



Neutrosophic subgroups and neutrosophic normal subgroups of groups

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Abstract

In this paper, we introduce the concepts of neutrosophic subgroups and neutrosophic normal subgroups of groups and investigate several properties. We investigate relations between neutrosophic subgroups (neutrosophic normal subgroups) and their neutrosophic level subsets of a group. We also look at the homomorphic image and inverse image of the neutrosophic subgroups and neutrosophic normal subgroups of groups, as well as some related properties.

Keywords: Group; Neutrosophic set; Neutrosophic subgroup; Neutrosophic normal subgroup

1 Introduction

The concept of fuzzy sets was proposed by Zadeh.¹² The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. One of the most important theorems in abstract algebra is Lagrange's theorem. This theorem is crucial in finite groups because it provides an overview of subgroup size. Lagrange's theorem has various applications in number theory. For further details, we refer to.⁷ Uncertainty is an unavoidable element of our lives. This universe is not built on assumptions or precise measures. It is not always feasible to make straightforward decisions. We face a significant problem in dealing with errors in decision-making situations. In 1965, Zadeh¹² proposed the fuzzy set as a way to deal with ambiguity in real-world problems. Following that, fuzzy sets have become a worldwide study trend. Rosenfeld⁶ was the first to examine the concept of fuzzy subgroup and its features in 1971. Ajmal and Prajapati¹ introduced the ideas of fuzzy cosets and fuzzy normal subgroups. Atanassov² established the intuitionistic fuzzy set in 1986 by attributing non-membership degrees to membership degrees. Yager¹¹ defined the Pythagorean fuzzy set in 2013 using this approach. This set provides a modern technique to model vagueness and uncertainty with high precision and accuracy compared to intuitionistic fuzzy sets. The notion of the neutrosophic set, which Smarandache developed,^{8,9} extends the notions of the classic set and fuzzy set, intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various parts (refer to the site <http://fs.gallup.unm.edu/neutrosophy.htm>).

In this paper, we introduce the concepts of neutrosophic subgroups and neutrosophic normal subgroups of groups and investigate several properties. We investigate relations between neutrosophic subgroups (neutrosophic normal subgroups) and their neutrosophic level subsets of a group. We also look at the homomorphic image and inverse image of the neutrosophic subgroups and neutrosophic normal subgroups of groups, as well as some related properties.

2 Preliminaries

Definition 2.1.⁸ Let X be a nonempty set. The *neutrosophic set* on X is defined to be a structure

$$A := \{ \langle x, \mu(x), \gamma(x), \psi(x) \rangle : x \in X \}, \quad (2.1)$$

where $\mu : X \rightarrow [0, 1]$ is a truth membership function, $\gamma : X \rightarrow [0, 1]$ is an indeterminate membership function, and $\psi : X \rightarrow [0, 1]$ is a false membership function. The neutrosophic fuzzy set in (2.1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

Definition 2.2.¹⁰ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a group (C, \circ) . Then A is a neutrosophic subgroup of C if

- (1) $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) \geq \gamma_A(x) \wedge \gamma_A(y)$, and $\gamma_A(x \circ y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in C$,
- (2) $\mu_A(x^{-1}) \geq \mu_A(x)$, $\gamma_A(x^{-1}) \geq \gamma_A(x)$, and $\gamma_A(x^{-1}) \leq \gamma_A(x)$ for all $x \in C$.

Proposition 2.3.¹⁰ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a group (C, \circ) . Then A is a neutrosophic subgroup of C if and only if $\mu_A(x \circ y^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y^{-1}) \geq \gamma_A(x) \wedge \gamma_A(y)$, and $\gamma_A(x \circ y^{-1}) \leq \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in C$.

Definition 2.4.¹⁰ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . Then for $x \in C$, the neutrosophic left coset of A is the neutrosophic set $xA = (x\mu_A, x\gamma_A, x\psi_A)$, defined by $(x\mu_A)(u) = \mu_A(x^{-1} \circ u)$, $(x\gamma_A)(u) = \gamma_A(x^{-1} \circ u)$, and $(x\psi_A)(u) = \psi_A(x^{-1} \circ u)$ and the neutrosophic right coset of A is the neutrosophic set $Ax = (\mu_Ax, \gamma_Ax, \psi_Ax)$, defined by $(\mu_Ax)(u) = \mu_A(u \circ x^{-1})$, $(\gamma_Ax)(u) = \gamma_A(u \circ x^{-1})$, and $(\psi_Ax)(u) = \psi_A(u \circ x^{-1})$ for all $u \in C$.

Definition 2.5.¹⁰ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . Then A is a neutrosophic normal subgroup on C if every neutrosophic left coset of A is a neutrosophic right coset of A on C . Equivalently, $xA = Ax$ for all $x \in C$.

Proposition 2.6.¹⁰ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . Then A is a neutrosophic normal subgroup on C if and only if $\mu_A(x \circ y) = \mu_A(y \circ x)$, $\gamma_A(x \circ y) = \gamma_A(y \circ x)$, and $\psi_A(x \circ y) = \psi_A(y \circ x)$ for all $x, y \in C$.

Proposition 2.7.¹⁰ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . Then A is a neutrosophic normal subgroup of C if and only if $\mu_A(k \circ u \circ k^{-1}) = \mu_A(u)$, $\gamma_A(k \circ u \circ k^{-1}) = \gamma_A(u)$, and $\psi_A(k \circ u \circ k^{-1}) = \psi_A(u)$ for all $u, k \in C$.

3 Main results

In this section, we define neutrosophic subgroups as extensions of fuzzy subgroups and neutrosophic subgroups. Now, we will check whether the union and intersection of two neutrosophic subgroups of a group (C, \circ) is a neutrosophic subgroup of C .

