^1 g(\tau)\tilde{L}_j(\tau)d\tau, \quad j = 0, 1, 2, \cdots. \tag{20}$$

To approximate the function using equation (19), we typically truncate after $(N + 1)$ terms of the SLPs as shown:

$$g_N^*(\tau) = \sum_{j=0}^{N} c_j \tilde{L}_j(\tau) = \mathbf{C}^T \mathbf{\Phi}_L(\tau), \tag{21}$$

In this context, the coefficient vector of the shifted Legendre, denoted as $\mathbf{C}$, is defined by $\mathbf{C}^T = [c_0, c_1, ..., c_N]$. Furthermore, the shifted Legendre vector, represented as $\mathbf{\Phi}_L(\tau)$, can be articulated in the following manner.

$$\mathbf{\Phi}_L(\tau) = [\tilde{L}_0(\tau), \tilde{L}_1(\tau), \cdots, \tilde{L}_N(\tau)]^T.$$

3.2 Symmetry Operational Matrix of Shifted Legendre Polynomials

In this subsection, the study present derivation process of a novel shifted Legendre symmetry operational (SLSO) matrix designed to address tempered fractional derivatives. The following theorem outlines our findings.

Theorem 3.3. *Let $\mathbf{\Phi}_{\tilde{L}}(\tau)$ be defined as the shifted Legendre vector, as indicated in equation (21). For the range $n-1<\alpha<n$, where ($n\in\mathbb{N}$), the following holds:*

$$ {}_0^T D_\tau^{(\alpha,\varphi)}\mathbf{\Phi}_{\tilde{L}}(\tau)=\mathbf{P}_{\tau;\tilde{L}}^{\alpha,\varphi}\mathbf{\Phi}(\tau), \tag{22}$$

where the matrix $\mathbf{P}_\tau^{\alpha,\varphi}$ represents an $N\times N$ symmetry operational matrix of TFD with order α,φ.

This matrix is defined as follows:

$$\mathbf{P}_\tau^{\alpha,\varphi}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum\limits_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\xi_{\lceil\alpha\rceil,0,k} & \sum\limits_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\xi_{\lceil\alpha\rceil,1,k} & \cdots & \sum\limits_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\xi_{\lceil\alpha\rceil,N-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{k=\lceil\alpha\rceil}^{i}\xi_{i,0,k} & \sum\limits_{k=\lceil\alpha\rceil}^{i}\xi_{i,1,k} & \cdots & \sum\limits_{k=\lceil\alpha\rceil}^{i}\xi_{i,N-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{k=\lceil\alpha\rceil}^{N-1}\xi_{N-1,0,k} & \sum\limits_{k=\lceil\alpha\rceil}^{N-1}\xi_{N-1,1,k} & \cdots & \sum\limits_{k=\lceil\alpha\rceil}^{N-1}\xi_{N-1,N-1,k} \end{bmatrix}, \tag{23}$$

with the variable $\xi_{i,j,k}$ defined as follows:

$$\begin{aligned}\xi_{i,j,k}=(2j+1)&\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!(k-\alpha)!}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2(1+k-\alpha+l)}\\ &\times {}_2\mathrm{F}_2(-\alpha,1+k-\alpha+l;\,k-\alpha+1,2+k-\alpha+l;\,-\varphi),\end{aligned} \tag{24}$$

for $i=\lceil\alpha\rceil,\cdots,N-1,\quad i=0,1,2,3,4,\cdots,N-1$. In this context, the variable i is defined as $\lceil\alpha\rceil,\cdots,N-1$ and the variable i takes on values from $0,1,2,\cdots,N-1$.

Proof. Utilizing equation (8) and setting $a=0$, we can derive the TFD for the expression τ^k as demonstrated below:

$$ {}_0^T D_\tau^{(\alpha,\varphi)}\tau^k=\tau^{k-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\,{}_1F_1(-\alpha;k-\alpha+1;-\varphi\tau). \tag{25}$$

Utilizing the expression presented in (25), we have derived the explicit formula for the TFD of the i-th degree shifted Legendre polynomial. This corresponds to the $(i+1)$-th element of the vector $\mathbf{\Phi}_L(\tau)$.

