A Generalized Directed Divergence of Fuzzy Entropy

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Abstract

In the present paper, we introduced a new generalized parametric measure of fuzzy directed divergence of order σ with the proof of its validity. The particular case and some elegant properties of fuzzy directed divergence measure are studied. Total ambiguity , fuzzy information improvement measure and reduction in improvement measure are given for the proposed measure. A comparative study of proposed measure with existing generalized fuzzy directed divergence measure is computed numerically and represented by using graphical representation. The application of proposed fuzzy directed divergence measure in multi criteria decision making problem is demonstrated by using numerical example.

Keywords: Entropy; Fuzzy Sets; Fuzzy Entropy; Directed Divergence; Fuzzy Directed Divergence measure; Multi Criteria Decision Making Problem

1 Introduction

In communication theory, the quantity of information is defined as, the amount of information transfered in an event , that depends on the probability of an event . Originally, the concept was given by Boltzmann [5], as a statistical measure for molecular disorder of a gas contained in the closed container. Shannon,[19] introduced the measure called entropy as, the average uncertainty in bits associated with the prediction of outcome in random experiment, which is useful in theory of communication. In real world applications the term uncertainty is not only restricted to the randomness alone but it is also due to the vagueness, ambiguity or a lack of information. This kind of uncertainty is known as fuzziness. To represent fuzziness through partial membership grade, a measure of fuzziness namely, fuzzy entropy was introduced by Lofti Zadeh,[16] which measures the fuzziness degree or uncertain information in fuzzy set theory. Kauffman[10] , Deluca and Termini[14] , Kosko[21] , Pal and Pal[22] defined different expressions for Fuzzy entropy. Several researchers contributed in the development of expression of Fuzzy entropy depends on the applications. Further, a measure of fuzzy information, which calculates the difference between two fuzzy sets was suggested, initially it was proposed by Kullback and Leibler.[12] This measure is popularly known as directed divergence measure which is used to measure the directed divergence between the two probability distributions. Later on, Bhandari and Pal[9] proposed a measure of directed divergence between the two fuzzy sets corresponding to Kullback and Leibler measure of directed divergence. Fan and Xie[17] introduced divergence measure based on exponential operator. Many researchers like Prakash,[11] Bajaj and Hooda,[5] Bhatiya and Singh[20] etc. proposed new generalized directed divergence measures of fuzzy entropy in various ways and gives their applications in different areas such as pattern recognition, speech theory, image processing, decision making, fuzzy clustering etc.
In recent years, applications of information and divergence measures between two fuzzy sets has extended considerably but still the development of better divergence measure is continued which can be applied to

several real life problems. As mentioned above , many researchers had presented their work and applications in several areas. Taking inspiration from their work, here we propose, a new parametric fuzzy divergence measure and study some of it's properties . The strength of the proposed generalized fuzzy directed divergence measure has been demonstrated by giving application in multi criteria decision making problem with numerical example.

The paper is organized as follows. In section 2, we introduce some basic concepts related to probability theory, fuzzy set theory , concept of entropy and fuzzy entropy. In section 3, we have proposed a new parametric generalized fuzzy divergence measure corresponding to our earlier work with particular case , studied and proved some of the properties. Also calculated the total ambiguity , improvement measure and reduction in improvemnt measure in case of proposed generalized fuzzy directed divergence measure. In section 4, given the comparative study of proposed measure with the existing fuzzy divergence measure using numerical computation and pictorial representation. In section 5, we present the application of proposed divergence measure in multi criteria decision making problem , demonstrated with example. Finally some concluding remarks are given in section 6.

2 Prelimanaries

In this section, we present some basic concepts of probability theory and fuzzy set theory which is required for the proposed work.

2.1 Probability Theory

Let X be a discrete random variable, with a set of all possible values$\{x_1, x_2, \ldots, x_n\}$ and the corresponding probabilities are $h(x_1), h(x_2), \ldots, h(x_n)$ respectively. Shannon defined entropy as the average information over all the events and it is given by

$$F(X) = -\sum_{k=1}^{n} h(x_k) log(h(x_k)) \tag{1}$$

Here, the minus sign is given, to make the Shannon entropy of an event A positive, as $log(h(x_k) < 1$. Shannon entropy satisfies three important properties which defines the entropy function as a valid entropy measure as:

1. The entropy function F(X) should be continuous.
2. Entropy function should remain unchanged when $h(x_1), h(x_2), \ldots, h(x_n)$ are interchanged with one another.
3. Entropy function has a maximum value, when all $h(x_i), i = 1, 2, \ldots, n$ are equal.
4. The entropy function satisfies additivity property

$F(h(x_1), h(x_2), ..., h(x_n); q(x_1), q(x_2), ...q(x_n))$
$= F((h(x_1), h(x_2), ..., h(x_{n-1})) + h(x_n)F(\frac{q(x_1)}{h(x_1)}, \frac{q(x_2)}{h(x_2)}, ..., \frac{q(x_n)}{h(x_n)}$
Where $h(x_n) = \sum_{i=1}^{m} q(x_i)$