Theorem 3.1. *The intersection of two neutrosophic subgroups of a group (C, \circ) is a neutrosophic subgroup of C .*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be two neutrosophic subgroups of a group (C, \circ) . Then $A \cap B = (\mu_{A \cap B}, \gamma_{A \cap B}, \psi_{A \cap B})$, where $\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$, $\gamma_{A \cap B}(x) = \gamma_A(x) \wedge \gamma_B(x)$, and $\psi_{A \cap B}(x) = \psi_A(x) \vee \psi_B(x)$ for all $x \in C$. Now, for all $x, y \in C$,

$$\begin{aligned} \mu_A(x \circ y^{-1}) &= \mu_A(x \circ y^{-1}) \wedge \mu_A(x \circ y^{-1}) \\ &\geq (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_A(x) \wedge \mu_A(y)) \\ &= (\mu_A(x) \wedge (\mu_A(y) \wedge \mu_A(y))) \\ &= \mu_A(x) \wedge \mu_A(y), \end{aligned}$$

$$\begin{aligned}
 \gamma_A(x \circ y^{-1}) &= \gamma_A(x \circ y^{-1}) \wedge \gamma_A(x \circ y^{-1}) \\
 &\geq (\gamma_A(x) \wedge \gamma_A(y)) \wedge (\gamma_A(x) \wedge \gamma_A(y)) \\
 &= (\gamma_A(x) \wedge \gamma_A(y)) \wedge (\gamma_A(y) \wedge \gamma_A(x)) \\
 &= \gamma_A(x) \wedge \gamma_A(y), \\
 \psi_A(x \circ y^{-1}) &= \psi_A(x \circ y^{-1}) \vee \psi_A(x \circ y^{-1}) \\
 &\leq (\psi_A(x) \vee \psi_A(y)) \vee (\psi_A(x) \vee \psi_A(y)) \\
 &= (\psi_A(x) \vee \psi_A(y)) \vee (\psi_A(y) \vee \psi_A(x)) \\
 &= \psi_A(x) \vee \psi_A(y).
 \end{aligned}$$

Therefore, $A \cap B = (\mu_{A \cap B}, \gamma_{A \cap B}, \psi_{A \cap B})$ is a neutrosophic subgroup of C . Hence, the intersection of two neutrosophic subgroups of a group is also a neutrosophic subgroup. □

Corollary 3.2. *The intersection of a family of neutrosophic subgroups of a group (C, \circ) is also a neutrosophic subgroup of C .*

Proof. Let A_1, A_2, \dots, A_p be neutrosophic subgroups of a group (C, \circ) . We have to show that $A = \bigcap_{i=1}^p A_i$ is a neutrosophic subgroup of C . Then $A = (\mu_A, \gamma_A, \psi_A)$ is given by $\mu(x) = \mu_{A_1}(x) \wedge \mu_{A_2}(x) \wedge \dots \wedge \mu_{A_p}(x)$, $\gamma(x) = \gamma_{A_1}(x) \wedge \gamma_{A_2}(x) \wedge \dots \wedge \gamma_{A_p}(x)$, and $\psi(x) = \psi_{A_1}(x) \vee \psi_{A_2}(x) \vee \dots \vee \psi_{A_p}(x)$ for all $x \in C$. Now, for all $x, y \in C$,

$$\begin{aligned}
 \mu_A(x \circ y^{-1}) &= \mu_{A_1}(x \circ y^{-1}) \wedge \mu_{A_2}(x \circ y^{-1}) \wedge \dots \wedge \mu_{A_p}(x \circ y^{-1}) \\
 &\geq (\mu_{A_1}(x) \wedge \mu_{A_1}(y)) \wedge (\mu_{A_2}(x) \wedge \mu_{A_2}(y)) \wedge \dots \wedge (\mu_{A_p}(x) \wedge \mu_{A_p}(y)) \\
 &= (\mu_{A_1}(x) \wedge \mu_{A_2}(x) \wedge \dots \wedge \mu_{A_p}(x)) \wedge (\mu_{A_1}(y) \wedge \mu_{A_2}(y) \wedge \dots \wedge \mu_{A_p}(y)) \\
 &= \mu_A(x) \wedge \mu_A(y), \\
 \gamma_A(x \circ y^{-1}) &= \gamma_{A_1}(x \circ y^{-1}) \wedge \gamma_{A_2}(x \circ y^{-1}) \wedge \dots \wedge \gamma_{A_p}(x \circ y^{-1}) \\
 &\geq (\gamma_{A_1}(x) \wedge \gamma_{A_1}(y)) \wedge (\gamma_{A_2}(x) \wedge \gamma_{A_2}(y)) \wedge \dots \wedge (\gamma_{A_p}(x) \wedge \gamma_{A_p}(y)) \\
 &= (\gamma_{A_1}(x) \wedge \gamma_{A_2}(x) \wedge \dots \wedge \gamma_{A_p}(x)) \wedge (\gamma_{A_1}(y) \wedge \gamma_{A_2}(y) \wedge \dots \wedge \gamma_{A_p}(y)) \\
 &= \gamma_A(x) \wedge \gamma_A(y), \\
 \psi_A(x \circ y^{-1}) &= \psi_{A_1}(x \circ y^{-1}) \vee \psi_{A_2}(x \circ y^{-1}) \vee \dots \vee \psi_{A_p}(x \circ y^{-1}) \\
 &\leq (\psi_{A_1}(x) \vee \psi_{A_1}(y)) \vee (\psi_{A_2}(x) \vee \psi_{A_2}(y)) \vee \dots \vee (\psi_{A_p}(x) \vee \psi_{A_p}(y)) \\
 &= (\psi_{A_1}(x) \vee \psi_{A_2}(x) \vee \dots \vee \psi_{A_p}(x)) \vee (\psi_{A_1}(y) \vee \psi_{A_2}(y) \vee \dots \vee \psi_{A_p}(y)) \\
 &= \psi_A(x) \vee \psi_A(y).
 \end{aligned}$$

Therefore, $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of C . Hence, the intersection of a family of neutrosophic subgroups of a group is also a neutrosophic subgroup. □

Remark 3.3. The union of two neutrosophic subgroups of a group may not be a neutrosophic subgroup.

