On the characterization of Harmonized fuzzy subgroups θ Open set

Eman A. AbuHijleh[1], Sara A. Khalil[2], Sana Abu-Ghurra[2], Ghada Alafifi[3]
[1]Basic sciences department, Al-Zarqa University College, Al-Balqa Applied University, Amman, Jordan
[2]Mathematics Department, Faculty of Science, Applied Science Private University, Amman, Jordan
[3]Mathematics Department, Science College, Philadelphia University, Amman, Jordan

Emails: emanhijleh@bau.edu.jo.; s_khalil@asu.edu.jo.; s_ abugurrah@asu.edu.jo.; gafifi@philadelphia.edu.jo.

Abstract

In this paper, we continue to discuss the concept of harmonized fuzzy subgroups. We present the harmonized fuzzy coset and harmonized fuzzy normal subgroup and their properties. We also study the effect of group homomorphism on harmonized fuzzy subgroups. Finally, we define and study the cartesian product of two harmonized fuzzy subgroups.

Keywords: Harmonized fuzzy coset subgroup; Harmonized fuzzy normal subgroup; Group homomorphism

1 Introduction

In daily life vagueness permeates almost every challenge we face. In 1965, as a development of the classical set to dominate vagueness in practical problems, Zadeh[1] introduced fuzzy set (FS). Building upon this foundation and put the limitations of FS, in handling vagueness and uncertainties, Atanassov introduced the intuitionistic fuzzy set (IFS)[2] (1986), and defined as a triple $(u, h(u), g(u))$, where $h(u)+g(u) \leq 1$. Atanassov showed that fuzzy sets do not fully capture the imprecision inherent in vagueness, i.e., Zadeh's FS definition. Atanassov added the degree of nonmembership with degree of membership of $u \in X$. Note that, for membership $(u, h(u))$ and its complement $g(u)$; $g(u) = 1 - h(u)$, i.e., nonmembership. However, in real-life scenarios, hesitations towards both membership and nonmembership exist. So that, he define the degree of indeterminacy; $\pi(u) = \sqrt{1 - (h(u) + g(u))}$ of $u \in X$. After nearly three decades, the researcher Yager (2013)[3] found that $h(u) + g(u)$ can be greater than 1, hence he defined Pythagorean fuzzy set (PFS) with triple $(u, h(u), g(u))$; $h^2(u) + g^2(u) \leq 1$, where the degree of indeterminacy is $\pi(u) = \sqrt{1 - (h^2(u) + g^2(u))}$. Hence, every IFS is PFS, but a PFS is not necessarily IFS. Moreover 2017,[4] Yager extended these sets to q-rung orthopair fuzzy set with $h^n(u) + g^n(u) \leq 1$ and the degree of indeterminacy is $\pi(u) = \sqrt[n]{1 - (g^n(u) + h^n(u))}$, $n \geq 1$. A specific case was introduced, by Senapati and Yager (2020),[5] called fermatean fuzzy set (FFS) with conditions $h^3(u) + g^3(u) \leq 1$ and $\pi(u) = \sqrt[3]{1 - (h^3(u) + g^3(u))}$, i.e. at $n = 3$.

As fuzzy set allows elements to be belong on set that have degrees vary within the interval $[0, 1]$, as far as, the general theory of the fuzzy group (FG) is concerned and look for the properties of algebraic structure of the FG. In 1971, Rosenfeld[6] introduced the novel concept of fuzzy subgroup.Later on 1981, Das[7] expanded upon this with fuzzy level subgroup, Mukherjee and Bhattacharya[8] (1984) presented the notion of fuzzy normal subgroups and fuzzy cosets. While Choudhury et.al.[9] in 1988, established the various properties of fuzzy subgroups and fuzzy homomorphisms. Subsequent developments, by Dixit et.al.[10] (1990), which explored fuzzy-level subgroups and the union of fuzzy subgroups. In addition, Ajmal and Prajapati[11] (1992) contributed insights into fuzzy normal subgroups, fuzzy coset, and fuzzy quotient subgroups. Further studies by Chakraborty and Khare[12] (1993) examined various properties of fuzzy homomorphisms on fuzzy subgroups.

Moreover, fuzzy subgroups (FSGs) not only studied in details with new properties, but improvement affected it by defining a new FSGs depending on it. Like as, an intuitionistic fuzzy subgroup (IFSG), defined by Biswas (1989).[13] Later on 2020, Pythagorean fuzzy subgroup (PFSG), defined by Bhunia.[14] In 2021 Silambarasan[15] with Onasanya et.al. in 2022,[16] both presented a definition of fermatean fuzzy subgroup (FFSG). Whereas, Onasanya et.al.,[16] added harmonized fuzzy subgroups (HFSG) with FFSG. Note that, HFSG is an improvement of q-rung orthopair fuzzy set to be q-rung ortopair fuzzy subgroup, where Onasanya et.al. paper gave a basic algebraic structure of HFSG. In this paper, we discover more results and investigate there properties.

This paper is divided into six sections. We have preliminaries in section two, which comes after the introduction section. The third section refers to harmonized fuzzy cosets and harmonized fuzzy normal subgroup. The homomorphism on harmonized fuzzy subgroup is covered in section four. Section five represents the cartesian product of two harmonized fuzzy subgroups. In section six there is a final general discussion.

2 Preliminaries

[1] A fuzzy set $\mathcal{A}$ in a given set $\mathcal{X}$ can be defined as a set of ordered pairs $\mathcal{A} = \{(u, h_{\mathcal{A}}(u)) : u \in \mathcal{X}\}$, where $h_{\mathcal{A}}(u)$ is the grade of membership of $u \in \mathcal{X}$ and $h_{\mathcal{A}} : \mathcal{X} \longrightarrow [0, 1]$ is the membership function.

[6] Let $(\mathcal{X}, \diamondsuit)$ be a group and h a fuzzy subset of $\mathcal{X}$. Then h is called a fuzzy subgroup of $\mathcal{X}$, if for any $u, v \in \mathcal{X}$,

1. $h(u\diamondsuit v) \geq \min\{h(u), h(v)\}$,
2. $h(u^{-1}) \geq h(u)$.

[2] Let $\mathcal{X}$ denote a crisp set. An intuitionistic fuzzy set (IFS) on $\mathcal{X}$ is defined as follows: $\mathcal{A} = \{(u, h_{\mathcal{A}}, g_{\mathcal{A}}) : u \in \mathcal{X}\}$, where $h_{\mathcal{A}} \in [0, 1]$ and $g_{\mathcal{A}}(u) \in [0, 1]$ are the degree of membership and non-membership of $u \in \mathcal{X}$, respectively, which satisfy the condition: $0 \leq h_{\mathcal{A}}(u) + g_{\mathcal{A}}(u) \leq 1$, for all $u \in \mathcal{X}$.