Some of the generalized entropies of order σ are listed as follows:

a) Renyi entropy[18] of order σ

$$F^{\sigma}(X) = \frac{1}{1-\sigma} log \sum_{k=1}^{n} h_k^{\sigma}, \sigma \neq 1, \sigma > 0 \tag{2}$$

b) Havrda-Charvat[15] entropy of order σ

$$F^{\sigma}(X) = \frac{1}{2^{1-\sigma} - 1} \sum_{k=1}^{n} h_k^{\sigma} - 1, \sigma \neq 1, \sigma > 0 \tag{3}$$

c) Tsallis entropy[4] of order σ

$$F^{\sigma}(X) = \frac{1}{1-\sigma}\sum_{k=1}^{n} h_k^{\sigma} - 1, \sigma \neq 1, \sigma > 0 \tag{4}$$

d) Mathai-Haubold entropy[6] of order σ

$$F^{\sigma}(X) = \frac{1}{\sigma-1}\sum_{k=1}^{n} h_k^{2-\sigma} - 1, \sigma \neq 1, -\infty < \sigma < 2 \tag{5}$$

$$F^{\sigma}(X) = \frac{1}{\sigma-1} log \sum_{k=1}^{n} h_k^{2-\sigma}, \sigma \neq 1, -\infty < \sigma < 2 \tag{6}$$

Kullback and Leibler [26] defined, the measure of directed divergence of probability distribution $U = \{u(x_1), u(x_2), \ldots, u(x_n)\}$ from another probability distribution $G = \{g(x_1), g(x_2), \ldots, g(x_n)\}$ as

$$D(U||G) = \sum_{i=1}^{n} u_i log(\frac{u_i}{g_i}) \tag{7}$$

$D(U||G) \neq D(G||U)$, Kullback suggested measure of symmetric divergence, which is stated as

$$J(U:G) = \sum_{i=1}^{n}(u(x_i) - g(x_i)) log \frac{u(x_i)}{g(x_i)} \tag{8}$$

Havrda and Charvat [20], defined the generalized directed divergence of degree σ as

$$I^{\sigma}(U,G) = \frac{1}{\sigma-1}\sum_{i=1}^{n}(u^{\sigma}(x_i) g^{1-\sigma}(x_i) - 1), \sigma > 0, \sigma \neq 1. \tag{9}$$

Based on this, the measure of symmetric divergence is defined as

$$J^{\sigma}(U,G) = I^{\sigma}(U,G) + I^{\sigma}(G,U) \tag{10}$$

2.2 Fuzzy set Theory and Fuzzy Entropy

When the representation of uncertainty level is required, the fuzziness term is used. Zadeh[16] defined the fuzzy set U as $\{(x, (\xi_U(x)))/(x \in X)\}$, where $\xi_U : X \to [0,1]$ is the membership function of U and X is the universe of discourse. Fuzzy set theory makes use of entropy to measure the fuzziness in the fuzzy set, this measure is known as fuzzy entropy. Fuzzy entropy measures the ambiguity and fuzziness of a fuzzy set. As the membership value for $\xi_U(x_i)$ and $(1 - \xi_U(x_i))$ are same, hence corresponding to equation (1) , De Luca and Termini[14] defined the measure of fuzzy entropy as

$$F(U) = -\sum_{i=1}^{n}(\xi_U(x_i) log(\xi_U(x_i)) + (1 - \xi_U(x_i)) log(1 - \xi_U(x_i)) \tag{11}$$

A measure of fuzzy entropy should have following properties:
P1: $F(U)$ is minimum if and only if $\xi_U(x) = 0$ or $\xi_U(x) = 1$, for all x.
P2: $F(U)$ is maximum if and only if $\xi_U(x) = 0.5$, for all x.

P3: $F(U) \geq (U^*)$, where U^* is sharpened version of U.
P4: $F(U) = F(U^{'})$, where $U^{'}$ is the complement of U.
Later on, Bhandari and Pal[9] developed a non- probabilistic measure of directed divergence corresponding to (2) as

$$I(U,V) = \sum_{i=1}^{n} \xi_U(x_i) log \frac{\xi_U(x_i)}{\xi_V(x_i)} + (1-\xi_U(x_i)) log \frac{(1-\xi_U(x_i))}{(1-\xi_V(x_i))} \tag{12}$$

Where U, V be any two standard fuzzy sets on universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ with the membership values $\xi_U(x_i)$ and $\xi_V(x_i)$, for all $i = 1, \ldots, n$. The symmetric divergence measure is given by $J(U,V) = I(U,V) + I(V,U)$ which on simplification gives,

$$J(U,V) = \sum_{i=1}^{n} (\xi_U(x_i) - \xi_V(x_i)) log \frac{\xi_V(x_i)(1-\xi_U(x_i))}{\xi_U(x_i)(1-\xi_V(x_i))} \tag{13}$$

Here if $V = U_E$, the fuzziest set then,

$$I(U,V) = nlog2 - nlog(F(U)) \tag{14}$$

Fan and Xie[17] , defined the discrimination measure with respect to exponential fuzzy entropy of Pal and Pal[22] as

$$I(U,V) = \frac{1}{(\sigma-1)} \sum_{i=1}^{n} (1 - (1-\xi_U(x_i)) e^{(\xi_U(x_i) - \xi_V(x_i))} - \xi_U(x_i) e^{(\xi_U(x_i) - \xi_V(x_i))}) \tag{15}$$

