Smarandache Fuzzy Semigroups Using (3, 2) Fuzzy Sets

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Abstract

This paper introduces the concept of $\breve{\mathcal{S}}$-(3,2) $f$uzzy semigroups within an $\breve{\mathcal{S}}$-semigroup and explore their characterizations. Various comparable conditions for $\breve{\mathcal{S}}$-(3,2) $f$uzzy normal subsemigroups are established. Additionally, the $\breve{\mathcal{S}}$-(3,2) $f$uzzy coset, $\breve{\mathcal{S}}$-(3,2) $f$uzzy ideal, $\breve{\mathcal{S}}$-(3,2) $f$uzzy symmetric semigroup and $\breve{\mathcal{S}}$-(3,2) $f$uzzy normal subsemigroups are defined. The idea of conjugate $\breve{\mathcal{S}}$-(3,2) $f$uzzy semigroups is also introduced, and the order of an $\breve{\mathcal{S}}$-(3,2) $f$uzzy semigroup is determined. The (3,2) $f$uzzy semigroup condition applied to decision making process also.

Keywords: $\breve{\mathcal{S}}$-semigroups[SSG]; $\breve{\mathcal{S}}$-$f$uzzy semigroups[SFSG] ; $\breve{\mathcal{S}}$-$f$uzzy normal subsemigroup[SFNSG]; $\breve{\mathcal{S}}$-(3,2) $f$uzzy semigroups[S(3,2)FSG]; $\breve{\mathcal{S}}$-non (3,2) $f$uzzy semigroups[SN(3,2)FSG]; $\breve{\mathcal{S}}$- strong (3,2) $f$uzzy semigroups[SS(3,2)FSG]; $\breve{\mathcal{S}}$-(3,2) $f$uzzy normal subsemigroups[S(3,2)FNSSG]; $\breve{\mathcal{S}}$-(3,2) $f$uzzy Ideals [S(3,2)FI]; $\breve{\mathcal{S}}$-(3.2) $f$uzzy hyper subsemigroup[S(3,2)FHSSG]; $\breve{\mathcal{S}}$-(3,2) $f$uzzy subsemigroup[S(3,2)FSSG] $\breve{\mathcal{S}}$-conjugate (3,2) $f$uzzy subsemigroups[SC(3,2)FSSG]; $\breve{\mathcal{S}}$-(3,2) $f$uzzy symmetric semigroup[S(3,2)FSYSG]; $\breve{\mathcal{S}}$-(3,2) $f$uzzy coset [S(3,2)FC]

1. Introduction

Fuzzy sets and their applications have led to new insights and approaches to problem resolution in the fields of logic and algebra. At the intersection of $f$uzzy logic and algebra, Smarandache $f$uzzy algebra emerges as a fascinating field that extends traditional concepts and explores novel mathematical structures. Traditional $f$uzzy set theory, developed by Lotfi Zadeh in the 1960s [1], offers a framework for dealing with uncertainty and partial truth, which classical binary logic cannot adequately address. Fuzzy sets allow elements to have degrees of membership, reflecting a more nuanced view of truth [2]. Smarandache $f$uzzy algebra builds on these ideas, incorporating elements of algebraic structures with $f$uzzy logic to create a richer and more versatile mathematical landscape [3]. W.B. Vasantha first developed Smarandache semigroup notions in 1999. W.B. Vasantha explored Smarandache $f$uzzy semigroups in 2003. Smarandache (3, 2)- $f$uzzy semigroups and Smarandache (3,2)- $f$uzzy normal semigroups are examined in this research along with some of its properties [5-7].

2. Preliminaries

The following ideas are gathered in this section in order to develop our major findings.