[13] Let $\mathcal{A} = \{(u, h_{\mathcal{A}}, g_{\mathcal{A}}) : u \in \mathcal{X}\}$, be an intuitionistic fuzzy subset (IFS) of a group $(\mathcal{X}, \diamondsuit)$. Then $\mathcal{A}$ is said to be an IFSG of $\mathcal{X}$ if the following conditions are satisfied:

(i) $h_{\mathcal{A}}(u\diamondsuit v) \geq \min\{h_{\mathcal{A}}(u), h_{\mathcal{A}}(v)\}$,

(ii) $h_{\mathcal{A}}(u^{-1}) \geq h_{\mathcal{A}}(u)$,

(iii) $g_{\mathcal{A}}(u\diamondsuit v) \leq \max\{g_{\mathcal{A}}(u), g_{\mathcal{A}}(v)\}$,

(iv) $g_{\mathcal{A}}(u^{-1}) \leq g_{\mathcal{A}}(u)$.

A generalized of FS and IFS are called q-rung orthopair fuzzy set $\mathcal{A}$ in $\mathcal{X}$, which was studied by.[4]

[4] Let $\mathcal{A} = \{(u, h_{\mathcal{A}}(u), g_{\mathcal{A}}(v)) : u \in \mathcal{X}\}$, where $h_{\mathcal{A}} : \mathcal{X} \longrightarrow [0, 1]$ and $g_{\mathcal{A}} : \mathcal{X} \longrightarrow [0, 1]$ are respectively, degree of membership and degree of non-membership, for every $u \in \mathcal{X}$ with

$$0 \leq h^n_{\mathcal{A}}(u) + g^n_{\mathcal{A}}(u) \leq 1,$$

and degree of indeterminacy of $u \in \mathcal{X}$ is $\pi(u) = \sqrt[n]{1 - h^n(u) - g^n(u)}$. Then $\mathcal{A}$ is called q-rung ortho-pair fuzzy set.

Note that, if $n = 2$ then we have Pythagorean fuzzy set (PFS),[3] and if $n = 3$ then we have fermatean fuzzy set (FFS).[5]

[4] Let $\mathcal{A}$ be a fuzzy set in $\mathcal{X}$. The complement of $\mathcal{A}$ is defined as $\mathcal{A}' = \{(u, g_{\mathcal{A}}(u)) : u \in \mathcal{X}\}$ where $g^n_{\mathcal{A}}(u) = 1 - h^n_{\mathcal{A}}(u)$ and $h_{\mathcal{A}}(u)$ is the grade of membership of $u \in \mathcal{A}$.

A q-rung orthopair fuzzy set in,[4] is denoted by Harmonized fuzzy set (HFS), in the paper of Onasanya et.al. 2022,[16] where they produced a Harmonized fuzzy subgroup (HFSG).

[16] Let $(\mathcal{X}, \diamondsuit)$ be a group and $\mathcal{A} = \{(u, h_{\mathcal{A}}(u), g_{\mathcal{A}}(u)) :\ u \in \mathcal{X}\}$ a HFS. Then, $\mathcal{A}$ is a Harmonized fuzzy subgroup (HFSG), if for any $u, v \in \mathcal{X}$, the following are satisfied:

(i) $h^n_{\mathcal{A}}(u \diamondsuit v) \geq \min\{h^n_{\mathcal{A}}(u), h^n_{\mathcal{A}}(v)\}$,

(ii) $h^n_{\mathcal{A}}(u^{-1}) \geq h^n_{\mathcal{A}}(u)$,

(iii) $g^n_{\mathcal{A}}(u \diamondsuit v) \leq \max\{g^n_{\mathcal{A}}(u), g^n_{\mathcal{A}}(v)\}$,

(iv) $g^n_{\mathcal{A}}(u^{-1}) \leq g^n_{\mathcal{A}}(u)$.

[16] Let $(\mathcal{X}, \diamondsuit)$ be a group and $\mathcal{A} = \{(u, h_{\mathcal{A}}(u), g_{\mathcal{A}}(u)) :\ u \in \mathcal{X}\}$ a HFSG. Then for every $u \in \mathcal{X}$:

(i) $h^n_{\mathcal{A}}(u^{-1}) = h^n_{\mathcal{A}}(u)$,

(ii) $g^n_{\mathcal{A}}(u^{-1}) = g^n_{\mathcal{A}}(u)$,

(iii) $h^n_{\mathcal{A}}(e) \geq g^n_{\mathcal{A}}(u)$,

(iv) $g^n_{\mathcal{A}}(e) \leq g^n_{\mathcal{A}}(u)$.

Note that, in HFS $\mathcal{A} = \{(u, h_{\mathcal{A}}(u), g_{\mathcal{A}}(u)) :\ u \in \mathcal{X}\}$, we have

$$0 \leq h^n_{\mathcal{A}} + g^n_{\mathcal{A}} \leq \cdots \leq h^3_{\mathcal{A}} + g^3_{\mathcal{A}} \leq h^2_{\mathcal{A}} + g^2_{\mathcal{A}} \leq h_{\mathcal{A}} + g_{\mathcal{A}} \leq 1, \tag{1}$$

where $h_{\mathcal{A}}$ and $g_{\mathcal{A}}$ are the membership and non-membership functions, respectively. There are some operations on the HFS $\mathcal{A} = \{(u, h_{\mathcal{A}}(u), g_{\mathcal{A}}(u)) :\ u \in \mathcal{X}\}$ that can be obtained from[4] and.[16]

Moreover, the HFG $\mathcal{A} = \{(u, h_{\mathcal{A}}(u), g_{\mathcal{A}}(u)) :\ u \in \mathcal{X}\}$ is an IFG if $n = 1$, where $h_{\mathcal{A}}$ is a fuzzy subgroup of $\mathcal{X}$ and $g_{\mathcal{A}}$ is antifuzzy subgroup.[2] If $n = 2$, it is a Pythagorean fuzzy group (PFG)[13] and it is a Fermatean fuzzy group (FFG) if $n = 3$[16] and[15] (both papers appear in the same tense), since the underlying fuzzy set is a Fermatean fuzzy set (FFS).