Kapur[8] , gave a generalised measure of fuzzy directed divergence as

$$I_\sigma(U,V) = \frac{1}{\sigma-1} \sum_{i=1}^{n} (\xi_U^{\sigma}(x_i) \xi_V^{1-\sigma}(x_i) + (1-\xi_U(x_i))^{\sigma} (1-\xi_V(x_i))^{1-\sigma}), \sigma \neq 1, \sigma > 0 \tag{16}$$

Bhatia and Singh[20] , proposed a measure of arithmetic-geometric fuzzy directed divergence measure as

$$\begin{aligned} I(U,V) = \sum_{i=1}^{n} (\frac{(\xi_U(x_i) + \xi_V(x_i))}{2} log \frac{(\xi_U(x_i) + \xi_V(x_i))}{2\sqrt{(\xi_U(x_i)\xi_V(x_i))}} \\ +(\frac{(2-\xi_U(x_i) + \xi_V(x_i))}{2} log \frac{(2-\xi_U(x_i) + \xi_V(x_i))}{2\sqrt{(1-\xi_U(x_i)(1-\xi_V(x_i)))}} \end{aligned} \tag{17}$$

Tomar and Ohlan[2] , defined the divergence measure based on Verma and Sharma[3] as

$$I(A,B) = \sum_{i=1}^{n} (1 - (1-\xi_U(x_i)) e^{(1-\xi_V(x_i))^{\sigma} - (1-\xi_U(x_i))^{\sigma})} - \xi_U(x_i) e^{\xi_V^{\sigma}(x_i) - \xi_U^{\sigma}(x_i)}) \tag{18}$$

Prakash[11] , defined measure of directed divergence as

$$\begin{aligned} I_\sigma(U,V) = \sum_{i=1}^{n} \xi_U(x_i) log \frac{\xi_U(x_i)}{\xi_V(x_i)} + (1-\xi_U(x_i)) log \frac{(1-\xi_U(x_i))}{(1-\xi_V(x_i))} \\ -\frac{1}{\sigma} \sum_{i=1}^{n} (1+\sigma\xi_U(x_i)) log \frac{(1+\sigma\xi_U(x_i))}{(1+\sigma\xi_V(x_i))} + (1+\sigma(1-\xi_U(x_i))) log \frac{(1+\sigma(1-\xi_U(x_i))}{(1+\sigma(1-\xi_V(x_i)))} \end{aligned} \tag{19}$$

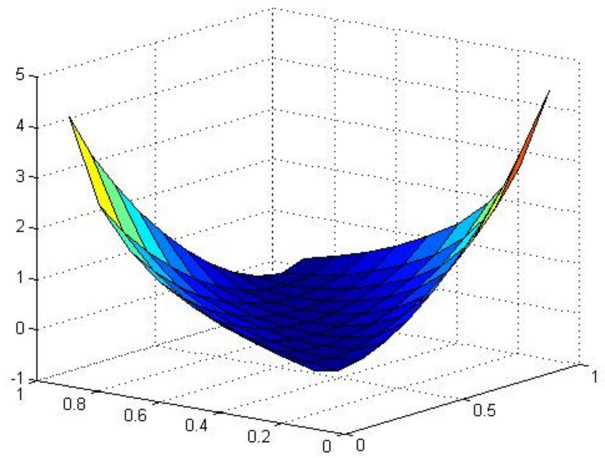


Figure 1: $J_\sigma(U,V)$ at different values of σ. (Refer Table 1)

3 FUZZY DIVERGENCE MEASURE

In this section, we propose a new fuzzy directed divergence of fuzzy entropy.

3.1 Fuzzy directed divergence measure

We propose a new fuzzy directed divergence measure between two fuzzy sets U and V corresponding to the entropy measure $I_\sigma(U)$

$$I_\sigma(U) = \frac{1}{n(2^{\sigma-1}-1)}\sum_{i=1}^{n}(\xi_U^{2-\sigma}(x_i) + (1-\xi_U(x_i))^{\sigma-1} - 1), \sigma \neq 1, 0 < \sigma < 2. \tag{20}$$

as

$$J_\sigma(U,V) = \frac{1}{n(2^{1-\sigma}-1)}\sum_{i=1}^{n}[\xi_U^{2-\sigma}(x_i)\xi_V^{\sigma-1}(x_i) + (1-\xi_U(x_i))^{2-\sigma}(1-\xi_V(x_i))^{\sigma-1} - 1], \sigma \neq 1, 0 < \sigma < 2 \tag{21}$$

where U and V be two fuzzy sets defined on a set $X = \{x_1, x_2, \ldots, x_n\}$ with the membership degree $\xi_U(x_i)$ and $\xi_V(x_i)$ respectively.