Definition 2.1. Let $\breve{X}$ be a set that is not empty. A function $\breve{\mathcal{A}}: \breve{X} \to [0,1]$ represents a $f$uzzy subset $\breve{\mathcal{A}}$ of the set

Definition 2.2. Given a set $\breve{X}$, let $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}$ be two $f$uzzy subsets of it. Next, we define

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(i) $\breve{\mathcal{A}} \subset \breve{\mathcal{B}}$ iff $\breve{\mathcal{A}}(x) \leq \breve{\mathcal{B}}(x)$, for all $x \in \breve{X}$

(ii) $\breve{\mathcal{A}} = \breve{\mathcal{B}}$ iff $\breve{\mathcal{A}} \subset \breve{\mathcal{B}}$ and $\breve{\mathcal{B}} \subset \breve{\mathcal{A}}$

(iii) $(\breve{\mathcal{A}} \cap \breve{\mathcal{B}})(x) = min\{\breve{\mathcal{A}}(x), \breve{\mathcal{B}}(x)\}$, for all $x \in \breve{X}$.

Definition 2.3. [8] A semigroup $\breve{\mathcal{S}}$ is known to as a $fuzzy$ semigroup if and only if its $fuzzy$ subset $\breve{\mathcal{A}}$ satisfies the condition $\breve{\mathcal{A}}(xy) \geq min\{\breve{\mathcal{A}}(x), \breve{\mathcal{A}}(y)\}$, for all $x, y \in \breve{\mathcal{S}}$.

Definition 2.4. A group $\breve{\mathcal{G}}$'s $fuzzy$ subset $\breve{\mathcal{A}}$ is referred to as its $fuzzy$ group if

(i) $\breve{\mathcal{A}}(xy) \geq min\{\breve{\mathcal{A}}(x), \breve{\mathcal{A}}(y)\}$

(ii) $\breve{\mathcal{A}}(x^{-1}) = \breve{\mathcal{A}}(x)$, for all $x, y \in \breve{\mathcal{G}}$

Definition 2.5. When a $fuzzy$ group of a group $\breve{\mathcal{G}}$ is $\breve{\mathcal{A}}: \breve{\mathcal{G}} \to [0,1]$, then

(i) $\breve{\mathcal{A}}(x) \leq \breve{\mathcal{A}}(e)$, where e is the identity element of $\breve{\mathcal{G}}$

(ii) $\breve{\mathcal{A}}(xy^{-1}) = \breve{\mathcal{A}}(e) \Rightarrow \breve{\mathcal{A}}(x) = \breve{\mathcal{A}}(y)$, for all $x, y \in \breve{\mathcal{G}}$

Definition 2.6. If group $\breve{\mathcal{G}}$ has a $fuzzy$ group $\breve{\mathcal{A}}: \breve{\mathcal{G}} \to [0,1]$, iff $\breve{\mathcal{A}}(xy^{-1}) \geq min\{\breve{\mathcal{A}}(x), \breve{\mathcal{A}}(y)\}$, for all $x, y \in \breve{\mathcal{G}}$.

Definition 2.7. Assume that x is a member of the group $\breve{\mathcal{G}}$ and that u is a $fuzzy$ subset of $\breve{\mathcal{G}}$.

$(xu)(y) = u(x^{-1}y), y \in \breve{\mathcal{G}}$ is the definition of a $fuzzy$ left coset of u, represented by xu. In a similar vein, $(ux)(y) = u(yx^{-1}), y \in \breve{\mathcal{G}}$ is the definition of a $fuzzy$ right coset of u, represented by ux.

Definition 2.8. [9] If a semigroup $\breve{\mathcal{H}}$ contains a proper subset $\breve{\mathfrak{S}}$ that forms a group under the operation of $\breve{\mathcal{H}}$, then $\breve{\mathcal{H}}$ is termed a Smarandache semigroup (*SSG*).

Definition 2.9. Assume that $\breve{\mathcal{S}}$ is an *SSG*. If $\breve{\mathcal{A}}: \breve{\mathcal{S}} \to [0,1]$ is such that $\breve{\mathcal{A}}$ is confined to a minimum of one proper subset $\breve{\mathcal{P}}$ of $\breve{\mathcal{S}}$, which is a group, and the map of constraints $\breve{\mathcal{A}}_{\mathcal{P}}: \breve{\mathcal{P}} \to [0,1]$ is a $fuzzy$ group, then $\breve{\mathcal{A}}: \breve{\mathcal{S}} \to [0,1]$ is a $SFSG$.