[16] Let $(\mathcal{X}, \diamondsuit)$ be a group and $\mathcal{A}$ be a FSG of $\mathcal{X}$. Then

$$FSG \Rightarrow\ IFSG \Rightarrow\ PFSG \Rightarrow\ FFSG \Rightarrow\ HFSG.$$

Proof. Since $0 \leq h_{\mathcal{A}} \leq h_{\mathcal{A}} + g_{\mathcal{A}} \leq 1$, then by previous inequality (1) result follows. □

3 Harmonization fuzzy coset and Harmonization fuzzy normal subgroups

As we mentioned in this paper, the study of HFSG started by Onasanya et.al[16] (2022). In their paper they discuss main properties of HFSG, whereas many group properties need to investigate. Henceforth, in the present section we discuss more properties, such as normality and cosets.

Let $\mathcal{A} = (h_{\mathcal{A}}, g_{\mathcal{A}})$ be HFSG of a group $(\mathcal{X}, \diamondsuit)$. Then for $\nu,\ u \in \mathcal{X}$, the harmonized fuzzy left coset of $\nu\mathcal{A} = (\nu h_{\mathcal{A}}, \nu g_{\mathcal{A}})$, is the HFS which defined by:
$(\nu h_{\mathcal{A}})^n(u) = h^n_{\mathcal{A}}(\nu^{-1}\diamondsuit u)$ for membership, and $(\nu g_{\mathcal{A}})^n(u) = g^n_{\mathcal{A}}(\nu^{-1}\diamondsuit u)$ for nonmembership.
The harmonized fuzzy right coset of $\mathcal{A}\nu = (h_{\mathcal{A}}\nu, g_{\mathcal{A}}\nu)$, is the HFS which defined by:
$(h_{\mathcal{A}}\nu)^n(u) = h^n_{\mathcal{A}}(u\diamondsuit\nu^{-1})$ for membership, and $(g_{\mathcal{A}}\nu)^n(u) = g^n_{\mathcal{A}}(u\diamondsuit\nu^{-1})$ for nonmembership.

Let $\mathcal{A} = (h_{\mathcal{A}}, g_{\mathcal{A}})$ be HFSG of a group $(\mathcal{X}, \diamondsuit)$. Then $\mathcal{A}$ is the harmonized fuzzy normal subgroup (HFNSG) of group $\mathcal{X}$, if $\nu\mathcal{A} = \mathcal{A}\nu$ for all $\nu \in \mathcal{X}$.

Let $\mathcal{A} = (h_{\mathcal{A}}, g_{\mathcal{A}})$ be HFSG of a group $(\mathcal{X}, \diamondsuit)$. Then $\mathcal{A}$ is HFNSG of $\mathcal{X}$ if and only if

$$h^n_{\mathcal{A}}(u\diamondsuit\nu) = h^n_{\mathcal{A}}(\nu\diamondsuit u)\ and\ g^n_{\mathcal{A}}(u\diamondsuit\nu) = g^n_{\mathcal{A}}(\nu\diamondsuit u), \tag{2}$$

for all $u, \nu \in \mathcal{X}$.

Proof. Assume that $\mathcal{A} = (h_{\mathcal{A}}, g_{\mathcal{A}})$ is HFNSG of $\mathcal{X}$. Then $\nu\mathcal{A} = \mathcal{A}\nu, \forall\nu \in \mathcal{X}$, i.e., $\nu h_{\mathcal{A}} = h_{\mathcal{A}}\nu$ and $\nu g_{\mathcal{A}} = g_{\mathcal{A}}\nu$. Now

$$\begin{aligned}(\nu h)^n(u) &= h^n(\nu^{-1}\diamondsuit u)\\ &= h^n(u\diamondsuit\nu^{-1})\end{aligned}. \tag{3}$$

Then we have,

$$\begin{aligned}h^n(\nu\diamondsuit u) &= h^n((\nu^{-1})^{-1})\diamondsuit u)\\ &= h^n(u\diamondsuit(\nu^{-1})^{-1}) = h^n(u\diamondsuit\nu)\end{aligned}, \tag{4}$$

for all $u, \nu \in \mathcal{X}$. Similarly, we can prove $g^n(\nu\diamondsuit u) = g^n(u\diamondsuit\nu)$.

Conversely, assume that $h^n_{\mathcal{A}}(u\diamondsuit\nu) = h^n_{\mathcal{A}}(\nu\diamondsuit u)$ and $g^n_{\mathcal{A}}(u\diamondsuit\nu) = g^n_{\mathcal{A}}(\nu\diamondsuit u)$, for all $u, \nu \in \mathcal{X}$. Then,
$h^n_{\mathcal{A}}(u\diamondsuit(\nu^{-1})^{-1}) = h^n_{\mathcal{A}}((\nu^{-1})^{-1}\diamondsuit u)$, and
$g^n_{\mathcal{A}}(u\diamondsuit(\nu^{-1})^{-1}) = g^n_{\mathcal{A}}((\nu^{-1})^{-1}\diamondsuit u)$. Assume $\nu^{-1} = w$, then we have

$$h^n(u\diamondsuit w^{-1}) = h^n(w^{-1}\diamondsuit u) \quad and \quad g^n(u\diamondsuit w^{-1}) = g^n(w^{-1}\diamondsuit u),$$

for all $u, w \in \mathcal{X}$. i.e.,

$$(h_{\mathcal{A}}w)^n(u) = (wh_{\mathcal{A}})^n(u) \quad and \quad (g_{\mathcal{A}}w)^n(u) = (wg_{\mathcal{A}})^n(u),$$

thus $w\mathcal{A} = \mathcal{A}w$, for all $w \in \mathcal{X}$, which means $\mathcal{A}$ is HFNSG. □

Let $\mathcal{X}$ be a group and $\mathcal{A}$ be a HFSG of $\mathcal{X}$. Then $\mathcal{A}$ is a HFNSG if and only if $h^n_{\mathcal{A}}(\nu\diamondsuit u\diamondsuit\nu^{-1}) = h^n_{\mathcal{A}}(u)$, and $g^n_{\mathcal{A}}(\nu\diamondsuit u\diamondsuit\nu^{-1}) = g^n_{\mathcal{A}}(u)$, for all $u, \nu \in \mathcal{X}$.

Proof. Assume that $\mathcal{A}$ is an HFNSG. Then

$$h^n_{\mathcal{A}}(u\diamondsuit\nu) = h^n_{\mathcal{A}}(\nu\diamondsuit u) \quad and \quad g^n_{\mathcal{A}}(u\diamondsuit\nu) = g^n_{\mathcal{A}}(\nu\diamondsuit u) \tag{5}$$

for all $u, \nu \in \mathcal{X}$. Now,

$$\begin{aligned} h^n(\nu \diamondsuit u \diamondsuit \nu^{-1}) &= h^n((\nu \diamondsuit u) \diamondsuit \nu^{-1}) \\ &= h^n(\nu^{-1} \diamondsuit (\nu \diamondsuit u)) = h^n((\nu^{-1} \diamondsuit \nu) \diamondsuit u)) \\ &= h^n(e \diamondsuit u) = h^n(u). \end{aligned} \tag{6}$$

In the same way, we can show that $g^n_{\mathcal{A}}(\nu \diamondsuit u \diamondsuit \nu^{-1}) = g^n_{\mathcal{A}}(u)$.