$$\begin{aligned} &{}_0^T D_\tau^{(\alpha,\varphi)}\tilde{L}_i(\tau)\\ &=\sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!^2}\left({}_0^T D_\tau^{(\alpha,\varphi)}x^k\right),\quad i=0,1,2,\cdots\\ &=\sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!^2}\left[\tau^{k-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\,{}_1F_1(-\alpha;k-\alpha+1;-\varphi\tau)\right]\\ &=\sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!}\left[\frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)}\,{}_1F_1(-\alpha;k-\alpha+1;-\varphi\tau)\right].\end{aligned} \tag{26}$$

The elements $\rho_{i,j}$ in the symmetry operational matrix $\mathbf{P}^{\alpha,\varphi}_{\tau;\tilde{L}}$ are calculated by taking the inner product of the TFD of SLPs, denoted as ${}_0^T D_\tau^{(\alpha,\varphi)} \tilde{L}_i(\tau)$, with the SLPs, $\tilde{L}_j(\tau)$. This is done for values of $i = 0, 1, 2, 3, \cdots, K-1$.

$$\begin{aligned}
{}_0^T D_\tau^{(\alpha,\varphi)} \tilde{L}_i(\tau) &= \sum_{j=0}^{N-1} \rho_{i,j} \tilde{L}_j(\tau) \\
\rho_{i,j} &= \left\langle {}_0^T D_\tau^{(\alpha,\varphi)} \tilde{L}_i(\tau), \tilde{L}_j(\tau) \right\rangle \\
&= \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!k!} \left[\frac{1}{\Gamma(k-\alpha+1)} \left\langle \tau^{k-\alpha} \, {}_1F_1(-\alpha; k-\alpha+1; -\varphi\tau), \tilde{L}_j(\tau) \right\rangle \right],
\end{aligned} \tag{27}$$

where the inner product can be calculated using the following method,

$$\begin{aligned}
&\left\langle \tau^{k-\alpha} \, {}_1F_1(-\alpha; k-\alpha+1; -\varphi\tau), \tilde{L}_j(\tau) \right\rangle \\
&= (2j+1) \int_0^1 \tau^{k-\alpha} \, {}_1F_1(-\alpha; k-\alpha+1; -\varphi\tau) \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2} \tau^l d\tau \\
&= (2j+1) \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2} \int_0^1 \tau^{k-\alpha} \, {}_1F_1(-\alpha; k-\alpha+1; -\varphi\tau) \tau^l d\tau \\
&= (2j+1) \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2} \left(\frac{{}_2\mathrm{F}_2(-\alpha, 1+k-\alpha+l;\, k-\alpha+1, 2+k-\alpha+l;\, -\varphi)}{1+k-\alpha+l} \right).
\end{aligned} \tag{28}$$

The integration presented in equation (28) can be achieved using the formula outlined in (9). By substituting equation (28) into (27), we arrive at the following result:

$$\begin{aligned}
\rho_{i,j} &= \left\langle {}_0^T D_\tau^{(\alpha,\varphi)} \tilde{L}_i(\tau), \tilde{L}_j(\tau) \right\rangle \\
&= \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!k!} \left[\frac{(2j+1)}{\Gamma(k-\alpha+1)} \right. \\
&\left. \times \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2} \left(\frac{{}_2\mathrm{F}_2(-\alpha, 1+k-\alpha+l;\, k-\alpha+1, 2+k-\alpha+l;\, -\varphi)}{1+k-\alpha+l} \right) \right] \\
&= (2j+1) \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!k!(k-\alpha)!} \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2(1+k-\alpha+l)} \\
&\times {}_2\mathrm{F}_2(-\alpha, 1+k-\alpha+l;\, k-\alpha+1, 2+k-\alpha+l;\, -\varphi).
\end{aligned} \tag{29}$$

Letting $\rho_{i,j} = \sum_{k=\lceil\alpha\rceil}^{i} \tau_{i,j,k}$

$$\begin{aligned}
\tau_{i,j,k} = (2j+1) \frac{(-1)^{i+k}(i+k)!}{(i-k)!k!(k-\alpha)!} \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2(1+k-\alpha+l)} \\
\times {}_2\mathrm{F}_2(-\alpha, 1+k-\alpha+l;\, k-\alpha+1, 2+k-\alpha+l;\, -\varphi).
\end{aligned}$$

Consequently, the elements of the SLO matrix for the tempered fractional derivatives have been derived. □