Definition 2.10. Assume that $\breve{\mathfrak{S}}$ is an SSG. A *SFNSG* of $\breve{\mathfrak{S}}$ is defined as an *SFSG* $\breve{\mathcal{A}}$ of $\breve{\mathfrak{S}}$ if, for all $x, y \in \breve{\mathcal{P}}, \breve{\mathcal{A}}(xy) = \breve{\mathcal{A}}(yx)$, where $\mathcal{P}$is a proper subset of $\breve{\mathfrak{S}}$, a group, and the restriction map $\mathcal{A}_{\mathcal{P}}: \mathcal{P} \to [0,1]$ is a $fuzzy$ group.

Definition 2.11. Let $\breve{\mathfrak{X}}$ be a set that isn't empty. The definition of the (3,2)- $fuzzy$ set on $\breve{\mathfrak{X}}$ is a structure

$$\breve{\mathfrak{S}}_{\breve{\mathfrak{X}}} = \{\langle x, \mathcal{M}(x), \mathcal{N}(x)\rangle \mid x \in \breve{\mathfrak{X}}\},$$

where $\mathcal{M}: \breve{\mathfrak{X}} \to [0,1]$ is the degree of membership of x to $\breve{\mathfrak{S}}$ and $\mathcal{N}: \breve{\mathfrak{X}} \to [0,1]$ is the degree of non-membership of x to $\breve{\mathfrak{S}}$ such that $0 \leq (\mathcal{M}(x))^3 + (\mathcal{N}(x))^2 \leq 1$.

Definition 2.12. A (3,2) $fuzzy$ set $\breve{\mathcal{A}} = (\mathcal{M}, \mathcal{N})$ in $\breve{\mathfrak{S}}$ is called a (3,2)- $fuzzy$ sub-semigroup of $\breve{\mathfrak{S}}$, if $\mathcal{M}^3(x,y) \geq \min\{\mathcal{M}^3(x), \mathcal{M}^3(y)\}$ and $\mathcal{N}^2(x,y) \leq max\{\mathcal{N}^2(x), \mathcal{N}^2(y)\}$ for all $x, y \in \breve{\mathfrak{S}}$.

3. Smarandache (3,2) $fuzzy$ semigroups

Definition 3.1. [10-14] Let $\breve{\mathfrak{S}}$ be a *SSG*. Let $\breve{\mathcal{A}}$ be a (3,2)- $fuzzy$ set of $\breve{\mathfrak{S}}$. Then $\breve{\mathcal{A}}$ of $\breve{\mathfrak{S}}$ is said to be a *S(3,2)FSG* if $\mathcal{M}_{\breve{\mathcal{A}}}: \breve{\mathfrak{S}} \to [0,1]$ and $\mathcal{N}_{\breve{\mathcal{A}}}: \breve{\mathfrak{S}} \to [0,1]$ is such that $\breve{\mathcal{A}}$ is confined to a minimum of one proper subset $\breve{\mathcal{P}}$of $\breve{\mathfrak{S}}$ which is a subgroup is a (3,2) $fuzzy$ group.

That is for all $x, y \in \breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$, it might fulfills the conditions

(i) $\mathcal{M}^3_{\breve{\mathcal{A}}}(xy) \geq \min\{\mathcal{M}^3_{\breve{\mathcal{A}}}(x), \mathcal{M}^3_{\breve{\mathcal{A}}}(y)\}; \mathcal{N}^2_{\breve{\mathcal{A}}}(xy) \leq max\{\mathcal{N}^2_{\breve{\mathcal{A}}}(x), \mathcal{N}^2_{\breve{\mathcal{A}}}(y)\}$ and

(ii) $\mathcal{M}^3_{\breve{\mathcal{A}}}(x) = \mathcal{M}^3_{\breve{\mathcal{A}}}(x^{-1})$; $\mathcal{N}^2_{\breve{\mathcal{A}}}(x) = \mathcal{N}^2_{\breve{\mathcal{A}}}(x^{-1})$.