Conversely, assume that $h^n_{\mathcal{A}}(\nu \diamondsuit u \diamondsuit \nu^{-1}) = h^n_{\mathcal{A}}(u)$ and $g^n_{\mathcal{A}}(\nu \diamondsuit u \diamondsuit \nu^{-1}) = g^n_{\mathcal{A}}(u)$ for all $u, \nu \in \mathcal{X}$. Then we have,

$$\begin{aligned} h^n_{\mathcal{A}}(u \diamondsuit \nu) &= h^n_{\mathcal{A}}(e \diamondsuit u \diamondsuit \nu) = h^n_{\mathcal{A}}(\nu^{-1} \diamondsuit \nu \diamondsuit u \diamondsuit \nu) \\ &= h^n_{\mathcal{A}}(\nu^{-1} \diamondsuit (\nu \diamondsuit u) \diamondsuit (\nu^{-1})^{-1}) = h^n_{\mathcal{A}}(\nu \diamondsuit u), \end{aligned} \tag{7}$$

Thus, we have $h^n_{\mathcal{A}}(u \diamondsuit \nu) = h^n_{\mathcal{A}}(\nu \diamondsuit u)$. In the same manner, we can prove that $g^n_{\mathcal{A}}(u \diamondsuit \nu) = g^n_{\mathcal{A}}(\nu \diamondsuit u)$. Therefore, by Proposition 3, $\mathcal{A}$ is HFNSG of the group $(\mathcal{X}, \diamondsuit)$. □

If $\mathcal{A} = (h_{\mathcal{A}}, g_{\mathcal{A}})$ is a HFNSG of a group $(\mathcal{X}, \diamondsuit)$. Then the set $\mathcal{U} = \{u \in \mathcal{X} : \ h^n_{\mathcal{A}}(e) = h^n_{\mathcal{A}}(u) \ \textit{and} \ g^n_{\mathcal{A}}(e) = g^n_{\mathcal{A}}(u)\}$ forms a normal subgroup of a group $\mathcal{X}$, where e is the identity element of $\mathcal{X}$.

Proof. Let $e \in \mathcal{U}$ so $\mathcal{U} \neq \emptyset$, then $\mathcal{U}$ is HFSG of $\mathcal{X}$ [Proposition 3.4[16]]. Consider $\nu \in \mathcal{X}$ and $u \in \mathcal{U}$, then

$$h^n(e) = h^n(u) \quad \textit{and} \quad g^n(e) = g^n(u). \tag{8}$$

Since $\mathcal{A}$ is HFNSG then, by Proposition 3 in this paper, we have:

$$h^n(\nu \diamondsuit u \diamondsuit \nu^{-1}) = h^n(e) \quad \textit{and} \quad g^n(\nu \diamondsuit u \diamondsuit \nu^{-1}) = g^n(e), \tag{9}$$

for all $u, \nu \in \mathcal{X}$. Hence, $\nu \diamondsuit u \diamondsuit \nu^{-1} \in \mathcal{U}$, so $\mathcal{U}$ is HFNSG of group $\mathcal{X}$. □

4 Homomorphism on Harmonized fuzzy subgroups

In this section, we discuss the effect of homomorphism on HFSG. For any two groups $\mathcal{X}$ and $\mathcal{Y}$, with HFSGs $\mathcal{S}$ and $\mathcal{T}$, respectively. Consider a homomorphism function $\mathbb{H} : \mathcal{X} \longrightarrow \mathcal{Y}$, where the image of set $\mathcal{S}$ is defined by,

$$\mathbb{H}(\mathcal{S})(\nu) = \{(\nu, \mathbb{H}(h_{\mathcal{S}})(\nu), \mathbb{H}(g_{\mathcal{S}})(\nu)), \forall \ \nu \in \mathcal{Y}\}, \tag{10}$$

for all $u \in \mathcal{X}$, such that

$$\mathbb{H}^n(h_{\mathcal{S}}) = \mathbb{H}(h^n_{\mathcal{S}}) = \begin{cases} \sup_{u \in \mathbb{H}^{-1}(u)} h^n_{\mathcal{S}}(u), & \mathbb{H}(u) = \nu \\ 0, & otherwise \end{cases}, \tag{11}$$

and

$$\mathbb{H}^n(g_{\mathcal{S}}) = \mathbb{H}(g^n_{\mathcal{S}}) = \begin{cases} \inf_{u \in \mathbb{H}^{-1}(u)} g^n_{\mathcal{S}}(u), & \mathbb{H}(u) = \nu \\ 1, & otherwise \end{cases}. \tag{12}$$

Also, the set of pre-image of $\mathcal{T}$,

$$\mathbb{H}^{-1}(\mathcal{T})(u) = \{(u, \mathbb{H}^{-1}(h_{\mathcal{T}})(u), \mathbb{H}^{-1}(g_{\mathcal{T}})(u), \forall u \in \mathcal{X}\}, \tag{13}$$

for all $\nu \in \mathcal{Y}$, such that,

$$< \mathbb{H}^{-1}(h^n_{\mathcal{T}})(u), \mathbb{H}^{-1}(g^n_{\mathcal{T}})(u) >=< (h_{\mathcal{T}})^n(\mathbb{H}(u)), (g_{\mathcal{T}})^n(\mathbb{H}(u)) > .$$

Let $(\mathcal{X}, \diamondsuit_1)$ and $(\mathcal{Y}, \diamondsuit_2)$ be two groups, and let $\mathcal{S}$ be HFSG of $\mathcal{X}$. If $\mathbb{H} : \mathcal{X} \longrightarrow \mathcal{Y}$ be an onto-homomorphism function, then $\mathbb{H}(\mathcal{S})$ is HFSG of $\mathcal{Y}$.

Proof. Assume that $\mathcal{S} = (h_{\mathcal{S}}, g_{\mathcal{S}})$ be HFSG, we want to show that $\mathbb{H}(\mathcal{S}) = (\mathbb{H}(h_{\mathcal{S}}), \mathbb{H}(g_{\mathcal{S}}))$ is HFSG.