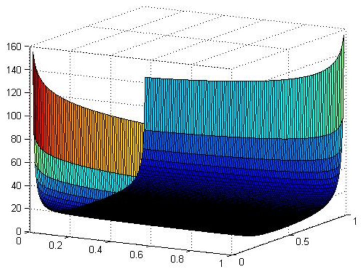
Theorem 3.1. *The directed divergence measure $J_\sigma(U,V)$ is a valid measure of divergence.*

Proof. To show that $J_\sigma(U,V)$ is a valid measure of divergence that means to show the following:
1. $J_\sigma(U,V) \geq 0$
2. $J_\sigma(U,V) = 0$, when the two fuzzy sets U and V coincides.
3. $J_\sigma(U,V) = 0$ is a convex function in $(0,1)$.
To check non-negativity of $J_\sigma(U,V) = 0$

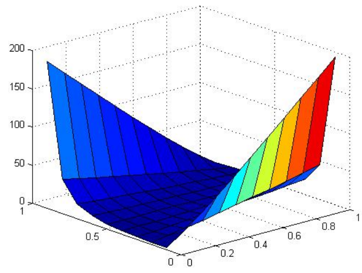
$$J_\sigma(U,V) = \frac{1}{n(2^{1-\sigma}-1)}\sum_{i=1}^{n}(\xi_U^{2-\sigma}(x_i)\xi_V^{\sigma-1}(x_i) + (1-\xi_U(x_i))^{2-\sigma}(1-\xi_V(x_i))^{\sigma-1} - 1), \sigma \neq 1, 0 < \sigma < 2 \tag{22}$$

Differentiating twice the equation no (2) with respect to ξ_U we get,

$$\frac{\partial J_\sigma}{\partial \xi_U} = \frac{(2-\sigma)}{n(2^{1-\sigma}-1)}\sum_{i=1}^{n}(\xi_U^{1-\sigma}(x_i)\xi_V^{\sigma-1}(x_i) - (1-\xi_U(x_i))^{1-\sigma}(1-\xi_V(x_i))^{\sigma-1}) \tag{23}$$



(a) $\frac{\partial^2 J_\sigma}{\partial \xi_U^2}$



(b) $\frac{\partial^2 J_\sigma}{\partial \xi_V^2}$

Figure 2: Second order partial derivative at different values of σ.

$$\frac{\partial^2 J_\sigma}{\partial \xi_U^2} = \frac{(2-\sigma)(1-\sigma)}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_U^{-\sigma}(x_i)\xi_V^{\sigma-1}(x_i) - (1-\xi_U(x_i))^{-\sigma}(1-\xi_V(x_i))^{\sigma-1}) \tag{24}$$

Similarly differentiating partially with respect to ξ_V we get,

$$\frac{\partial J_\sigma}{\partial \xi_V} = \frac{(\sigma-1)}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_U^{2-\sigma}(x_i)\xi_V^{\sigma-2}(x_i) - (1-\xi_U(x_i))^{2-\sigma}(1-\xi_V(x_i))^{\sigma-2}) \tag{25}$$

$$\frac{\partial^2 J_\sigma}{\partial \xi_V^2} = \frac{(\sigma-1)(\sigma-2)}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_U^{2-\sigma}(x_i)\xi_V^{\sigma-3}(x_i) - (1-\xi_U(x_i))^{2-\sigma}(1-\xi_V(x_i))^{\sigma-3}) \tag{26}$$

Therefore for $\sigma \neq 1$, $0 < \sigma < 2$,$\frac{\partial^2 J_\sigma}{\partial \xi_U^2} > 0$ and $\frac{\partial^2 J_\sigma}{\partial \xi_V^2} > 0$ (Refer figure 2)
Hence, $J_\sigma(U,V)$ is a non-negative and convex function. If $\xi_U(x_i) = \xi_V(x_i)$, then the value of $J_\sigma(U,V)$ becomes
$J_\sigma(U,V) = \frac{1}{n(2^{(\sigma-1)}-1)} \sum_{i=1}^{n} (\xi_U^{(2-\sigma)}(x_i)\xi_V^{(\sigma-1)}(x_i) + (1-\xi_U(x_i))^{(2-\sigma)}(1-\xi_V(x_i))^{(\sigma-1)} - 1)$
$= 0$
Since all properties are true hence the divergence measure $J_\sigma(U,V)$ is a valid fuzzy divergence measure. □

The proposed divergence measure is not symmetric that is $J_\sigma(U,V) \neq J_\sigma(V,U)$. The fuzzy symmetric divergence measure is defined as $D(U,V) = J_\sigma(U,V) + J_\sigma(V,U)$. In particular, if set $V = U_F$ is almost fuzzy set that means the membership value $\xi_V = 0.5$, then it gives relationship between $J_\sigma(U,V)$ and $I_\sigma(U)$ as
$J_\sigma(U,V) = -I_\sigma(U) + \frac{1+2^{(\sigma-1)}}{1-2^{(\sigma-1)}}$.

3.2 Some results on Fuzzy Divergence Measure

Let us see some results of the proposed fuzzy divergence measure $J_\sigma(U,V)$.