4 The Tempered Fractional Differential Equations (TFDE) analyzed through the Shifted Legendre Symmetry Operational (SLSO) Matrix Technique involving another Function φ

In this section, we will examine the specific benefits of the additional function (φ), in the context of the SLO Matrix process, and we will numerically test the proposed method. To develop a new algorithm that includes a different function denoted as φ, we applied the a scheme to analyze the behaviour of the fractional-order Tempered Fractional Differential Equation.

$$\sum_{r=1}^{l} q_r{}^T D_{\tau}^{(\alpha_r,\varphi_r)} f(\tau) = h(\tau)$$
$$f^{(i)}(0) = d_i, \quad i = 0, 1, \cdots, m-1. \tag{30}$$

We outline the comprehensive steps for addressing multi-order TFDE, as exemplified in equation (30), utilizing the shifted Legendre symmetry operational matrix with another function φ

The solution, called $f(\tau)$, is estimated using simplified SLPs,

$$f(\tau) \approx f_N(\tau) \approx \mathbf{C}^T \tilde{\mathbf{L}}(\tau), \tag{31}$$

The vector $\mathbf{C}$ is written as $\mathbf{C}^T = [c_0, c_1, c_2, \ldots, c_N]$. We use the SLO matrix to estimate the tempered fractional derivatives, as described in (22) through (24) and outlined in equation (30).

$${}^T D_{\tau}^{(\alpha,\varphi)} f(\tau) \approx \mathbf{C}^T \mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)} \mathbf{\Phi}_{\tilde{L}}(\tau). \tag{32}$$

Remark:

The function $h(\tau)$ presented on the right side of (30) may also be approximated using truncated SLPs, as detailed below:

$$h(\tau) \approx \mathbf{H}^T \mathbf{\Phi}_{\tilde{L}}(\tau), \tag{33}$$

In this context, let $\mathbf{H} = [h_i]^T$. The coefficient h_i are calculated via (20). To improve the efficiency and accuracy of the scheme and make the calculations more efficient, we can apply a collocation scheme directly to these functions.

We obtained that:

$$\sum_{r=1}^{l} q_r \mathbf{\Phi}_{\tilde{L}}^T(\tau)(\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})^T \mathbf{C} = h(\tau). \tag{34}$$

After conducting a series of algebraic manipulations, we arrive at the equation $\mathbf{\Phi}_{\tilde{L}}^T(\tau) \left(\sum_{r=1}^{l} q_r (\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})^T \mathbf{C} \right) - h(\tau) = 0.$. Consequently, we define the residual as follows:

$$\mathcal{R}(\tau) = \mathbf{\Phi}_{\tilde{L}}^T(\tau) \left(\sum_{r=1}^{l} q_r (\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})^T \mathbf{C} \right) - h(\tau) = 0. \tag{35}$$

The analysis is based on the application of a set of SLPs as the foundational basis.

$\mathbf{\Phi}_{\tilde{L}}^T(\tau) = [\tilde{L}_0(\tau) \;\; \tilde{L}_1(\tau) \;\; \cdots \;\; \tilde{L}_N(\tau)]$ is linearly independent, we obtain

$$\mathbf{\Phi}_{\tilde{L}}^T(\tau) \sum_{r=1}^{l} q_r (\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})^T \mathbf{C} = h(\tau). \tag{36}$$

We create a set of $K + 1$ algebraic equations based on equation (36). We also estimate the initial condition described in equation (30) using SLPs.

$$f^{(i)}(0) = d_i, \; i = 0, \cdots, m-1$$
$$\mathbf{\Phi}_{\tilde{L}}^T(0)(\mathbf{P}_{0+;\tilde{L}}^{i})^T \mathbf{C} = d_i. \tag{37}$$

By considering $(K - m)$ equations from (36) and combining them with the initial conditions from (37) will produce a system of N linear equations for $\mathbf{C}$. You can use any numerical method you prefer to solve this system. After that, use the following equation to find the approximate solution.

$$f^*(\tau) = \mathbf{C}^T \mathbf{\Phi}_{\tilde{L}}(\tau). \tag{38}$$

4.1 Error Analysis involving another Function, φ.

In our analysis of the error associated with our method, we adopt the approach outlined in.[42] This process uses alternating LP to create an symmetry operational matrix. This matrix helps solve fractional differential equations based on the classical Caputo definition.