Example 3.4. Let us take the group $(\mathbb{Z}, +)$, the group of integers under usual addition, and let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be two neutrosophic subgroups of $(\mathbb{Z}, +)$ defined by

$$\begin{aligned}
 \mu_A(a) &= \begin{cases} 0.3 & \text{when } a \in 5\mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}, \quad \gamma_A(a) = \begin{cases} 0.2 & \text{when } a \in 5\mathbb{Z} \\ 0 & \text{elsewhere} \end{cases} \\
 \psi_A(a) &= \begin{cases} 0 & \text{when } a \in 5\mathbb{Z} \\ 0.5 & \text{elsewhere} \end{cases}, \quad \mu_B(a) = \begin{cases} 0.15 & \text{when } a \in 3\mathbb{Z} \\ 0 & \text{elsewhere} \end{cases} \\
 \gamma_B(a) &= \begin{cases} 0.1 & \text{when } a \in 3\mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}, \quad \psi_B(a) = \begin{cases} 0.2 & \text{when } a \in 3\mathbb{Z} \\ 0.3 & \text{elsewhere.} \end{cases}
 \end{aligned}$$

Then $A \cup B = (\mu_{A \cup B}, \gamma_{A \cup B}, \psi_{A \cup B})$, where

$$\mu_{A \cup B}(a) = \begin{cases} 0.3 & \text{when } a \in 5\mathbb{Z} \\ 0.15 & \text{when } a \in 3\mathbb{Z} - 5\mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}, \quad \gamma_{A \cup B}(a) = \begin{cases} 0.2 & \text{when } a \in 5\mathbb{Z} \\ 0.1 & \text{when } a \in 3\mathbb{Z} - 5\mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}$$

$$\psi_{A \cup B}(a) = \begin{cases} 0 & \text{when } a \in 5\mathbb{Z} \\ 0.2 & \text{when } a \in 3\mathbb{Z} - 5\mathbb{Z} \\ 0.3 & \text{elsewhere.} \end{cases}$$

Here $\mu_A(5 + (-3)) = \mu_A(2) = 0$, but $\mu_A(5) \wedge \mu_A(-3) = \min\{0.3, 0.15\} = 0.15$. So, $\mu_A(5 + (-3)) \not\geq \mu_A(5) \wedge \mu_A(-3)$. Also, $\gamma_A(5 + (-3)) = \gamma_A(2) = 0$, but $\gamma_A(5) \wedge \gamma_A(-3) = \min\{0.2, 0.1\} = 0.1$. So, $\gamma_A(5 + (-3)) \not\geq \gamma_A(5) \wedge \gamma_A(-3)$. Again, $\psi_A(5 + (-3)) = \psi_A(2) = 0.3$, but $\psi_A(5) \vee \psi_A(-3) = \max\{0, 0.2\} = 0.2$. So, $\psi_A(5 + (-3)) \not\leq \psi_A(5) \vee \psi_A(-3)$. Hence, $A \cup B = (\mu_{A \cup B}, \gamma_{A \cup B}, \psi_{A \cup B})$ is not a neutrosophic subgroup of \mathbb{Z} .

Proposition 3.5. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group (C, \circ) , then $\mu_A(x^k) \geq \mu_A(x)$, $\gamma_A(x^k) \geq \gamma_A(x)$, and $\psi_A(x^k) \leq \psi_A(x)$ for all $x \in C$ and $k \in \mathbb{N}$. Here $x^k = x \circ x \circ \dots \circ x$ (k times).*

Proof. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group (C, \circ) , we have $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) \geq \gamma_A(x) \vee \gamma_A(y)$, and $\psi_A(x \circ y) \leq \psi_A(x) \vee \psi_A(y)$ for all $x, y \in C$. So, $\mu_A(x) = \mu_A(x \circ x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$, $\gamma_A(x) = \gamma_A(x \circ x) \leq \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x)$, and $\psi_A(x) = \psi_A(x \circ x) \geq \psi_A(x) \wedge \psi_A(x) = \psi_A(x)$ for all $x \in C$. Thus, by induction, we can show that $\mu_A(x^k) \geq \mu_A(x)$, $\gamma_A(x^k) \geq \gamma_A(x)$, and $\psi_A(x^k) \leq \psi_A(x)$ for all $x \in C$ and $k \in \mathbb{N}$. \square

Proposition 3.6. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . If $\mu(x) \neq \mu(y)$, $\gamma(x) \neq \gamma(y)$, and $\psi(x) \neq \psi(y)$, then $\mu_A(x \circ y) = \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) = \gamma_A(x) \wedge \gamma_A(y)$, and $\psi_A(x \circ y) = \psi_A(x) \vee \psi_A(y)$, respectively, for all $x, y \in C$.*

Proof. Let us assume that $\mu(x) \neq \mu(y)$, $\gamma(x) \neq \gamma(y)$, and $\psi(x) \neq \psi(y)$. So, $\mu_A(x) > \mu_A(y)$, $\gamma_A(x) > \gamma_A(y)$, and $\psi_A(x) < \psi_A(y)$ for all $x, y \in C$. Now,

$$\begin{aligned} \mu_A(y) &= \mu_A(x^{-1} \circ x \circ y) \\ &\geq \mu_A(x^{-1}) \wedge \mu_A(x \circ y) \\ &= \mu_A(x) \wedge \mu_A(x \circ y) \\ &\geq \mu_A(x \circ y), \end{aligned}$$

otherwise $\mu_A(y) \geq \mu_A(x)$, a contradiction. This shows that $\mu_A(y) \geq \mu_A(x \circ y)$. Again, $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(y)$. Therefore, $\mu_A(x \circ y) \geq \mu_A(y)$. So, $\mu_A(x \circ y) = \mu_A(y) = \mu_A(x) \wedge \mu_A(y)$ for all $x, y \in C$. Similarly, when $\mu(x) < \mu(y)$, this result also holds. Again,

$$\begin{aligned} \gamma_A(y) &= \gamma_A(x^{-1} \circ x \circ y) \\ &\geq \gamma_A(x^{-1}) \wedge \gamma_A(x \circ y) \\ &= \gamma_A(x) \wedge \gamma_A(x \circ y) \\ &\geq \gamma_A(x \circ y), \end{aligned}$$