At first, since $\mathbb{H}(\mathcal{X}) = \mathcal{Y}$; $\mathbb{H}$ is onto-homomorphism function, then $\exists\, \nu_1, \nu_2 \in \mathcal{Y}$ such that $\nu_1 = \mathbb{H}(u_1)$ and $\nu_2 = \mathbb{H}(u_2)$, for some $u_1, u_2 \in \mathcal{X}$. Now to consider:

$$\begin{aligned}
\mathbb{H}^n(h_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2) &= \{\mathbb{H}(h_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2)\}^n \\
&= \{\sup\{h_{\mathcal{S}}(x) | x \in \mathcal{X}; \mathbb{H}(x) = \nu_1 \diamondsuit_2 \nu_2\}\}^n \\
&= \sup\{h^n_{\mathcal{S}}(x) | x \in \mathcal{X}; \mathbb{H}(x) = \nu_1 \diamondsuit_2 \nu_2\} \\
&\geq \sup\{h^n_{\mathcal{S}}(u_1 \diamondsuit_1 u_2) | u_1, u_2 \in \mathcal{X}; \mathbb{H}(x) = \nu_1 \diamondsuit_2 \nu_2\} \\
&\geq \sup\{h^n_{\mathcal{S}}(u_1) \bigwedge h^n_{\mathcal{S}}(u_2) | u_1, u_2 \in \mathcal{X} \\
\textit{and } \mathbb{H}(u_1) = \nu_1, \mathbb{H}(u_2) = \nu_2\} &= \sup\{h^n_{\mathcal{S}}(u_1) | u_1, \in \mathcal{X} \textit{ and } \mathbb{H}(u_1) = \nu_1\} \\
&\bigwedge \sup\{h^n_{\mathcal{S}}(u_2) | u_2, \in \mathcal{X} \textit{ and } \mathbb{H}(u_2) = \nu_2\} \\
&= \{\mathbb{H}(h_{\mathcal{S}})(\nu_1)\}^n \bigwedge \{\mathbb{H}(h_{\mathcal{S}})(\nu_2)\}^n \\
&= \mathbb{H}^n(h_{\mathcal{S}})(\nu_1) \bigwedge \mathbb{H}^n(h_{\mathcal{S}})(\nu_2),
\end{aligned} \tag{14}$$

i.e., $\mathbb{H}^n(h_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2) \geq \mathbb{H}^n(h_{\mathcal{S}})(\nu_1) \bigwedge \mathbb{H}^n(h_{\mathcal{S}})(\nu_2)$ for all $\nu_1, \nu_2 \in \mathcal{Y}$. Moreover, one can prove that, $\mathbb{H}^n(g_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2) \leq \mathbb{H}^n(g_{\mathcal{S}})(\nu_1) \bigvee \mathbb{H}^n(g_{\mathcal{S}})(\nu_2)$.

In addition, for all $\nu \in \mathcal{Y}$, we have:

$$\begin{aligned}
\mathbb{H}^n(h_{\mathcal{S}})(\nu^{-1}) &= \{\mathbb{H}(h_{\mathcal{S}})(\nu^{-1})\}^n \\
&= \{\sup\{h_{\mathcal{S}}(u) | u \in \mathcal{X} \textit{ and } \mathbb{H}(u) = \nu^{-1}\}\}^n, \\
&= \{\sup\{h_{\mathcal{S}}(u^{-1}) | u^{-1} \in \mathcal{X} \textit{ and } \mathbb{H}(u^{-1}) = \nu\}\}^n, \\
&= \mathbb{H}^n(h_{\mathcal{S}})(\nu).
\end{aligned} \tag{15}$$

In the same manner, we can show that $\mathbb{H}^n(g_{\mathcal{S}})(\nu^{-1}) = \mathbb{H}^n(g_{\mathcal{S}})(\nu)$, for all $\nu \in \mathcal{Y}$. Hence, $\mathbb{H}(\mathcal{S}) = (\mathbb{H}(h_{\mathcal{S}}), \mathbb{H}(g_{\mathcal{S}}))$ is HFSG of $(\mathcal{Y}, \diamondsuit_2)$. □

Let $(\mathcal{X}, \diamondsuit_1)$ and $(\mathcal{Y}, \diamondsuit_2)$ be two groups. Assume that, $\mathbb{H}$ is bijection homomorphism function from $(\mathcal{X}, \diamondsuit_1)$ to $(\mathcal{Y}, \diamondsuit_2)$ and $\mathcal{T} = (h_{\mathcal{T}}, g_{\mathcal{T}})$ is HFSG of $(\mathcal{Y}, \diamondsuit_2)$. Then,

$$\mathbb{H}^{-1}(\mathcal{T}) = (\mathbb{H}^{-1}(h_{\mathcal{T}}), \mathbb{H}^{-1}(g_{\mathcal{T}})), \tag{16}$$

is HFSG of $(\mathcal{X}, \diamondsuit_1)$.

Proof. Let u_1, u_2 be two arbitrary elements of $\mathcal{X}$, then $u_1 \diamondsuit_1 u_2 \in \mathcal{X}$. We have $\mathbb{H}^{-1}(\mathcal{T}) = \{(u, (\mathbb{H}^{-1}(h_{\mathcal{T}})(u), \mathbb{H}^{-1}(g_{\mathcal{T}})(u)) | u \in \mathcal{X}\}$. Now

$$\begin{aligned}
(\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1 \diamondsuit_1 u_2) &= (\mathbb{H}^{-1}(h_{\mathcal{T}})(u_1 \diamondsuit_1 u_2))^n \\
&= \{h_{\mathcal{T}}(\mathbb{H}(u_1 \diamondsuit_1 u_2))\}^n \\
&= \{h_{\mathcal{T}}\mathbb{H}(u_1) \diamondsuit_2 h_{\mathcal{T}}\mathbb{H}(u_2)\}^n \\
&= h^n_{\mathcal{T}}\{(\mathbb{H}(u_1) \diamondsuit_2 \mathbb{H}(u_2))\} \\
&\geq h^n_{\mathcal{T}}\mathbb{H}(u_1) \bigwedge h^n_{\mathcal{T}}\mathbb{H}(u_2) \\
&= (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1) \bigwedge (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_2),
\end{aligned} \tag{17}$$

therefore, we have

$$(\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1 \diamondsuit_1 u_2) \geq (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1) \bigwedge (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_2). \tag{18}$$

Similarly, we can prove that

$$(\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u_1 \diamondsuit_1 u_2) \leq (\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u_1) \bigvee (\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u_2), \tag{19}$$

for all $u_1, u_2 \in \mathcal{X}$. Moreover; consider,

$$\begin{aligned}(\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1^{-1}) &= (\mathbb{H}^{-1}(h_{\mathcal{T}})(u^{-1}))^n \\ &= (h_{\mathcal{T}}(\mathbb{H}(u^{-1}))^n \\ &= h_{\mathcal{T}}^n(\mathbb{H}(u^{-1})) \\ &= h_{\mathcal{T}}^n(\mathbb{H})(u) \\ &= (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u).\end{aligned} \tag{20}$$