Theorem 3.2. *For any three fuzzy sets R, S and T of universal set X, the following properties are holds.*
$(i) J_\sigma(R\cup S, R) + J_\sigma(R\cap S, R) = J_\sigma(S, R)$

$(ii) J_\sigma(R\cup S, T) + J_\sigma(R\cap S, T) = J_\sigma(R, T) + J_\sigma(S, T)$

$(iii) J_\sigma(R\cup S, R\cap S) = J_\sigma(R\cup S, S) + J_\sigma(S, R\cap S)$

$(iv) J_\sigma(R, R\cup S) = J_\sigma(R\cap S, S)$

Proof. Let X_1 and X_2 be two fuzzy sets defined as
$X_1 = \{x/x \in X_1, \xi_R(x_i) \geq \xi_S(x_i)\}$ and
$X_2 = \{x/x \in X_2, \xi_S(x_i) > \xi_R(x_i)\}$.
In the set X_1 ,
$\xi_{R\cup S}(x) = max(\xi_R(x), \xi_S(x)) = \xi_R(x)$ and $\xi_{R\cap S}(x) = min(\xi_R(x), \xi_S(x)) = \xi_S(x)$.
In the set X_2 ,
$\xi_{R\cup S}(x) = max(\xi_R(x), \xi_S(x)) = \xi_S(x)$ and $\xi_{R\cap S}(x) = min(\xi_R(x), \xi_S(x)) = \xi_R(x)$.
Consider,
$J_\sigma(R \cup S, R) + J_\sigma(R \cap S, R)$

$= \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_{R\cup S}^{2-\sigma}(x_i)\xi_R^{\sigma-1}(x_i) + (1-\xi_{R\cup S}(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1) + \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_{R\cap S}^{2-\sigma}(x_i)\mu_R^{\sigma-1}(x_i) + (1-\xi_{R\cap S}(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1)$

$= \frac{1}{n(2^{1-\sigma}-1)} \{\sum_{X_1} (\xi_R^{2-\sigma}(x_i)\xi_R^{\sigma-1}(x_i) + (1-\xi_R(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1)\} + \{\sum_{X_2} (\xi_S^{2-\sigma}(x_i)\xi_R^{\sigma-1}(x_i) + (1-\xi_S(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1)\} + \frac{1}{n(2^{1-\sigma}-1)} \{\sum_{X_1} (\xi_S^{2-\sigma}(x_i)\xi_R^{\sigma-1}(x_i) + (1-\xi_S(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1)\} + \{\sum_{X_2} (\xi_R^{2-\sigma}(x_i)\xi_R^{\sigma-1}(x_i) + (1-\xi_R(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1)\}.$

$= \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_S^{2-\sigma}(x_i)\xi_R^{\sigma-1}(x_i) + (1-\xi_S(x_i))^{2-\sigma}(1-\xi_R(x_i))^{\sigma-1}-1)$

$= J_\sigma(S, R)$
Hence proved.
$(ii) J_\sigma(R \cup S, T) + J_\sigma(R \cap S, T)$
In the set X_1, $\xi_{R\cup S}(x) = \xi_R(x)$ and $\xi_{R\cap S}(x) = \xi_S(x)$.
Also in X_2 , $\xi_{R\cup S}(x) = \xi_S(x)$ and $\xi_{R\cap S}(x) = \xi_R(x)$.

$J_\sigma(R \cup S, T) + J_\sigma(R \cap S, T)$

$= \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_R^{2-\sigma}(x_i)\xi_T^{\sigma-1}(x_i) + (1-\xi_R(x_i))^{2-\sigma}(1-\xi_T(x_i))^{\sigma-1}-1) + \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (\xi_S^{2-\sigma}(x_i)\xi_T^{\sigma-1}(x_i) + (1-\xi_S(x_i))^{2-\sigma}(1-\xi_T(x_i))^{\sigma-1}-1)$

$= J_\sigma(R, T) + J_\sigma(S, T)$
Hence proved.
Using earlier method one can prove the remaining properties. □

Theorem 3.3. *Let R and S be any two fuzzy sets of X, then the following holds*
$(i) J_\sigma(R', S') = J_\sigma(R, S)$

$(ii) J_\sigma(R, S') = J_\sigma(R', S)$

$(iii) J_\sigma(R, S) + J_\sigma(R', S) = J_\sigma(R', S') + J_\sigma(R, S')$

Proof. $(i) J_\sigma(R', S')$
$= \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} (1-\xi_R(x_i)^{2-\sigma}(1-\xi_S(x_i)^{\sigma-1} + (1-(1-\xi_R(x_i))^{2-\sigma}(1-(1-\xi_S(x_i))^{\sigma-1}-1)$

$= J_\sigma(R, S)$

$(ii) J_\sigma(R, S')$

$= \frac{1}{n(2^{1-\sigma}-1)} \sum_{i=1}^{n} \xi_R^{2-\sigma}(x_i)(1-\xi_S(x_i))^{\sigma-1} + (1-\xi_R(x_i))^{2-\sigma}(1-(1-\xi_S(x_i))^{\sigma-1}-1)$

$= J_\sigma(R', S)$

(iii) It can be easily proved just by adding the earlier (i) and (ii) properties. □