otherwise $\gamma_A(y) \geq \gamma_A(x)$, a contradiction. This shows that $\gamma_A(y) \geq \gamma_A(x \circ y)$. Again, $\gamma_A(x \circ y) \geq \gamma_A(x) \wedge \gamma_A(y) = \gamma_A(y)$. Therefore, $\gamma_A(x \circ y) \geq \gamma_A(y)$. So, $\gamma_A(x \circ y) = \gamma_A(y) = \gamma_A(x) \wedge \gamma_A(y)$ for all $x, y \in C$. Similarly, when $\gamma(x) < \gamma(y)$, this result also holds. Also,

$$\begin{aligned} \psi_A(y) &= \psi_A(x^{-1} \circ x \circ y) \\ &\leq \psi_A(x^{-1}) \vee \psi_A(x \circ y) \\ &= \psi_A(x) \vee \psi_A(x \circ y) \\ &\leq \psi_A(x \circ y), \end{aligned}$$

otherwise $\psi_A(y) \leq \psi_A(x)$, a contradiction. Thus, $\psi_A(y) \leq \psi_A(x \circ y)$. Again, $\psi_A(x \circ y) \leq \psi_A(x) \vee \psi_A(y) = \psi_A(y)$. Therefore, $\psi_A(x \circ y) \leq \psi_A(y)$. Therefore, $\psi_A(x \circ y) = \psi_A(y) = \psi_A(x) \vee \psi_A(y)$ for all $x, y \in C$. Similarly, for $\psi(x) > \psi(y)$, this result also holds. Hence, $\mu_A(x \circ y) = \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) = \gamma_A(x) \wedge \gamma_A(y)$, and $\psi_A(x \circ y) = \psi_A(x) \vee \psi_A(y)$, respectively, for all $x, y \in C$. \square

Proposition 3.7. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) with e as the identity element and $x \in C$. Then*

$$(\forall y \in C) \begin{pmatrix} \mu_A(x) = \mu_A(e) \Rightarrow \mu_A(x \circ y) = \mu_A(y) \\ \gamma_A(x) = \gamma_A(e) \Rightarrow \gamma_A(x \circ y) = \gamma_A(y) \\ \psi_A(x) = \psi_A(e) \Rightarrow \psi_A(x \circ y) = \psi_A(y) \end{pmatrix}.$$

Proof. Let us assume that $\mu_A(x) = \mu_A(e)$, $\gamma_A(x) = \gamma_A(e)$, and $\psi_A(x) = \psi_A(e)$. So, $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(e) \wedge \mu_A(y) = \mu_A(y)$, since $\mu_A(e) \geq \mu_A(y)$ for all $y \in C$. Also,

$$\begin{aligned} \mu_A(y) &= \mu_A(x^{-1} \circ x \circ y) \\ &\geq \mu_A(x^{-1}) \wedge \mu_A(x \circ y) \\ &= \mu_A(x) \wedge \mu_A(x \circ y) \\ &= \mu_A(e) \wedge \mu_A(x \circ y) \\ &= \mu_A(x \circ y), \text{ since } \mu_A(e) \geq \mu_A(y) \text{ for all } y \in C. \end{aligned}$$

Therefore, $\mu_A(x \circ y) = \mu_A(y)$ for all $y \in C$, when $\mu_A(x) = \mu_A(e)$. Again, $\gamma_A(x \circ y) \geq \gamma_A(x) \wedge \gamma_A(y) = \gamma_A(e) \wedge \gamma_A(y) = \gamma_A(y)$, since $\gamma_A(e) \geq \gamma_A(y)$ for all $y \in C$. Also,

$$\begin{aligned} \gamma_A(y) &= \gamma_A(x^{-1} \circ x \circ y) \\ &\geq \gamma_A(x^{-1}) \wedge \gamma_A(x \circ y) \\ &= \gamma_A(x) \wedge \gamma_A(x \circ y) \\ &= \gamma_A(e) \wedge \gamma_A(x \circ y) \\ &= \gamma_A(x \circ y), \text{ since } \gamma_A(e) \geq \gamma_A(y) \text{ for all } y \in C. \end{aligned}$$

Therefore, $\gamma_A(x \circ y) = \gamma_A(y)$ for all $y \in C$, when $\gamma_A(x) = \gamma_A(e)$. Also, $\psi_A(x \circ y) \leq \psi_A(x) \vee \psi_A(y) = \psi_A(e) \vee \psi_A(y) = \psi_A(y)$, since $\psi_A(e) \leq \psi_A(y)$ for all $y \in C$. Now,

$$\begin{aligned} \psi_A(y) &= \psi_A(x^{-1} \circ x \circ y) \\ &\leq \psi_A(x^{-1}) \vee \psi_A(x \circ y) \\ &= \psi_A(x) \vee \psi_A(x \circ y) \\ &= \psi_A(e) \vee \psi_A(x \circ y) \\ &= \psi_A(x \circ y), \text{ since } \psi_A(e) \leq \psi_A(y) \text{ for all } y \in C. \end{aligned}$$

Hence, $\psi_A(x \circ y) = \psi_A(y)$ for all $y \in C$, when $\psi_A(x) = \psi_A(e)$. □

Theorem 3.8. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . Then the set $N = \{x \in C : \mu_A(e) = \mu_A(x), \gamma_A(e) = \gamma_A(x), \psi_A(e) = \psi_A(x)\}$ forms a subgroup of C , where e is the identity element of C .