Lemma 4.1. *Suppose that $f(x) \in C^{n+1}(\Theta)$ and $f(x) \approx f_N(x) \approx C_T \tilde{L}(x)$ represents its expansion in terms of shifted Legendre polynomials, Then*

$$\|f(\tau) - f_N(\tau)\|_2 \leq \frac{M}{(N+1)!2^{2N+1}},$$

with M being a constant satisfying the condition $|f^{(N+1)}(\tau)| \leq M$.

Proof:
Consider the polynomial $s_n(\tau)$, which serves as the interpolating polynomial for a function $f(\tau)$ at the nodes τ_j, defined as the solution of shifted Chebyshev polynomials of degree $n+1$. We have the following expression for the error in interpolation:

$$f(\tau) - s_n(\tau) = \frac{f^{(n+1)}(\varsigma_\tau)}{(n+1)!} \prod_{j=0}^{n} (\tau - \tau_j),\ \varsigma_\tau \in \Theta = [0, 1].$$

According to the proof provided in,[42] this approximation yields a bounded error defined by the inequality:

$$|f(\tau) - s_n(\tau)| \leq \frac{M}{(n+1)!2^{2n+1}},\ \forall \tau \in \Theta.$$

This establishes a quantifiable measure of the accuracy of the interpolation method employed.

$$\begin{aligned}
\|f(\tau) - f_N(\tau)\|_2^2 &\leq \|f(\tau) - s_N(\tau)\|_2^2 \\
&= \int_0^1 \|f(\tau) - s_N(\tau)\|^2 d\tau \\
&= \int_0^1 \left(\frac{M}{(N+1)!2^{2N+1}}\right)^2 d\tau \\
&= \left(\frac{M}{(N+1)!2^{2N+1}}\right)^2.
\end{aligned}$$

However, taking the square root of both sides yields the inequality

$$\|f(\tau) - f_N(\tau)\|_2 \leq \frac{M}{(N+1)!2^{2N+1}}.$$

This concludes the proof.

To estimate errors in the new numerical scheme, we will use a method called residual correction.. Referring to equation (36), we define the residual as follows: $\mathcal{R}(\tau) = \mathbf{\Phi}_{\tilde{L}}^T(\tau) \left(\sum_{r=1}^{l} q_r (\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})^T \mathbf{C} \right) - h(\tau) = 0$, hence

$$\mathbf{\Phi}_{\tilde{L}}^T(\tau) \left(\sum_{r=1}^{l} q_r (\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})^T \mathbf{C} \right) - h(\tau) = 0. \quad (39)$$

As N approaches infinity, we can utilize the symmetry operational matrix based on SLPs in approximating the tempered fractional derivatives with another function, φ. Additionally, we can express $h(\tau)$ through the framework of SLPs. This methodology offers a robust approach to our analysis as:

$$|\mathbf{\Phi}_{\tilde{L}}^{T}(\tau)\sum_{r=1}^{l} q_r(\mathbf{P}_{\tau;\tilde{L}}^{(\alpha_r,\varphi_r)})_{(N\times N)}^{T} - \sum_{r=1}^{l} {q_r}^{T} D_{\tau}^{(\alpha_r,\varphi_r)} f(\tau)| \approx 0,\ N \to \infty. \tag{40}$$

In our proposed methodology, we consider N to be finite. Consequently, when utilizing m terms of the SLP, it is acknowledged that a small error, denoted as e_m, may be unavoidable.

$$\sum_{r=1}^{l} |D_{\tau}^{(\alpha_r,\varphi_r)} f(\tau) - \mathbf{P}_{\tau;L}^{(\alpha_r,\varphi_r)})|_2 = e_m, N = m. \tag{41}$$

Let e_m^* be the approximate solution obtained using the SLO matrix method. If the condition $||e_m - e_m^*|| < \epsilon$ holds true, it means the difference is small enough. In this case, we can accurately estimate the absolute errors e_m using e_m^*. As a result, we can determine the best value for m (that is, K).