Hence,

$$(\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1^{-1}) = (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u).$$

Also, we can prove that

$$(\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u_1^{-1}) = (\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u), \tag{21}$$

for all $u \in \mathcal{X}$, by similar method. Therefore, $\mathbb{H}^{-1}(\mathcal{T})$ is HFSG. □

Let $(\mathcal{X}, \diamondsuit_1)$ and $(\mathcal{Y}, \diamondsuit_2)$ be two groups, and $\mathbb{H}$ be onto-homomorphism function from $(\mathcal{X}, \diamondsuit_1)$ to $(\mathcal{Y}, \diamondsuit_2)$, with $\mathcal{S} = (h_{\mathcal{S}}, g_{\mathcal{S}})$ is HFNSG on $\mathcal{X}$. Then,

$$\mathbb{H}(\mathcal{S}) = (\mathbb{H}(h_{\mathcal{S}}), \mathbb{H}(g_{\mathcal{S}})) \tag{22}$$

is HFNSG of $\mathcal{Y}$.

Proof. Since $\mathbb{H}$ is onto-homomorphism, then by Theorem 4, $\mathbb{H}$ is HFSG. In addition, by Proposition 3,

$$h_{\mathcal{S}}^n(u_1 \diamondsuit_1 u_2) = h_{\mathcal{S}}^n(u_2 \diamondsuit_1 u_1),$$

and

$$g_{\mathcal{S}}^n(u_1 \diamondsuit_1 u_2) = g_{\mathcal{S}}^n(u_2 \diamondsuit_1 u_1),$$

for all $u_1, u_2 \in \mathcal{X}$. Let ν_1, ν_2 be two elements of $\mathcal{Y}$, then $\exists\, u_1, u_2 \in \mathcal{X}$ such that $\nu_1 = \mathbb{H}(u_1)$ and $\nu_2 = \mathbb{H}(u_2)$. Hence, take

$$\begin{aligned}\mathbb{H}^n(h_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2) &= \{\mathbb{H}(h_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2)\}^n \\ &= \{h_{\mathcal{S}}(u) |\, u \in \mathcal{X} \textit{ and } \mathbb{H}(u) = (\nu_1 \diamondsuit_2 \nu_2)\}^n \\ &= (\sup\{h_{\mathcal{S}}(u_1 \diamondsuit_1 u_2) |\, u_1, u_2 \in \mathcal{X} \textit{ and } \mathbb{H}(u_1) = \nu_1, \mathbb{H}(u_2) = \nu_2\})^n \\ &= (\sup\{h_{\mathcal{S}}(u_2 \diamondsuit_1 u_1) |\, u_1, u_2 \in \mathcal{X} \textit{ and } \mathbb{H}(u_1) = \nu_1, \mathbb{H}(u_2) = \nu_2\})^n \\ &= \{h_{\mathcal{S}}(u) |\, u \in \mathcal{X} \textit{ and } \mathbb{H}(u) = (\nu_2 \diamondsuit_2 \nu_1)\}^n \\ &= \{\mathbb{H}(h_{\mathcal{S}})(\nu_2 \diamondsuit_2 \nu_1)\}^n \\ &= \mathbb{H}^n(h_{\mathcal{S}})(\nu_2 \diamondsuit_2 \nu_1)\end{aligned} \tag{23}$$

Hence, we have $\mathbb{H}^n(h_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2) = \mathbb{H}^n(h_{\mathcal{S}})(\nu_2 \diamondsuit_2 \nu_1)$. After that, by similar method, we can show that $\mathbb{H}^n(g_{\mathcal{S}})(\nu_1 \diamondsuit_2 \nu_2) = \mathbb{H}^n(g_{\mathcal{S}})(\nu_2 \diamondsuit_2 \nu_1)$. □

Let $(\mathcal{X}, \diamondsuit_1)$ and $(\mathcal{Y}, \diamondsuit_2)$ be two groups. Also, $\mathbb{H}$ be a bijection homomorphism function from $(\mathcal{X}, \diamondsuit_1)$ to $(\mathcal{Y}, \diamondsuit_2)$, and $\mathcal{T} = (h_{\mathcal{T}}, g_{\mathcal{T}})$ be a HFNSG on $\mathcal{Y}$. Then $\mathbb{H}^{-1}(\mathcal{T}) = (\mathbb{H}^{-1}(h_{\mathcal{T}}), \mathbb{H}^{-1}(g_{\mathcal{T}}))$, is a HFSNSG of $\mathcal{X}$.

Proof. Based on Theorem 4, we have $\mathbb{H}^{-1}(\mathcal{T}) = (\mathbb{H}^{-1}(h_{\mathcal{T}}), \mathbb{H}^{-1}(g_{\mathcal{T}}))$, is HFSG on $\mathcal{X}$. Also, since $\mathcal{T} = (h_{\mathcal{T}}, g_{\mathcal{T}})$ is HFSNG on $\mathcal{Y}$ and by Proposition 3, we have,

$$h_{\mathcal{T}}^n(\nu_1 \diamondsuit_2 \nu_2) = h_{\mathcal{T}}^n(\nu_2 \diamondsuit_2 \nu_1), \quad \textit{and} \quad g_{\mathcal{T}}^n(\nu_1 \diamondsuit_2 \nu_2) = g_{\mathcal{T}}^n(\nu_2 \diamondsuit_2 \nu_1), \tag{24}$$

for all $\nu_1, \nu_2 \in \mathcal{Y}$. Now for all $u_1, u_2 \in \mathcal{X}$,

$$\begin{aligned} (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1 \diamondsuit_1 u_2) &= (\mathbb{H}^{-1}(h_{\mathcal{T}})(u_1 \diamondsuit_1 u_2))^n \\ &= \{h_{\mathcal{T}}(\mathbb{H}(u_1 \diamondsuit_1 u_2))\}^n \\ &= h^n_{\mathcal{T}}(\mathbb{H}(u_1 \diamondsuit_1 u_2,) \\ &= h^n_{\mathcal{T}}(\mathbb{H}(u_1) \diamondsuit_2 \mathbb{H}(u_2)) \\ &= h^n_{\mathcal{T}}(\mathbb{H}(u_2) \diamondsuit_2 \mathbb{H}(u_1)) \\ &= h^n_{\mathcal{T}}(\mathbb{H}(u_2 \diamondsuit_1 u_1)) = (\mathbb{H}^{-1}(h_{\mathcal{T}})(u_2 \diamondsuit_1 u_1))^n \\ &= (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_2 \diamondsuit_1 u_1). \end{aligned} \tag{25}$$