3.3 Total Ambiguity and Improvement Measure

Let U and V be two fuzzy sets of Universe of discourse X. The total ambiguity is defined as the sum of fuzzy entropy present in fuzzy set U and fuzzy directed divergence of U from V for the two fuzzy sets U and V. Using Havrda and Charvat, Kapur defined the total ambiguity as
Total ambiguity=$\frac{1}{\sigma-1}\sum_{i=1}^{n}(\xi_U^{\sigma}(x_i)(1-\xi_V^{1-\sigma}(x_i))+(1-\xi_U(x_i))^{\sigma}(1-(1-\xi_V(x_i))^{1-\sigma}$

Analogously, we have defined the measure of total ambiguity from equations (20) and (21) as:

Total ambiguity =$I_\sigma(U)+J_\sigma(U,V)$

$$=\frac{1}{n(2^{1-\sigma}-1)}\sum_{i=1}^{n}\{\xi_U^{2-\sigma}(x_i)(1+\xi_V^{\sigma-1}(x_i)+(1-\xi_U(x_i))^{\sigma-1}(1+(1-\xi_U(x_i))^{-2\sigma+3}(1-\xi_V(x_i))^{\sigma-1}\}$$

Note that the measure of total ambiguity is not symmetric. In case of fuzzy directed divergence, the reduction in ambiguity is also called as the generalized fuzzy information improvement measure of order σ. Theil's [31] defined the measure of information improvement as

$$D(U:V)-D(U:W)=\sum_{i=1}^{n}u_i log(\frac{w_i}{v_i}).$$

Where U and V be observed and predicated distributions of a random variable respectively and W be the revised probability distribution of V . If a fuzzy set U is correct and original estimated fuzzy set V is now corrected as fuzzy set W , then the reduction in ambiguity is given as

$$I(U,V)-I(U,W)=\sum_{i=1}^{n}\{\xi_U(x_i)log(\frac{\xi_W(x_i)}{\xi_V(x_i)})+(1-\xi_U(x_i))log(\frac{1-\xi_W(x_i)}{1-\xi_V(x_i)})\}$$

In case of proposed fuzzy divergence, the reduction of measure improvement is defined as,
$J_\sigma(U,V)-J_\sigma(U,W)=$

$$\frac{1}{n(2^{1-\sigma}-1)}\sum_{i=1}^{n}\{\xi_U^{2-\sigma}(x_i)(\xi_V^{\sigma-1}(x_i)-\xi_W^{\sigma-1}(x_i))+(1-\xi_U(x_i))^{2-\sigma}[(1-\xi_V(x_i))^{\sigma-1}-(1-\xi_W(x_i))^{\sigma-1}]\}$$

4 Comparative Study

In this section, we have compared the efficiency of the proposed directed divergence measure with the existing fuzzy divergence measures given by Tomar,[2] Rakesh Bajaj and Hooda,[5] Kapur,[8] Prakash.[11] Let us consider the fuzzy sets $U=\{0.2,0.4,0.7,0.9,0.1\}$ and $V=\{0.5,0.2,0.3,0.8,0.6\}$. Numerically evaluated the fuzzy divergence measure for the existing measures of divergences for different parametric values is given in Table 1. Here it is observed that , the numerical discrimination value of proposed measure is less than the existing fuzzy divergence measures (Refer Figure-3).

Table 1: Comparative study at different values of σ

	Fuzzy directed divergence measure				
σ	$J_{Tomar}(U,V)$	$J_{Bajaj}(U,V)$	$J_{Kapur}(U,V)$	$J_{Prakash}(U,V)$	$J_\sigma(U,V)$
0.9	0.4994	0.4907	1.1100	0.3820	0.3728
1.2	0.5678	0.6047	1.4533	0.3501	0.3078
1.3	0.5863	0.6375	1.5702	0.3407	0.2818
1.5	0.6179	0.6966	1.8110	0.3234	0.2214
1.9	0.6636	0.7925	2.3339	0.2938	0.0547

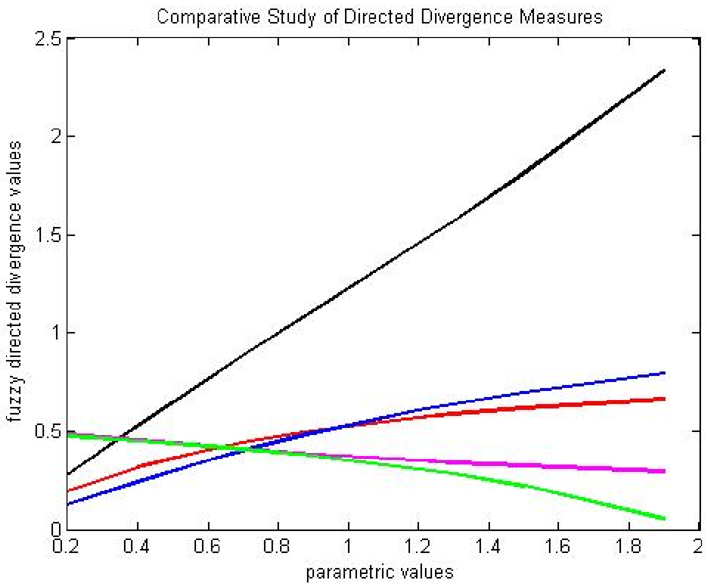


Figure 3: $J_\sigma(U,V)$ at different values of σ. (Refer Table 1)