Remark 3.2. A (3,2) $fuzzy$ subset $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ might occasionally be an *S(3,2)FSG*, but it also might not be. Therefore, we name as *SN(3,2)FSG* any $fuzzy$ subsets from $\breve{\mathcal{S}}$ to [0, 1] that are not *S(3,2)FSG* when $\breve{\mathfrak{S}}$ is an *SSG*.

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Definition 3.3. Consider the (3,2) $fuzzy$ set $\breve{\mathcal{A}}$. We define to the $fuzzy$ subset $\breve{\mathcal{P}}$ as an *S(3,2)FSG* if $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ such that there exists at least one subset $\breve{\mathcal{P}}_i \subset \breve{\mathfrak{S}}$, which is a group, and $\breve{A}_{\breve{\mathcal{P}}_i}$ is a (3,2) $fuzzy$ subgroup.

Definition 3.4. A *SS(3,2)FSG* occurs when the (3,2) $fuzzy$ subset $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ is such that $\breve{A}_{\breve{\mathcal{P}}_i}: \breve{\mathcal{P}}_i \to [0,1]$ for $i = 1,2,\dots,n$ are (3,2) $fuzzy$ subgroups.

Note 3.5. In simple terms, a (3,2) $fuzzy$ subset $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ where $\breve{\mathfrak{S}}$ is a *SSG* can be either an *S(3,2)FSG* or a *SS(3,2)FSG* or *SN(3,2)FSG*

Example 3.6. Consider a *SSG* $\breve{\mathfrak{S}} = \{a, b, c, d, e, f\}$ with the following Cayley table.

$\mathfrak{S}$	a	b	c	d	e	f
a	a	b	c	d	c	f
b	b	a	a	a	a	a
c	c	a	b	a	a	f
d	d	b	b	d	d	f
e	e	c	d	c	e	f
f	f	e	c	d	e	f

A (3, 2) $fuzzy$ set $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ is defined as follows.

$\mathfrak{S}$	a	b	c	d	e	f
$\mathcal{M}_{\mathcal{A}}$	0.89	0.61	0.42	0.61	0.72	0.9
$\mathcal{N}_{\mathcal{A}}$	0.33	0.51	0.72	0.59	0.54	0.41

Clearly $\breve{\mathcal{A}}$ is a *S(3,2)FSG* . Simple verification reveals that $\breve{\mathcal{A}}$ is limited to the subset

$$\breve{\mathcal{P}} = \{a, d\}$$

which is a subgroup in $\breve{\mathcal{A}}$ is a (3,2) $fuzzy$ subgroup. i.e. $\breve{\mathcal{A}}_{\breve{\mathcal{P}}}: \breve{\mathfrak{S}} \to [0,1]$ is a (3,2) $fuzzy$ subgroup.

But $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$is not a (3,2) $fuzzy$ subgroup.

For if $x = a$ and $y = e$

$$\mathcal{M}^3_{\breve{\mathcal{A}}}(ae) = \mathcal{M}^3_{\breve{\mathcal{A}}}(c) \ngeq \min\{(0.89)^3, (0.72)^3\}$$

$$\text{That is, } (0.42)^3 \ngeq (0.72)^3$$

and

$$\mathcal{N}^2_{\breve{\mathcal{A}}}(ae) = \mathcal{N}^2_{\breve{\mathcal{A}}}(c) \nleq \max\{(0.33)^2, (0.54)^2\}$$

$$\text{That is, } (0.72)^2 \nleq (0.54)^2$$

Thus $\breve{\mathcal{A}}$ is only a *S(3,2)FSG* and not a *SS(3,2)FSG* .

We call these *S(3,2)FSG* as level I *S(3,2)FSG* .