Therefore, $(\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_1 \diamondsuit_1 u_2) = (\mathbb{H}^{-1}(h_{\mathcal{T}}))^n(u_2 \diamondsuit_1 u_1)$, for all $u_1, u_2 \in \mathcal{X}$. Similarly, we can prove that $(\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u_1 \diamondsuit_1 u_2) = (\mathbb{H}^{-1}(g_{\mathcal{T}}))^n(u_2 \diamondsuit_1 u_1)$, for all $u_1, u_2 \in \mathcal{X}$. Hence, by Proposition 3, $\mathbb{H}^{-1}(\mathcal{T})$ is HFNSG of $\mathcal{X}$. □

5 Product of Two Harmonization Fuzzy Subgroups

In this section, we redefine(modify) the definition of the cartesian product of complex intuitionistic fuzzy subgroup (CIFSG)[17] to be for HFSG as follows.

Let $W = (h_W, g_W)$ and $V = (h_V, g_V)$ be two HFSs of two groups $(\mathcal{H}_1, \diamondsuit_1)$ and $(\mathcal{H}_2, \diamondsuit_2)$, respectively. Then the cartesian product of HFS W and V is:

$$W \times V = \{((w, v), h^n_{W\times V}(w, v), g^n_{W\times V}(w, v))\}, \tag{26}$$

such that

$$\begin{aligned} h^n_{W\times V}(w, v) &= \min\{h^n_W(w), h^n_V(v)\}, \\ g^n_{W\times V}(w, v) &= \max\{g^n_W(w), g^n_V(v)\}. \end{aligned} \tag{27}$$

Whereas, $h^n_{W\times V}(w, v)$ define the membership and $g^n_{W\times V}(w, v)$ define non-membership functions of cartesian product of HFS $W \times V$.

The following theorem leads us to note that the cartesian product of two HFS is HFS.

Let $W = (h_W, g_W)$ and $V = (h_V, g_V)$ be two HFSGs of $(\mathcal{H}_1, \diamondsuit_1)$ and $(\mathcal{H}_2, \diamondsuit_2)$, respectively. Then $W \times V$ is HFSG of $(\mathcal{H}_1 \times \mathcal{H}_2, \diamondsuit_3)$.

Proof. For $w_1,\ w_2 \in \mathcal{H}_1$ and $v_1,\ v_2 \in \mathcal{H}_2$, then

$$\begin{aligned} h^n_{W\times V}((w_1, v_1)\diamondsuit_3(w_2, v_2)) &= h^n_{W\times V}(w_1 \diamondsuit_1 w_2, v_1 \diamondsuit_2 v_2) \\ &= \min\{h^n_W(w_1 \diamondsuit_1 w_2), h^n_V(v_1 \diamondsuit_2 v_2)\} \\ &\geq \min\{\min\{h^n_W(w_1), h^n_W(w_2)\}, \min\{h^n_V(v_1), h^n_V(v_2)\}\} \\ &= \min\{\min\{h^n_W(w_1), h^n_V(v_1)\}, \min\{h^n_W(w_2), h^n_V(v_2)\}\} \\ &= \min\{h^n_{W\times V}(w_1, v_1), h^n_{W\times V}(w_2, v_2)\}. \end{aligned} \tag{28}$$

$$\begin{aligned} h^n_{W\times V}(w^{-1}, v^{-1}) &= \min\{h^n_W(w^{-1}), h^n_V(v^{-1})\} \\ &\geq \min\{h^n_W(w), h^n_V(v)\}. \end{aligned} \tag{29}$$

$$\begin{aligned} g^n_{W\times V}((w_1, v_1)\diamondsuit_3(w_2, v_2)) &= g^n_{W\times V}(w_1 \diamondsuit_1 w_2, v_1 \diamondsuit_2 v_2) \\ &= \max\{g^n_W(w_1 \diamondsuit_1 w_2), g^n_V(v_1 \diamondsuit_2 v_2)\} \\ &\leq \max\{\max\{g^n_W(w_1), g^n_W(w_2)\}, \max\{g^n_V(v_1), g^n_V(v_2)\}\} \\ &= \max\{\max\{g^n_W(w_1), g^n_V(v_1)\}, \max\{g^n_W(w_2), g^n_V(v_2)\}\} \\ &= \max\{g^n_{W\times V}(w_1, v_1), g^n_{W\times V}(w_2, v_2)\}. \end{aligned} \tag{30}$$

$$\begin{aligned} g^n_{W\times V}(w^{-1}, v^{-1}) &= \max\{g^n_W(w^{-1}), g^n_V(v^{-1})\} \\ &\leq \max\{g^n_W(w), g^n_V(v)\}. \end{aligned} \tag{31}$$

By inequalities (28) - (31) and Definition 2 the result follows. □

If we have more than two groups, say $k \geq 3$, then by induction we can show the following corollary.

Let $W_1, W_2, \ldots, W_k$, be HFSGs of $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$, respectively. Then $W_1 \times W_2 \times \cdots \times W_k$, is HFSG of $\mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_k$. The converse of Theorem 5 is not true.

Let W and V be HFSs of groups $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Whereas, $W \times V$ and V are HFSG but W is not;

$$\mathcal{H}_1 =< e_1, a_1, b_1;\ a_1^2 = b_1 \text{ and } a_1 = b_1^{-1} >, \tag{32}$$

$$\mathcal{H}_2 =< e_2, a_2;\ a_2^2 = e_2 > . \tag{33}$$

Then,

$$\mathcal{H}_1 \times \mathcal{H}_2 = \{(e_1, e_2), (e_1, a_2), (a_1, e_2), (a_1, a_2), (b_1, e_2), (b_1, a_2)\} \tag{34}$$

Consider,

$$\begin{aligned} W &= \{(e_1, 0.85, 0.8), (a_1, 0.7, 0.75), (b_1, 0.7, 0.75)\}, \text{ and} \\ V &= \{(e_2, 0.87, 0.83), (a_2, 0.83, 0.85)\}. \end{aligned} \tag{35}$$

So that, W with $n \geq 4$ and V with $n \geq 5$ are HFSs. Then for $n \geq 5$, we can check that V is HFSG but W is not:

$$g_V^5(a_1b_1) = g_V^5(e_1) \nleq \max\{g_V^5(a_1), g_V^5(b_1)\} \quad i.e., \quad 0.8^5 \nleq 0.75^5. \tag{36}$$

Moreover, by Definition 5,

$$\begin{aligned} W \times V = \{&((e_1, e_2), 0.85, 0.83), ((e_1, a_2), 0.83, 0.85), \\ &((a_1, e_2), 0.7, 0.83), ((b_1, e_2), 0.7, 0.83), ((a_1, a_2), 0.7, 0.85), ((b_1, a_2), 0.7, 0.85)\}, \end{aligned} \tag{37}$$

By Definition 2, one can check that $W \times V$ satisfied conditions of HFSG.