Lemma 4.2. *Let $f_N^*(\tau)$ and $f_{N+1}^*(\tau)$ represent the numerical solutions for the unknown function $f(\tau)$ as derived from definition 1, corresponding to orders k and $k+1$, respectively. The error estimation, defined as $\chi_N = \|f_N^*(\tau) - f_{N+1}^*(\tau)\|_2$, demonstrates convergent behavior.*

Proof.

$$\begin{aligned}\chi_N &= ||f_N^*(\tau) - f_{N+1}^*(\tau)||_2 \\ &= ||f_N^*(\tau) - f(\tau) + f(\tau) - f_{N+1}^*(\tau)||_2 \\ &\le ||f_N^*(\tau) - f(\tau)||_2 + ||f(\tau) - f_{N+1}^*(\tau)||_2 \\ &= \psi_N + \psi_{N+1}.\end{aligned} \tag{42}$$

The sequences χ_N and χ_{N+1} are both limited by the upper bound established in the Lemma 4.1. Because of this constraint, we can conclude that each of these sequences is convergent. Consequently, it follows that the sequence χ_N exhibits convergence as well. This result is significant because it confirms the stability of the sequence under the given conditions. □

5 Numerical Application of Tempered Fractional Differential Equations with Respect to Another Function

This part of the study will explore the specific advantages of the additional function, denoted as φ, in relation to Tempered Fractional Differential Equations. We will also conduct numerical tests of the proposed method. To develop a new algorithm that incorporates this function φ, we applied the proposed method to analyze the behavior of Tempered Fractional Differential Equations (TFDE).

Example 5.1. Consider the following TFDE with another function φ from.[44]

$${}_{0}^{T}D_{\tau}^{(\alpha,\varphi)} y(\tau) = e^{-\varphi\tau}\left(\frac{\Gamma(6)}{\Gamma(6-\alpha)}\tau^{5-\alpha} - e^{-\varphi\tau}\tau^{10} + e^{\varphi\tau}y^2(\tau)\right),\ \ 0<\alpha<1, \tag{43}$$

Given the initial condition that the derivative of $y(0)$ with respect to τ equals 0, we find the exact solution to be $y(\tau) = e^{-\varphi\tau}\tau^5$. You can confirm this by following the steps outlined in equations (5) and (7).

$${}_{0}^{T}D_{\tau}^{(\alpha,\varphi)} e^{-\varphi\tau}\tau^{\mu} = \frac{\mu+1}{\Gamma(\mu-\alpha+1)} e^{-\varphi\tau}\tau^{p-\alpha},\ \alpha > -1. \tag{44}$$

Solution: To check how accurate the symmetry operational matrix from Theorem 3.3 is, we use it to estimate the TFD for example 5.1.

For

$$G(i,\tau) = \sum_{\mu=0}^{i} \frac{(-1)^{i+\mu}\,(i+\mu)!\,\tau^{\mu}}{(i-\mu)!\,(\mu!)^2} \tag{45}$$

So that,

$$GM(4,\tau) = \begin{bmatrix} 1 \\ -1+2\tau \\ 6\tau^2-6\tau+1 \\ 20\tau^3-30\tau^2+12\tau-1 \end{bmatrix} \tag{46}$$

We derive the operational matrix $\mathbf{P}_x^{\alpha,\varphi}$ for the parameters $\alpha = \frac{1}{2}$ and $\varphi = \frac{1}{2}$.

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 1.64986529929430 & 1.08772094320343 & -0.182838844891694 & 0.0953947499138880 \\ 1.08772094320345 & 1.79613637520739 & 1.59535401363030 & -0.376813850426124 \\ 1.46702645440274 & -0.5076330704248 & 2.16568030629547 & 2.05540113836993 \end{bmatrix}. \tag{47}$$

In Figure 1, we evaluate the accuracy of the approximation by comparing the exact solution of the tempered fractional derivative for two cases: when $N = 6$ and $N = 8$. This comparison is made against another function, denoted as φ. The figure also showcases the approximation obtained through the symmetry operational matrix method, as outlined in equation (47). By analyzing these results, we can gain insights into how the differences in N impact the precision of the approximation and its effectiveness in capturing the behavior of the fractional derivative relative to the chosen function φ. Figure 1 shows the comparison of the solutions for $N = 4$ and $N = 6$, respectively.