Proof. Suppose $N = \{x \in C : \mu_A(e) = \mu_A(x), \gamma_A(e) = \gamma_A(x), \psi_A(e) = \psi_A(x)\}$. Clearly, N is nonempty, as $e \in N$. To show that N is a subgroup of C , for $x \circ y^{-1} \in N$ for all $x, y \in N$. Let $x, y \in N$. Then $\mu_A(x) = \mu_A(e) = \mu_A(y)$, $\gamma_A(x) = \gamma_A(e) = \gamma_A(y)$, and $\psi_A(x) = \psi_A(e) = \psi_A(y)$. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of C , we have

$$\begin{aligned} \mu_A(x \circ y^{-1}) &\geq \mu_A(x) \wedge \mu_A(y^{-1}) \\ &= \mu_A(x) \wedge \mu_A(y) \\ &= \mu_A(e) \wedge \mu_A(e) \\ &= \mu_A(e), \\ \gamma_A(x \circ y^{-1}) &\geq \gamma_A(x) \wedge \gamma_A(y^{-1}) \\ &= \gamma_A(x) \wedge \gamma_A(y) \\ &= \gamma_A(e) \wedge \gamma_A(e) \\ &= \gamma_A(e), \\ \psi_A(x \circ y^{-1}) &\leq \psi_A(x) \vee \psi_A(y^{-1}) \\ &= \psi_A(x) \vee \psi_A(y) \\ &= \psi_A(e) \vee \psi_A(e) \\ &= \psi_A(e). \end{aligned}$$

Again, we have $\mu_A(e) \geq \mu_A(x \circ y^{-1})$, $\gamma_A(e) \geq \gamma_A(x \circ y^{-1})$, and $\psi_A(e) \leq \psi_A(x \circ y^{-1})$. Therefore, $\mu_A(x \circ y^{-1}) = \mu_A(e)$, $\gamma_A(x \circ y^{-1}) = \gamma_A(e)$, and $\psi_A(x \circ y^{-1}) = \psi_A(e)$. So, $x \circ y^{-1} \in N$. Hence, N is a subgroup of C . □

Definition 3.9. Let C be a crisp set. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on the set C . For $\alpha, \beta, \delta \in [0, 1]$, the set $\psi_{(\alpha, \beta, \delta)} = \{x \in C : \mu_A(x) \geq \alpha, \gamma_A(x) \geq \beta, \psi_A(x) \leq \delta\}$ is called a neutrosophic level subset of the neutrosophic set A of C , where $0 \leq \alpha + \beta + \delta \leq 1$.

Proposition 3.10. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be two neutrosophic sets on the universal set C . Then

- (1) $A_{(\alpha, \beta, \delta)} \subseteq A_{(\epsilon, \omega, \sigma)}$ if $\epsilon \leq \alpha$, $\omega \leq \beta$ and $\delta \leq \sigma$ for $\epsilon, \beta, \alpha, \omega, \delta, \sigma \in [0, 1]$,
- (2) $A \subseteq B \Rightarrow A_{(\alpha, \beta, \delta)} \subseteq B_{(\alpha, \beta, \delta)}$ for $\alpha, \beta, \delta \in [0, 1]$.

Proof. (1) Let $x \in A_{(\alpha, \beta, \delta)}$. Then $\mu_A(x) \geq \alpha$, $\gamma_A(x) \geq \beta$, and $\psi_A(x) \leq \delta$. We have $\epsilon \leq \alpha$, $\omega \leq \beta$, and $\delta \leq \sigma$. So, $\epsilon \leq \alpha \leq \mu_A(x)$, $\gamma_A(x) \geq \beta \geq \omega$, and $\psi_A(x) \leq \delta \leq \sigma$. Therefore, $x \in A_{(\epsilon, \omega, \sigma)}$. Hence, $\epsilon \leq \alpha$, $\omega \leq \beta$, and $\delta \leq \sigma \Rightarrow A_{(\alpha, \beta, \delta)} \subseteq A_{(\epsilon, \omega, \sigma)}$.

(2) Since $A \subseteq B$, we have $\mu_A(x) \leq \mu_B(x)$, $\gamma_A(x) \leq \gamma_B(x)$, and $\psi_A(x) \geq \psi_B(x)$ for all $x \in C$. Let $x \in A_{(\alpha, \beta, \delta)}$. This implies that $\mu_A(x) \geq \alpha$, $\gamma_A(x) \geq \beta$, and $\psi_A(x) \leq \delta$. So, $\alpha \leq \mu_A(x) \leq \mu_B(x)$, $\beta \leq \gamma_A(x) \leq \gamma_B(x)$, and $\psi_B(x) \leq \psi_A(x) \leq \delta$. This shows that $\alpha \leq \mu_B(x)$, $\beta \leq \gamma_B(x)$, and $\psi_B(x) \leq \delta$. Therefore, $x \in B_{(\alpha, \beta, \delta)}$. Hence, $A_{(\alpha, \beta, \delta)} \subseteq B_{(\alpha, \beta, \delta)}$. □

Proposition 3.11. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group (C, \circ) . Then, the neutrosophic level subset $A_{(\alpha, \beta, \delta)}$ forms a subgroup of C , where $\alpha \leq \mu_A(e)$, $\beta \leq \gamma_A(e)$, and $\delta \geq \psi_A(e)$, and e is the identity element of C .

Proof. Here $A_{(\alpha, \beta, \delta)} = \{x \in C : \mu_A(x) \geq \alpha, \gamma_A(x) \geq \beta, \psi_A(x) \leq \delta\}$. Clearly, $A_{(\alpha, \beta, \delta)}$ is nonempty, as $e \in A_{(\alpha, \beta, \delta)}$. For $A_{(\alpha, \beta, \delta)}$ is a subgroup of C , we have to show that for all $x, y \in A_{(\alpha, \beta, \delta)}$, $x \circ y^{-1} \in A_{(\alpha, \beta, \delta)}$. Let $x, y \in A_{(\alpha, \beta, \delta)}$. Then $\mu_A(x) \geq \alpha$, $\gamma_A(x) \geq \beta$, and $\psi_A(x) \leq \delta$ and $\mu_A(y) \geq \alpha$, $\gamma_A(y) \geq \beta$, and $\psi_A(y) \leq \delta$. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of C , we have

$$\begin{aligned} \mu_A(x \circ y^{-1}) &\geq \mu_A(x) \wedge \mu_A(y^{-1}) \\ &= \mu_A(x) \wedge \mu_A(y) \\ &\geq \alpha \wedge \alpha \\ &= \alpha, \\ \gamma_A(x \circ y^{-1}) &\geq \gamma_A(x) \wedge \gamma_A(y^{-1}) \\ &= \gamma_A(x) \wedge \gamma_A(y) \\ &\geq \beta \wedge \beta \\ &= \beta, \\ \psi_A(x \circ y^{-1}) &\leq \psi_A(x) \vee \psi_A(y^{-1}) \\ &= \psi_A(x) \vee \psi_A(y) \\ &\leq \delta \vee \delta \\ &= \delta. \end{aligned}$$