Definition 3.7. Let the *SSG* be $\breve{\mathfrak{S}}$. A (3,2) $fuzzy$ semigroup $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ is called the *S(3,2)FSG* of level II if, for some subset $\breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$, where $\breve{\mathcal{P}}$ forms a group, the restriction of $\breve{\mathcal{A}}$ to $\breve{\mathcal{P}}$ results in a $fuzzy$ group.

Theorem 3.8. Every level II *S(3,2)FSG* is also a level I *S(3,2)FSG* .

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Proof. Straightforward by definition.

Note 3.9. Of the above theorem, the reverse is false.

Theorem 3.10. Every level I *S(3,2)FSG* need not be a level II *S(3,2)FSG*

Proof: Followed by a counter example.

Example 3.11. Consider a *SSG* $\breve{\mathcal{S}}(3)$.

The map $\mathcal{M}_{\breve{\mathcal{A}}} : \mathfrak{S} \rightarrow [0,1]$ and $\mathcal{N}_{\breve{\mathcal{A}}} : \mathfrak{S} \rightarrow [0,1]$ are defined by

$$\mathcal{M}_{\breve{\mathcal{A}}}(x) = \begin{cases} 0.6 & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.5 & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ 0.7 & otherwise \end{cases}$$

$$\mathcal{N}_{\breve{\mathcal{A}}}(x) = \begin{cases} 0.4 & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.3 & if\ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ 0.8 & otherwise \end{cases}$$

Clearly $\breve{\mathcal{A}}$ is a *S(3,2)FSG* I and is not a *S(3,2)FSG* II.

Definition 3.12. Consider $\mathfrak{S}$ be a *SSG*. Let $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \mathfrak{S} \rightarrow [0,1]$ be a (3,2) $f$uzzy semigroup. $\breve{\mathcal{A}}$ can be described as *S(3,2)FSG* of level III if $\breve{\mathcal{Q}} \subset \breve{\mathcal{A}}$ such that $\mathcal{M}_{\breve{\mathrm{Q}}}^3 \leq \mathcal{M}_{\breve{\mathcal{A}}}^3$ *and* $\mathcal{N}_{\breve{\mathrm{Q}}}^2 \geq \mathcal{N}_{\breve{\mathcal{A}}}^2$ and $\breve{\mathcal{Q}}$ is a $f$uzzy subgroup.

4. Properties and substructures of *S(3,2)FSG*

Definition 4.1. Consider $\mathfrak{S}$ be an *SSG* . $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \mathfrak{S} \rightarrow [0,1]$ is said to be a *S(3,2)FI* of the *S(3,2)FSG* $\mathfrak{S}$ if $\mathcal{M}_{\breve{\mathcal{A}}}^3(x) = \mathcal{M}_{\breve{\mathcal{A}}}^3(yxy^{-1})$ and $\mathcal{N}_{\breve{\mathcal{A}}}^2(x) = \mathcal{N}_{\breve{\mathcal{A}}}^2(yxy^{-1})$ for all $x, y \in \breve{\mathcal{A}}$ where $\breve{\mathcal{A}} \subset \mathfrak{S}$ and $\breve{\mathcal{A}}$ is a subgroup of $\mathfrak{S}$.

Definition 4.2. $\mathfrak{S}$ should be an *SSG* and $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \mathfrak{S} \rightarrow [0,1]$ be a *S(3,2)FSG* of $\mathfrak{S}$. The Smarandache level semigroup ($\breve{\mathcal{S}}$-level semigroup) of the *S(3,2)FSG* $\breve{\mathcal{A}}$, where $\breve{\mathcal{A}} \subset \mathfrak{S}$ and $\breve{\mathcal{A}}$ is the subgroup of the *SSG* $\mathfrak{S}$, is defined as follows for $t \leq \mathcal{M}(e)$ the set $\mathcal{M}_t = \{x \in \breve{\mathcal{A}} \mid \mathcal{M}(x) \geq t\}$ and for $t \geq \mathcal{N}(e)$, the set $\mathcal{N}_t = \{x \in \breve{\mathcal{A}} \mid \mathcal{N}(x) \leq t\}$ and $t \in [0,1]$.