For two groups $(\mathcal{H}_1, \diamondsuit_1)$ and $(\mathcal{H}_2, \diamondsuit_2)$, with HFSs W and V, respectively. If $W \times V$ is HFSG of $(\mathcal{H}_1 \times \mathcal{H}_2, \diamondsuit_3)$., then either W or V is HFSG.

Proof. Assume that neither W nor V is HFSG, but $W \times V$ is HFSG. Now, if W and V are not HFSGs, then at least one condition of Definition 2 is not correct. So that, $\exists\, w_1, w_2 \in W \textit{ and } v_1, v_2 \in V$, whereas;

(i) $h_W^n(w_1 \diamondsuit_1 w_2) \leq \min\{h_W^n(w_1), h_W^n(w_2)\}$, and

(ii) $h_V^n(v_1 \diamondsuit_2 v_2) \leq \min\{h_v^n(v_1), h_V^n(v_2)\}$

Then for $W \times V$, we have

$$\begin{aligned} h_{W\times V}^n((w_1, v_1)\diamondsuit_3(w_2, v_2)) &= h_{W\times V}^n((w_1\diamondsuit_1 w_2), (v_1\diamondsuit_2 v_2)), \\ &= \min\{h_W^n(w_1\diamondsuit_1 w_2), h_V^n(v_1\diamondsuit_2 v_2)\}, \\ &\leq \min\{\min\{h_W^n(w_1), h_W^n(w_2)\}, \min\{h_V^n(v_1), h_V^n(v_2)\}\}, \\ &= \min\{\min\{h_W^n(w_1), h_V^n(v_1)\}, \min\{h_W^n(w_2), h_V^n(v_2)\}\}, \\ &= \min\{h_{W\times V}^n(w_1, v_1), h_{W\times V}^n(w_2, v_2)\}. \end{aligned} \tag{38}$$

Hence, $h_{W\times V}^n((w_1, v_1)\diamondsuit_3(w_2, v_2)) \leq \min\{h_{W\times V}^n(w_1, v_1), h_{W\times V}^n(w_2, v_2)\}$ and this condratics that $W \times V$ is HFSG. Therefore, at least one of HFSs W or V of $\mathcal{H}_1$ or $\mathcal{H}_2$, respectively, must be HFSG. □

Let W and V be HFSs of groups $(\mathcal{H}_1, \diamondsuit_1)$ and $(\mathcal{H}_2, \diamondsuit_2)$, respectively, and $h_V^n(e_V) \geq h_W^n(w)$, $g_V^n(e_V) \leq g_W^n(w), \forall w \in \mathcal{H}_1$; e_V is identity of $\mathcal{H}_2$. If $W \times V$ is HFSG of $(\mathcal{H}_1 \times \mathcal{H}_2, \diamondsuit_3)$, then W is HFSG of $\mathcal{H}_1$.

Proof. Let $(w_1, e_V), (w_2, e_V) \in \mathcal{H}_1 \times \mathcal{H}_2$. Then

$$\begin{aligned} h^n_W(w_1 \diamondsuit_1 w_2) &= \min\{h^n_W(w_1 \diamondsuit_1 w_2), h^n_V(e_V \diamondsuit_2 e_V)\} \\ &= h^n_{W\times V}((w_1, e_V)\diamondsuit_3(w_2, e_V)) \\ &\geq \min\{h^n_{W\times V}(w_1, e_V), h^n_{W\times V}(w_2, e_V)\} \\ &= \min\{\min\{h^n_W(w_1), h^n_V(e_V)\}, \min\{h^n_W(w_2), h^n_V(e_V)\}\} \\ &= \min\{h^n_W(w_1), h^n_W(w_2)\}. \end{aligned} \tag{39}$$

$$\begin{aligned} g^n_W(w_1 \diamondsuit_1 w_2) &= \max\{g^n_W(w_1 \diamondsuit_1 w_2), g^n_V(e_V \diamondsuit_2 e_V)\} \\ &= g^n_{W\times V}((w_1, e_V)\diamondsuit_3(w_2, e_V)) \\ &\leq \max\{g^n_{W\times V}(w_1, e_V), g^n_{W\times V}(w_2, e_V)\} \\ &= \max\{\max\{g^n_W(w_1), g^n_V(e_V)\}, \max\{g^n_W(w_2), g^n_V(e_V)\}\} \\ &= \max\{g^n_W(w_1), g^n_W(w_2)\}. \end{aligned} \tag{40}$$

$$\begin{aligned} h^n_W(w^{-1}) &= \min\{h^n_W(w^{-1}), h^n_V(e_V^{-1})\} \\ &= h^n_{W\times V}(w^{-1}, e_V^{-1}) \geq h^n_{W\times V}(w, e_V) \\ &= \min\{h^n_W(w), h^n_V(e_V)\} = h^n_W(w). \end{aligned} \tag{41}$$

$$\begin{aligned} g^n_W(w^{-1}) &= \max\{g^n_W(w^{-1}), g^n_V(e_V^{-1})\} \\ &= g^n_{W\times V}(w^{-1}, e_V^{-1}) \leq g^n_{W\times V}(w, e_V) \\ &= \max\{g^n_W(w), g^n_V(e_V)\} = g^n_W(w). \end{aligned} \tag{42}$$

By (39)-(42), c.f. Definition 2, W is HFSG of $\mathcal{H}_1$. □

6 Discussion

This paper is continuation of "Harmonization of fuzzy subgroups",[16] where we study cosets and normality of HFSG with their properties. Besides, study the effect of homomorphism on HFSG. In addition, define a cartesian product on two HFSGs, by modifying the definition presented in the paper of CIFSGs.[17] Hence, this paper presented cartesian product of HFSs, at $n \geq 2$, for first time.

In another side, as a continuation for this paper, one can investigate more properties and types of subgroup, for example, quotient subgroup, T-product groups,. . . , ect. In addition, one can improve papers such as[18] and.[19] Another direction is to improve the HFSG to be complex harmonization fuzzy subgroup (CHFSG), and to study and discuss the HFSG properties at complex manner.

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