Therefore, $x \circ y^{-1} \in A_{(\alpha, \beta, \delta)}$. Hence, $A_{(\alpha, \beta, \delta)}$ is a subgroup of C . □

Definition 3.12. The subgroup $A_{(\alpha, \beta, \delta)}$ of the group (C, \circ) is called the neutrosophic level subgroup of the neutrosophic subgroup $A = (\mu_A, \gamma_A, \psi_A)$.

Proposition 3.13. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a group (C, \circ) . If the neutrosophic level subgroup $A_{(\alpha, \beta, \delta)}$ is a subgroup of C , where $\alpha \leq \mu_A(e)$, $\beta \leq \gamma_A(e)$, and $\delta \geq \psi_A(e)$, then $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of C .

Proof. Let $x, y \in C$. Given that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic set of C and $A_{(\alpha, \beta, \delta)}$ is a subgroup of C . Let us assume that $\mu_A(x) = \alpha_1$, $\mu_A(y) = \alpha_2$ with $\alpha_1 < \alpha_2$, $\gamma_A(x) = \beta_1$, $\gamma_A(y) = \beta_2$ with $\beta_1 < \beta_2$, and $\psi_A(x) = \delta_1$, $\psi_A(y) = \delta_2$ with $\delta_1 > \delta_2$. This implies that $x \in A_{(\alpha_1, \beta_1, \delta_1)}$ and $y \in A_{(\alpha_2, \beta_2, \delta_2)}$. Since $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, and $\delta_1 > \delta_2$, we have $A_{(\alpha_2, \beta_2, \delta_2)} \subseteq A_{(\alpha_1, \beta_1, \delta_1)}$. So, $y \in A_{(\alpha_1, \beta_1, \delta_1)}$. Now, $x \in A_{(\alpha_1, \beta_1, \delta_1)}$ and $y \in A_{(\alpha_1, \beta_1, \delta_1)}$. So, $x \circ y \in A_{(\alpha_1, \beta_1, \delta_1)}$, since $A_{(\alpha_1, \beta_1, \delta_1)}$ is a subgroup of C . Therefore, $\mu_A(x \circ y) \geq \alpha_1$, $\gamma_A(x \circ y) \geq \beta_1$, and $\psi_A(x \circ y) \leq \delta_1 \Rightarrow \mu_A(x \circ y) \geq \alpha_1 \wedge \alpha_2$, $\gamma_A(x \circ y) \geq \beta_1 \wedge \beta_2$, and $\psi_A(x \circ y) \leq \delta_1 \vee \delta_2$, since $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, and $\delta_1 > \delta_2 \Rightarrow \mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) \geq \gamma_A(x) \wedge \gamma_A(y)$, and $\psi_A(x \circ y) \leq \psi_A(x) \vee \psi_A(y)$. Again, $x \in A_{(\alpha_1, \beta_1, \delta_1)} \Rightarrow x^{-1} \in A_{(\alpha_1, \beta_1, \delta_1)}$, since $A_{(\alpha_1, \beta_1, \delta_1)}$ is a subgroup of C . Also, $\mu_A(x^{-1}) \geq \alpha_1$, $\gamma_A(x^{-1}) \geq \beta_1$, and $\psi_A(x^{-1}) \leq \delta_1 \Rightarrow \mu_A(x^{-1}) \geq \mu_A(x)$, $\gamma_A(x^{-1}) \geq \gamma_A(x)$, and $\psi_A(x^{-1}) \leq \psi_A(x)$. Since $x, y \in C$ are arbitrary, $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) \geq \gamma_A(x) \wedge \gamma_A(y)$, and $\psi_A(x \circ y) \leq \psi_A(x) \vee \psi_A(y)$ for all $x, y \in C$, and $\mu_A(x^{-1}) \geq \mu_A(x)$, $\gamma_A(x^{-1}) \geq \gamma_A(x)$, $\psi_A(x^{-1}) \leq \psi_A(x)$ for all $x \in C$. Hence, $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of C . □

Proposition 3.14. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of a group (C, \circ) . Then, the neutrosophic level subset $A_{(\alpha, \beta, \delta)}$ forms a normal subgroup of C , where $\alpha \leq \mu_A(e)$, $\beta \leq \gamma_A(e)$ and $\delta \geq \psi_A(e)$, and e is the identity element in C .

Proof. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of C , we have for all $u, k \in C$, $\mu_A(k \circ u \circ k^{-1}) = \mu_A(u)$, $\gamma_A(k \circ u \circ k^{-1}) = \gamma_A(u)$, and $\psi_A(k \circ u \circ k^{-1}) = \psi_A(u)$. By Proposition 3.11, $A_{(\alpha, \beta, \delta)} = \{x \in G : \mu_A(x) \geq \alpha, \gamma_A(x) \geq \beta, \psi_A(x) \leq \delta\}$ is a subgroup of C . Let $k \in G$ and $u \in A_{(\alpha, \beta, \delta)}$. Then $\mu_A(k \circ u \circ k^{-1}) = \mu_A(u) \geq \alpha$, $\gamma_A(k \circ u \circ k^{-1}) = \gamma_A(u) \geq \beta$, and $\psi_A(k \circ u \circ k^{-1}) = \psi_A(u) \leq \delta$. Thus, $k \circ u \circ k^{-1} \in A_{(\alpha, \beta, \delta)}$. Hence, $A_{(\alpha, \beta, \delta)}$ is a normal subgroup of C . \square

In this part, we will discuss the effect of group homomorphisms on neutrosophic subgroups.