Theorem 4.3. $\mathfrak{S}$ should be an *SSG* with $\breve{\mathcal{A}} \subset \mathfrak{S}$,which is a subgroup of $\mathfrak{S}$. $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \mathfrak{S} \rightarrow [0,1]$ is a *S(3,2)FSG* if and only if $\mathcal{M}_{\breve{\mathcal{A}}}^3(xy^{-1}) \geq \min\left(\mathcal{M}_{\breve{\mathcal{A}}}^3(x), \mathcal{M}_{\breve{\mathcal{A}}}^3(y)\right)$ and $\mathcal{N}_{\breve{\mathcal{A}}}^2(\mathrm{xy}^{-1}) \leq \max(\mathcal{N}_{\breve{\mathcal{A}}}^2(\mathrm{x}), \mathcal{N}_{\breve{\mathcal{A}}}^2(\mathrm{y}))$ for all $x, y \in \breve{\mathcal{A}}$.

Proof**.** Based on the definitions of (3,2) $f$uzzy subgroup, *SSG*, and *S(3,2)FSG*, the outcome is naturally obvious.

Theorem 4.4. $\mathfrak{S}$ should be an *SSG*. Depending on how many subgroups $\breve{\mathcal{A}}$ there are in $\mathfrak{S}$, we can have multiple $\breve{\mathcal{S}}$-level semigroups for $t \leq \mathcal{M}(e)$ and $t \geq \mathcal{N}(e)$, where and $t \in [0,1]$.

Definition 4.5. $\mathfrak{S}$ should be an *SSG*. When $\breve{\mathcal{A}}$ contains the largest subgroup of $\mathfrak{S}$ and $\breve{\mathcal{A}} \subset \mathfrak{S}$, a subsemigroup of $\mathfrak{S}$, we refer to the $f$uzzy subset as $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \mathfrak{S} \rightarrow [0,1]$ to be a *S(3,2)FHSSG*. Provided that $\breve{\mathcal{A}}$ is limited to $\breve{\mathcal{P}}$, or $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathcal{P}} \rightarrow [0,1]$. The $f$uzzy subgroup of $\mathfrak{S}$ is $\breve{\mathcal{P}} \rightarrow [0,1]$, $\breve{\mathcal{P}} \subset \breve{\mathcal{A}}$, and $\breve{\mathcal{P}}$ is the upmost subgroup of $\breve{\mathcal{A}}$.

Example 4.6. Let $\breve{\mathcal{S}}(3)$ be a *SSG*. Clearly

$$\breve{\mathcal{A}} = \left\{\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\right), \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\right\} \cup \breve{\mathcal{S}}_3$$

This set is a semigroup. Now, consider a map, $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathcal{S}}_3 \rightarrow [0,1]]$. When this map is restricted to the subset $\breve{\mathcal{S}}_3$ it forms a (3,2) $f$uzzy group. Therefore, the map $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}})$ represents an *S(3,2)FHSSG*.

Theorem 4.7. [15] A *S(3,2)FSSG* of $\mathfrak{S}$ does not necessarily have to be an *S(3,2)FHSSG* of any *SSG* $\mathfrak{S}$.

Proof**.** Obtained from the definition itself.

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Corollary 4.8. All *S(3,2)FHSSG* of an *SSG* $\breve{\mathfrak{S}}$ is a *S(3,2)FSSG* of $\breve{\mathfrak{S}}$.

Definition 4.9. Let $\breve{\mathfrak{S}}$ be a *SSG*. We say $\breve{\mathfrak{S}}$ is a Smarandache (3,2) $f$uzzy simple semigroup ($\mathcal{S}$ - (3,2) $f$uzzy simple semigroup) if $\breve{\mathfrak{S}}$ has no $S(3,2)FHSSG$.