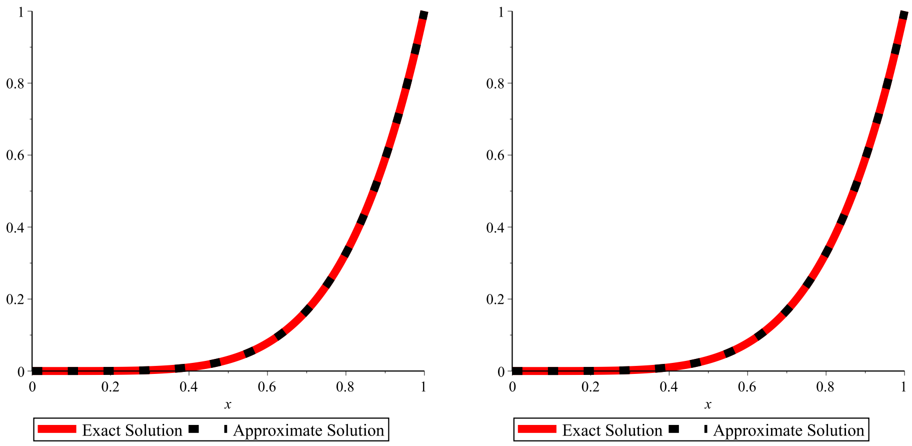

Figure 1: Comparing the results of approximate and exact solutions for Example 5.1

Table 1 provides a detailed comparison between the exact and approximate solutions, highlighting the associated errors for each case. Upon conducting a thorough analysis alongside various established methods, it becomes evident that the new method not only excels in accuracy but also achieves a significant reduction in computational costs. This reduction is particularly notable when compared to the with Jacobi-predictor corrector algorithm methods outlined in,[44] underscoring the efficiency and effectiveness of our approach in solving the problem at hand. Figure 2 shows the comparison of the solutions for $N = 4$ and $N = 6$,

Table 1: The comparing the results for approximate and exact solutions

τ	Exact solutions	Approximate solutions	Error
0.1	0.0000439341	0.0000100000	3.39341E-05
0.2	0.0003276233	0.0003200000	7.62329E-06
0.3	0.0024203799	0.0024300000	9.62008E-06
0.4	0.0102394808	0.0102400000	5.19193E-07
0.5	0.0312699358	0.0312500000	1.99358E-05
0.6	0.0777842563	0.0777600000	2.42563E-05
0.7	0.1680622233	0.1680700000	7.77668E-06
0.8	0.3276106564	0.3276800000	6.93436E-05
0.9	0.5903831815	0.5904900000	1.06818E-04
1.0	1.0000000000	1.0000000000	1.00000E-49

Example 5.2. Consider the tempered fractional differential equation with another function φ as follows:

$$ {}_0^T D_\tau^{(\alpha_1,\varphi_1)} y(\tau) + {}_0^T D_\tau^{(\alpha_2,\varphi_2)} y(\tau) = \frac{16\,\mathrm{e}^{-t/2} t^{3/2}\left(4\sqrt[4]{t}\sqrt{\pi} + 7\,\Gamma\left(3/4\right)\right)}{21\sqrt{\pi}\Gamma\left(3/4\right)}, \tag{48} $$

The parameters are defined as follows: $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$, and $\varphi_1 = \varphi_2 = \frac{1}{2}$, with the initial condition specified as $y(0) = 0$. The exact solution for the equation is expressed as $y(\tau) = 2e^{-\tau/2}\tau^2$.

Solution: By conducting a series of numerical experiments with the values of $N = 6, 8, 10$, we have generated the results summarized in Figure and Table . The findings highlight the relative absolute errors, which are indicative of the accuracy achieved. Specifically, our approach involves utilizing a finite number of terms derived from the SLPs in conjunction with their operational matrix. This methodology has proven to deliver satisfactory results when applied to tempered fractional differential equations, particularly in relation to the additional function incorporated into the analysis. The outcomes suggest that this technique effectively balances complexity and precision in approximating solutions.