Theorem 3.15. Let (C_1, \circ_1) and (C_2, \circ_2) be two groups. Let $f : C_1 \rightarrow C_2$ be a surjective homomorphism and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of C_1 . Then $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic subgroup of C_2 .

Proof. Since $f : C_1 \rightarrow C_2$ is a surjective homomorphism, we have $f(C_1) = C_2$. Let $x_2, y_2 \in C_2$. Suppose $x_2 = f(x_1)$ and $y_2 = f(y_1)$ for some $x_1, y_1 \in C_1$. Now,

$$\begin{aligned} (f(\mu_A))(x_2 \circ_2 y_2) &= \bigvee \{ \mu_A(t) : t \in C_1, f(t) = x_2 \circ_2 y_2 \} \\ &= \bigvee \{ \mu_A(t) : t \in C_1, f(t) = x_2 \circ_2 y_2 \} \\ &\geq \bigvee \{ \mu_A(x_1 \circ_1 y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2 \} \\ &\geq \bigvee \{ \mu_A(x_1) \wedge \mu_A(y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2 \} \\ &= \bigvee \{ \mu_A(x_1) : x_1 \in C_1, f(x_1) = x_2 \} \wedge \\ &\quad \bigvee \{ \mu_A(y_1) : y_1 \in C_1, f(y_1) = y_2 \} \\ &= (f(\mu_A))(x_2) \wedge (f(\mu_A))(y_2), \\ \\ (f(\gamma_A))(x_2 \circ_2 y_2) &= \bigvee \{ \gamma_A(t) : t \in C_1, f(t) = x_2 \circ_2 y_2 \} \\ &= \bigvee \{ \gamma_A(t) : t \in C_1, f(t) = x_2 \circ_2 y_2 \} \\ &\geq \bigvee \{ \gamma_A(x_1 \circ_1 y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2 \} \\ &\geq \bigvee \{ \gamma_A(x_1) \wedge \gamma_A(y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2 \} \\ &= \bigvee \{ \gamma_A(x_1) : x_1 \in C_1, f(x_1) = x_2 \} \wedge \\ &\quad \bigvee \{ \gamma_A(y_1) : y_1 \in C_1, f(y_1) = y_2 \} \\ &= (f(\gamma_A))(x_2) \wedge (f(\gamma_A))(y_2), \\ \\ (f(\psi_A))(x_2 \circ_2 y_2) &= \bigwedge \{ \psi_A(t) : t \in C_1, f(t) = x_2 \circ_2 y_2 \} \\ &= \bigwedge \{ \psi_A(t) : t \in C_1, f(t) = x_2 \circ_2 y_2 \} \\ &\leq \bigwedge \{ \psi_A(x_1 \circ_1 y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2 \} \\ &\leq \bigwedge \{ \psi_A(x_1) \vee \psi_A(y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2 \} \\ &= \bigwedge \{ \psi_A(x_1) : x_1 \in C_1, f(x_1) = x_2 \} \vee \\ &\quad \bigwedge \{ \psi_A(y_1) : y_1 \in C_1, f(y_1) = y_2 \} \\ &= (f(\psi_A))(x_2) \vee (f(\psi_A))(y_2). \end{aligned}$$

Again,

$$\begin{aligned} (f(\mu_A))(x_2^{-1}) &= \bigvee \{ \mu_A(x) : x \in C_1, f(x) = (x_2^{-1}) \} \\ &= \bigvee \{ \mu_A(x^{-1}) : x^{-1} \in C_1, f(x^{-1}) = x_2 \} \\ &= (f(\mu_A))(x_2), \\ \\ (f(\gamma_A))(x_2^{-1}) &= \bigvee \{ \gamma_A(x) : x \in C_1, f(x) = (x_2^{-1}) \} \\ &= \bigvee \{ \gamma_A(x^{-1}) : x^{-1} \in C_1, f(x^{-1}) = x_2 \} \\ &= (f(\gamma_A))(x_2), \\ \\ (f(\psi_A))(x_2^{-1}) &= \bigwedge \{ \psi_A(x) : x \in C_1, f(x) = (x_2^{-1}) \} \\ &= \bigwedge \{ \psi_A(x^{-1}) : x^{-1} \in C_1, f(x^{-1}) = x_2 \} \\ &= (f(\psi_A))(x_2). \end{aligned}$$

Hence, $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic subgroup of C_2 . \square

Theorem 3.16. Let (C_1, \circ_1) and (C_2, \circ_2) be two groups. Let f be a bijective homomorphism from C_1 to C_2 and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of C_2 . Then $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic subgroup of C_1 .

Proof. Let $x_1, y_1 \in C_1$. Now,

$$\begin{aligned} (f^{-1}(\mu_A))(x_1 \circ_1 y_1) &= \mu_A(f(x_1 \circ_1 y_1)) \\ &= \mu_A(f(x_1) \circ_2 f(y_1)) \\ &\geq \mu_A(f(x_1)) \wedge \mu_A(f(y_1)) \\ &= (f^{-1}(\mu_A))(x_1) \wedge (f^{-1}(\mu_A))(y_1), \\ (f^{-1}(\gamma_A))(x_1 \circ_1 y_1) &= \gamma_A(f(x_1 \circ_1 y_1)) \\ &= \gamma_A(f(x_1) \circ_2 f(y_1)) \\ &\geq \gamma_A(f(x_1)) \wedge \gamma_A(f(y_1)) \\ &= (f^{-1}(\gamma_A))(x_1) \wedge (f^{-1}(\gamma_A))(y_1), \\ (f^{-1}(\psi_A))(x_1 \circ_1 y_1) &= \psi_A(f(x_1 \circ_1 y_1)) \\ &= \psi_A(f(x_1) \circ_2 f(y_1)) \\ &\leq \psi_A(f(x_1)) \vee \psi_A(f(y_1)) \\ &= (f^{-1}(\psi_A))(x_1) \vee (f^{-1}(\psi_A))(y_1). \end{aligned}$$