5 Application in Multi- criteria decision making

In this section, we present the application of TOPSIS methods for multi criteria decision making problem by using the proposed fuzzy directed divergence measure. Decision making process involves n-number of courses of actions under the imprecise environment. To take the decision , a person needs to choose the best course of action amongst all the courses of action with respect to criterians or goals. Multi criterian decision making problem is a well established branch of decision making which allows to find the best alternative as well as the ranks to all available courses of actions. Here we discuss the application of proposed generalized divergence measure in the multi criteria decision making problem which is illustrated by using the following example.

Example 5.1. Suppose that a customer wants to buy an internet connection. There are five service Providers are available nearby say $M_1, M_2, ..., M_5$ on the basis of network quality K_1, internet speed K_2, Tarrif plan K_3, optic fibre K_4, customer service K_5 be the five criterians. The decision maker forms five fuzzy set $M_1, ..., M_5$ as follows.

$M_1 = \{(K_1, 0.5), (K_2, 0.7), (K_3, 0.2), (K_4, 0.8), (K_5, 0.9)\}$

$M_2 = \{(K_1, 0.2), (K_2, 0.8), (K_3, 0.5), (K_4, 0.3), (K_5, 0.6)\}$

$M_3 = \{(K_1, 0.1), (K_2, 0.3), (K_3, 0.6), (K_4, 0.5), (K_5, 0.8)\}$

$M_4 = \{(K_1, 0.6), (K_2, 0.4), (K_3, 0.7), (K_4, 0.2), (K_5, 0.9)\}$

$M_5 = \{(K_1, 0.5), (K_2, 0.7), (K_3, 0.4), (K_4, 0.2), (K_5, 0.1)\}$

Now the question is to choose the best service provider which fulfil all the criteria of the customer.

To solve stepwise the above multi criteria fuzzy decision making problem , initially compute the positive ideal and negative ideal solution by choosing maximum and minimum of membership values respectively for each criteria.

$M^+ = \{(K_1, 0.6), (K_2, 0.8), (K_3, 0.7), (K_4, 0.5), (K_5, 0.9)\}$

$M^- = \{(K_1, 0.1), (K_2, 0.3), (K_3, 0.4), (K_4, 0.2), (K_5, 0.1)\}$

First of all calculating the value of $J^\sigma(M^+, M_i)$ as well as $J^\sigma(M^-, M_i)$ at $\sigma = 0.1, 0.3, 0.7, 1.1, 1.5, 1.7$ using proposed fuzzy divergence measure (Refer table 2 and 3). The value of relative fuzzy divergence measure $J_\alpha(M_i)$ of alternatives M_i with respect to M^+ and M^- are calculated by using the formula,

$$J_\sigma(M_i) = \frac{J^\sigma(M^+, M_i)}{J^\sigma(M^+, M_i) + J^\sigma(M^-, M_i)}$$

According to the calculated numerical values of the relative divergences for different values of σ, ranking of the alternatives are as follows.

For $\sigma = 0.1$, $M_4 > M_1 > M_2 > M_3 > M_5$

For $\sigma = 0.3$, $M_4 > M_1 > M_2 > M_3 > M_5$

Table 2: Numerical calculation of $J^{\sigma}(M^{+}, M_i)$

	σ					
$J^{\alpha}(M^{+}, M_i)$	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.7$	$\sigma = 1.1$	$\sigma = 1.5$	$\sigma = 1.7$
M_1	0.422353	0.382908	0.3053711	0.2251056	0.1359106	0.085652
M_2	0.334930	0.3137935	0.2665394	0.2085683	0.1333215	0.086399
M_3	0.7599915	0.6618088	0.49635	0.3524054	0.2093279	0.131774
M_4	0.2390935	0.2247186	0.1913915	0.1495244	0.09506612	0.06135937
M_5	1.472973	1.215043	0.8421455	0.577572	0.3480352	0.2250639

Table 3: Numerical calculation of $J^{\alpha}(M^{-}, M_i)$

	σ					
$J^{\sigma}(M^{-}, M_i)$	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.7$	$\sigma = 1.1$	$\sigma = 1.5$	$\sigma = 1.7$
M_1	2.023034	1.717152	1.254173	0.896888	0.5554206	0.3618148
M_2	0.5316625	0.4936353	0.415658	0.3270175	0.2136077	0.1408817
M_3	0.6813991	0.6141245	0.4946385	0.3804628	0.2488375	0.1657961
M_4	1.570403	1.312335	0.9371514	0.6646516	0.4145896	0.2725922
M_5	0.2751748	0.2635763	0.234064	0.1919004	0.1290967	0.08602783