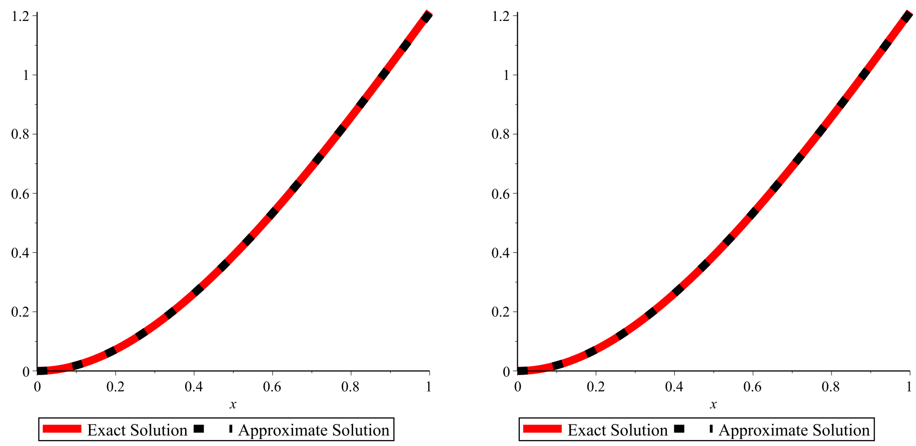

Figure 2: Comparing the results of approximate and exact solutions for Example 5.2

Table 2 provides a detailed comparison between the approximate and exact solutions, focusing specifically on the associated error rates. A thorough analysis, combined with established methodologies in the field, demonstrates that the proposed approach not only significantly reduces computational costs but also offers enhanced efficiency when contrasted with the solutions found in both the approximate and exact frameworks. This reduction in computational expense is critical for applications where resources and time are limited.

Example 5.3. We look at the TFDE with another function as follows:

$$ \begin{aligned} & {}_0^T D_\tau^{(\alpha_1,\varphi)} y(\tau) + {}_0^T D_\tau^{(\alpha_1,\varphi)} y(\tau) \\ & = 2\,\frac{1/2\left(14\sqrt{t} - 5\,t^{3/2} + 3\,t^{5/2} + 1/2\,t^{7/2}\right)\sqrt{\pi} erf\sqrt{t} + t\mathrm{e}^{-t/4}\left(t^2 + 4\,t - 10\right)}{\sqrt{t}\sqrt{\pi}}, \end{aligned} \tag{49} $$

Table 2: The comparing the results for exact solutions and the absolute error for Example 5.2

τ	Exact solutions	Absolute error ($N = 6$)	Absolute error ($N = 8$)
0.1	0.0189253955	9.91930E-05	1.41125E-04
0.2	0.0724998314	1.12838E-04	6.82926E-05
0.3	0.1550926034	1.65168E-04	6.60785E-06
0.4	0.2620077886	1.39477E-05	1.38474E-05
0.5	0.3892448842	1.55507E-04	5.43589E-05
0.6	0.5332594459	1.29673E-04	2.34626E-05
0.7	0.6907237265	1.29399E-04	2.77916E-05
0.8	0.8582873146	2.77656E-04	4.22536E-05
0.9	1.0323377730	6.19833E-04	9.35034E-06
1.0	1.2087612772	4.30004E-03	1.92141E-03

The parameters are defined as follows: $\alpha_1 = \alpha_2 = \frac{1}{2}$, and $\varphi_1 = \frac{1}{2}, \varphi_2 = \frac{1}{4}$, with initial condition $y(0) = 0$. The exact solution is given as $y(\tau) = 2\tau^3 + 4\tau$.

Solution:

Figure 3 presents a thorough comparison between the precise solution derived in Example 5.3 for the tempered fractional derivative at two different values of N, specifically $N = 6$ and $N = 8$. This comparison is made with respect to another predefined function characterized by the parameters $\varphi_1 = \frac{1}{2}$ and $\varphi_2 = \frac{1}{4}$. Additionally, the figure incorporates the approximation results derived from the symmetry operational matrix, as outlined in equation (47). It is noteworthy that the precision of this approximation can be significantly improved by increasing the value of N, which effectively enhances the accuracy of the results and provides a more reliable representation of the underlying function behavior.