Again,

$$\begin{aligned} (f^{-1}(\mu_A))(x_1^{-1}) &= \mu_A(f(x_1^{-1})) \\ &= \mu_A(f(x_1)^{-1}) \\ &= \mu_A(f(x_1)) \\ &= (f^{-1}(\mu_A))(x_1), \\ (f^{-1}(\gamma_A))(x_1^{-1}) &= \gamma_A(f(x_1^{-1})) \\ &= \gamma_A(f(x_1)^{-1}) \\ &= \gamma_A(f(x_1)) \\ &= (f^{-1}(\gamma_A))(x_1), \\ (f^{-1}(\psi_A))(x_1^{-1}) &= \psi_A(f(x_1^{-1})) \\ &= \psi_A(f(x_1)^{-1}) \\ &= \psi_A(f(x_1)) \\ &= (f^{-1}(\psi_A))(x_1). \end{aligned}$$

Hence, $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic subgroup of C_1 . □

Theorem 3.17. Let (C_1, \circ_1) and (C_2, \circ_2) be two groups. Let f be a surjective homomorphism from C_1 to C_2 and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of C_1 . Then $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic normal subgroup of C_2 .

Proof. In view of Theorem 3.15, we can state that $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic subgroup of C_2 . Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of C_1 and by Proposition 2.6, we have $\mu_A(x_1 \circ_1 y_1) = \mu_A(y_1 \circ_1 x_1)$, $\gamma_A(x_1 \circ_1 y_1) = \gamma_A(y_1 \circ_1 x_1)$, and $\psi_A(x_1 \circ_1 y_1) = \psi_A(y_1 \circ_1 x_1)$ for all $x_1, y_1 \in C_1$. Let x_2 and y_2 be two elements of C_2 . Suppose that there exist unique x_1 and $y_1 \in C_1$ such that $x_2 = f(x_1)$ and $y_2 = f(y_1)$. Now,

$$\begin{aligned} f(\mu_A)(x_2 \circ_2 y_2) &= \vee\{\mu_A(l) : l \in C_1, f(l) = x_2 \circ_2 y_2\} \\ &= \vee\{\mu_A(x_1 \circ_1 y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2\} \\ &= \vee\{\mu_A(l) : l \in C_1, f(l) = y_2 \circ_2 x_2\} \\ &= (f(\mu_A))(y_2 \circ_2 x_2), \\ f(\gamma_A)(x_2 \circ_2 y_2) &= \vee\{\gamma_A(l) : l \in C_1, f(l) = x_2 \circ_2 y_2\} \\ &= \vee\{\gamma_A(x_1 \circ_1 y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2\} \\ &= \vee\{\gamma_A(l) : l \in C_1, f(l) = y_2 \circ_2 x_2\} \\ &= (f(\gamma_A))(y_2 \circ_2 x_2), \\ f(\psi_A)(x_2 \circ_2 y_2) &= \wedge\{\psi_A(l) : l \in C_1, f(l) = x_2 \circ_2 y_2\} \\ &= \wedge\{\psi_A(x_1 \circ_1 y_1) : x_1, y_1 \in C_1, f(x_1) = x_2, f(y_1) = y_2\} \\ &= \wedge\{\psi_A(l) : l \in C_1, f(l) = y_2 \circ_2 x_2\} \\ &= (f(\psi_A))(y_2 \circ_2 x_2). \end{aligned}$$

Hence, by Proposition 2.6, we have $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic normal subgroup of C_2 . □

Theorem 3.18. Let (C_1, \circ_1) and (C_2, \circ_2) be two groups. Let f be a bijective homomorphism from C_1 to C_2 and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of C_2 . Then $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic normal subgroup of C_1 .

Proof. In view of Theorem 3.16, we can state that $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic subgroup of C_1 . Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of C_2 and by Proposition 2.6, we have $\mu_A(x_2 \circ_2 y_2) = \mu_A(y_2 \circ_2 x_2)$, $\gamma_A(x_2 \circ_2 y_2) = \gamma_A(y_2 \circ_2 x_2)$, and $\psi_A(x_2 \circ_2 y_2) = \psi_A(y_2 \circ_2 x_2)$ for all $x_2, y_2 \in C_2$. Let x_1 and y_1 be two elements of C_1 . Then

$$\begin{aligned} f^{-1}(\mu_A)(x_1 \circ_1 y_1) &= \mu_A(f(x_1 \circ_1 y_1)) \\ &= \mu_A(f(x_1) \circ_2 f(y_1)) \\ &= \mu_A(f(y_1 \circ_1 x_1)) \\ &= (f^{-1}(\mu_A))(y_1 \circ_1 x_1), \end{aligned}$$

$$\begin{aligned} f^{-1}(\gamma_A)(x_1 \circ_1 y_1) &= \gamma_A(f(x_1 \circ_1 y_1)) \\ &= \gamma_A(f(x_1) \circ_2 f(y_1)) \\ &= \gamma_A(f(y_1 \circ_1 x_1)) \\ &= (f^{-1}(\gamma_A))(y_1 \circ_1 x_1), \end{aligned}$$

$$\begin{aligned} f^{-1}(\psi_A)(x_1 \circ_1 y_1) &= \psi_A(f(x_1 \circ_1 y_1)) \\ &= \psi_A(f(x_1) \circ_2 f(y_1)) \\ &= \psi_A(f(y_1 \circ_1 x_1)) \\ &= (f^{-1}(\psi_A))(y_1 \circ_1 x_1). \end{aligned}$$

Hence, by Proposition 2.6, we have $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic normal subgroup of C_1 . \square

4 Conclusion

This paper introduced neutrosophic subgroups and neutrosophic normal subgroups, extending traditional subgroup structures using neutrosophic sets. We established fundamental properties, showing that while intersections of neutrosophic subgroups remain subgroups, unions do not necessarily do so. We explored conditions for neutrosophic normal subgroups, characterized their behavior under cosets and homomorphisms, and examined neutrosophic level subsets. Our findings contribute to neutrosophic algebra, enhancing its relevance in uncertainty modeling and information sciences. Future research may explore computational implementations and broader algebraic integrations.

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