Table 4: Numerical value of relative fuzzy divergence measure $J^{\sigma}(M_i)$

	σ					
$J^{\sigma} M_i$	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.7$	$\sigma = 1.1$	$\sigma = 1.5$	$\sigma = 1.7$
M_1	0.1727142	0.1823319	0.1958079	0.20063	0.1965926	0.1914154
M_2	0.3864912	0.388633	0.3907071	0.3894209	0.3842903	0.3801434
M_3	0.5272627	0.518686	0.5008635	0.4808578	0.4568829	0.4428335
M_4	0.1321326	0.1462009	0.1695917	0.1836512	0.1865301	0.1837374
M_5	0.8425906	0.8217416	0.7825108	0.7506078	0.7294319	0.7234648

For $\sigma = 0.7$, $M_4 > M_1 > M_2 > M_3 > M_5$
For $\sigma = 1.1$, $M_4 > M_1 > M_2 > M_3 > M_5$
For $\sigma = 1.5$, $M_4 > M_1 > M_2 > M_3 > M_5$
For $\sigma = 1.7$, $M_4 > M_1 > M_2 > M_3 > M_5$
Here we found that for all different values of σ , the best choice is M_4. Therefore M_4 is the best alternative.

6 Conclusion

We have proposed a new generalized fuzzy directed divergence measure of order σ and also the validity measure conditions are proved and pictorially represented. Discussed the concept of total ambiguity measures, fuzzy information improvement measure and reduction in ambiguity measure for the proposed measure of fuzzy divergence. Further the comparative study of parametric divergence is studied and discussed application of proposed divergence measure in decision making. The multi criteria decision making problem is illustrated by using numerical example. Here we propose that the proposed divergence measure provides greater flexibility in multi criteria decision making problem. Further we propose the generalization with two parameter fuzzy divergence measure and related improvement measure and it's applications in decision making real life problems.

References

[1] A. Bhat, N. Siddiqui, I. Mageed, S. Alkhazaleh, V. Das and M. Baig, *Generalization of Renyi Entropy and its applications in source coding*, Applied Mathematics and Information Sciences, 17(5), 941-948, 2023.

[2] V. Tomar, A. Ohlan, *New parametric generalized exponential fuzzy divergence measure*, Journal of Uncertainty Analysis and Applications,2 , 24, 1-14,2014.

[3] R. Verma, B. Sharma, *On Generalized Exponential Fuzzy Entropy*, World Academy of Science , Engineering and Technology, 60, 2011.

[4] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, Journal of Statistical Physics, 52 (1–2): 479–487,1988.

[5] R. Bajaj, D. Hooda , *On Some Generalized Measures of Fuzzy Information*, World Academy of Science , Engineering and Technology, 62, 2010.

[6] A. Mathai, H. Haubold, *Pathway models, Tsallis statistics, superstatistics and a generalized measure of fuzzy entropy*, Physics A. ,375, 110-122, 2006.

[7] D. Hooda , *On Generalized Measures of Fuzzy Entropy*, Mathematica Slovaca, 3, 315-325, 2004.

[8] J. Kapur, *Measures of fuzzy information*,Mathematical Sciences Trust Society, New Delhi, 1997.

[9] D. Bandari and N. Pal, *Some new information measures for fuzzy sets*, Information Science, 67, 202-228,1993.

[10] A. Kaufmann ,*Fuzzy subsets: Fundamental Theoretical Elements*, Academic Press, New Yor ,3 ,1980.

[11] O. Prakash, P. Sharma, S. Kumar, *Two new measures of fuzzy divergence and their properties*, SQU Journal for Science, 11, 69-77, 2006.

[12] S. Kullback, R. Leibler, *On Information and Sufficiency* , The Annala of Mathematical Statistics, 22(1), 79-86 , 1951.

[13] B. Sharma and D. Mittal, *New non-additive measures of entropy for discrete probability distributions*, J. Math. Sci, 10, 28-40 , 1975.

[14] De Luca and S. Termini, *A definition of non-probabilistic entropy in the setting of fuzzy set theory*, Information and Control, 20, 301-312 , 1972.

[15] J. Havrda and F. Charvat, *Quantification method of classification processes.*, Concept of structural a-entropy, Kybernetika, 3(1), 30-35 , 1967.

[16] L. Zadeh, *Fuzzy sets*, Information and Control, 8, 338-353 , 1965.

[17] J. Fan, W. Xie, *Distance measure and induced fuzzy entropy*, Fuzzy Sets Systems, 104, 305–314 , 1999.

[18] A. Renyi, *On measure of entropy and information, Proceeding Fourth Berkley Symposium on Mathematical Statistics and Probability*, University of California, Press 1, 546-561 , 1961.

[19] C. Shannon, *A mathematical theory of communication*, Bell System Technical Journal, 27, 379-423, 623-659 , 1948.

[20] P. Bhatia , S. Singh, *Three families of generalized fuzzy directed divergence*, AMO- Advanced Modelling and Optimization, 14(3) , 599-614 , 2012.

[21] B. KOSKO, *Neural Networks and Fuzzy Systems*, Prentice Hall, Englewood Cliffs, N J, 1992.

[22] N. Pal, S. Pal , *Object-background segmentation using new definitions of entropy*, IEE Proc. E Computer Digit. Tech. , 136, 284–295 , 1989.