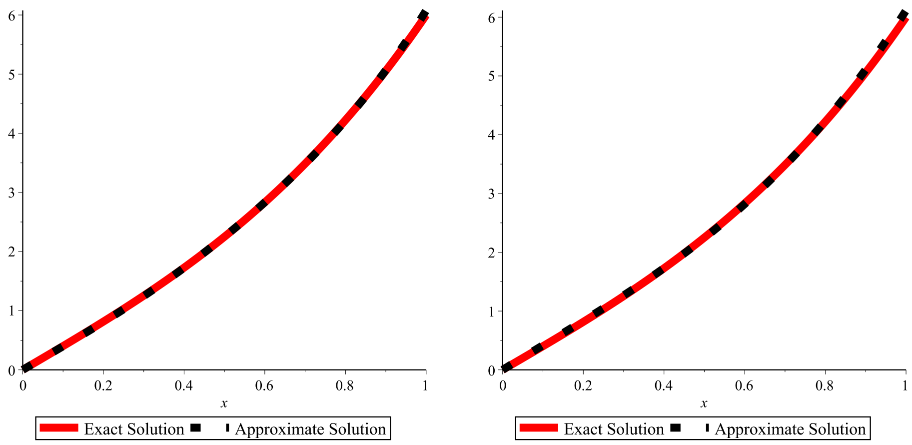

Figure 3: The results of the comparison between the approximate and exact solutions for Example 5.3

Table 3 presents a comprehensive comparison between the approximate and exact solutions, with a focus on the associated error metrics for each method. The analysis, conducted in conjunction with established methodologies, demonstrates that the proposed approach significantly enhances computational efficiency when compared to the existing solutions within both the approximate and exact frameworks.

Utilizing the parameters $N = 6$ and $N = 8$, the numerical results are meticulously outlined in Table 3 and illustrated in Figure 3. By employing a selective number of terms from the SLPs, we have achieved promising results for the tempered fractional differential equation under examination. This methodology exemplifies a judicious balance between computational cost and solution accuracy, laying a solid foundation for further exploration in this field.

Table 3: The comparison between the exact solutions and the absolute error for Example 5.3

τ	Exact solutions	Absolute error ($N = 6$)	Absolute error ($N = 8$)
0.1	0.4181661239	2.94670E-03	1.02401E-02
0.2	0.8347809897	6.98366E-03	2.80772E-02
0.3	1.2665442475	6.24524E-03	4.37026E-02
0.4	1.2665442475	1.77259E-03	4.95866E-02
0.5	2.2423145396	2.08354E-03	4.04925E-02
0.6	2.8197208743	1.25992E-03	1.34647E-02
0.7	3.4790742016	4.42193E-03	3.22097E-02
0.8	4.2370741717	7.65861E-03	9.50645E-02
0.9	5.1104204347	9.93234E-03	1.71547E-01
1.0	6.1158126407	8.14096E-02	12.56136E-01

6 Conclusion

In our study, we examine the specific benefits of the additional function, φ, in the context of SLPs. We have built upon and expanded the research utilizing symmetry operational matrix methods as outlined in.[44] Our work involves employing a new numerical algorithm to analyze the behavior of Tempered Fractional Differential Equations. Unlike previous studies that focused solely on the parameter α, our novel approach incorporates both α and φ. We compared our results with those obtained using the Jacobi-predictor corrector algorithm approach, which was previously simulated in the study referenced in.[44] Additionally, we provide a detailed comparison between the approximate and exact solutions, with a specific focus on the associated error rates. Our thorough analysis, combined with established methodologies in the field, demonstrates that our proposed approach significantly reduces computational costs while enhancing efficiency compared to solutions found within both the approximate and exact frameworks. This reduction in computational expense is crucial for applications where resources and time are constrained. However, future work could focus on extending the SLPs technique to address the Tempered fractional problem more comprehensively. This could involve investigating the cases where the method has already been successfully applied and using this information to develop a more refined approach. Additionally, researchers could explore other ways to enhance its effectiveness in conjunction with the SLPs technique to further enhance its effectiveness. By pursuing these avenues of research, it may be possible to develop a more robust and comprehensive solution for the Tempered fractional problem. This, in turn, could have significant implications for various applications in fields such as physics, engineering, and finance.

Data availability

Not applicable.

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Conflicts of interest

The authors declare that they have no conflict of interest.

Author contributions

Conceptualization, M.A.A; methodology, M.A.A. and A.A; software, A.A validation, M.A.A. and A.A; formal analysis, M.A.A. and A.A; investigation, M.A.A. and A.A; resources, M.A.A. and A.A; writing—original draft preparation, M.A.A. and A.A; writing—review and editing, M.A.A. and A.A; visualization, A.A.; supervision, M.A.A.; funding acquisition, M.A.A. and A.A. All authors contributed equally and agreed with the publication of this version of the paper.

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