Theorem 4.10. If an *S(3,2)FSG* of an *SSG* $\breve{\mathfrak{S}}$ is $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \rightarrow [0,1]$ with respect to a group $\breve{\mathcal{P}}$ that is a proper subset of $\breve{\mathfrak{S}}$, then

(i) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) \leq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e)$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) \geq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e)$, where e is the identity component of $\breve{\mathcal{P}}$

(ii) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e)$ implies $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$ and

$\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e)$ implies $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$ for all $x, y \in \breve{\mathcal{P}}$.

Proof. Assume that $\breve{\mathcal{A}}$ is an *S(3,2)FSG* of an *SSG* $\breve{\mathfrak{S}}$ relative to a group $\breve{\mathcal{P}}$ in $\breve{\mathfrak{S}}$.

(i) By the definition of *S(3,2)FSG*,

$\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e) \geq \min\left\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})\right\} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$ and

$\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e) \leq \max\left\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{A}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})\right\} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$ for all $x \in \breve{\mathcal{P}}$.

(ii) Let $x, y \in \breve{\mathcal{P}}$.

Case 1: If we assume that $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e)$,

then $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}y)$

$= \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}\big((xy^{-1})y\big)$

$\geq \min\left\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Therefore, $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) \geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Similarly, $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) \geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$.

Thus $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Case 2: If we assume that $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(e)$,

then $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}y)$

$= \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}\big((xy^{-1})y\big)$

$\leq \max\left\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Therefore, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) \leq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Similarly, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) \leq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$.

Thus $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

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Theorem 4.11. $\breve{\mathcal{A}}$ is an *S(3,2)FSG* if $\breve{\mathcal{A}}$ is an *SFSG* of an *SSG* $\breve{\mathfrak{S}}$.

Proof: $\breve{\mathcal{A}}$ is limited to a proper subset $\breve{\mathcal{P}}$ of $\breve{\mathfrak{S}}$, which is a group such that the constraint map $\breve{\mathcal{A}}_{\breve{\mathcal{P}}}: \breve{\mathfrak{S}} \to [0,1]$ is a $f$uzzy group, because $\breve{\mathcal{A}}$ is an *SFSG* of $\breve{\mathfrak{S}}$.

Let $x, y \in \breve{\mathcal{P}}$. Now,

(i) $\mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) \geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) \geq \min\left\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\} \geq \min\left\{\mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\}$ and $\mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) \leq$

$\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) \leq max\left\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\} \leq \max\{\mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\}$

(ii) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})$; $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})$.

Theorem 4.12. Assume that $\breve{\mathfrak{S}}$ is an *SSG* and that $\breve{\mathcal{P}}$ is a proper subset of $\breve{\mathfrak{S}}$, a group. Then $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ is an *S(3,2)FSG* of $\breve{\mathfrak{S}}$ relative to $\breve{\mathcal{P}}$ iff

$\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}) \geq \min\left\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\}$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) \leq max\left\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\}$ for all $x, y \in \breve{\mathcal{P}}$.

Proof. Given that $\breve{\mathfrak{S}}$ is an *SSG* and that $\breve{\mathcal{P}}$ is a proper subset of $\breve{\mathfrak{S}}$, a group.

$\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathfrak{S}} \to [0,1]$ is an *S(3,2)FSG* iff $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathcal{P}} \to [0,1]$ is a $f$uzzy group iff $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy^{-1}) \geq \min\left\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\}$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) \leq max\left\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\right\}$ for all $x, y \in \breve{\mathcal{P}}$.

Definition 4.13. If $\mathcal{M}^3_{\breve{\mathcal{A}}}(\mathrm{xy}) = \mathcal{M}^3_{\breve{\mathcal{A}}}(\mathrm{yx})$ and $\mathcal{N}^2_{\breve{\mathcal{A}}}(\mathrm{xy}) = \mathcal{N}^2_{\breve{\mathcal{A}}}(\mathrm{yx})$ for all $x, y \in \breve{\mathcal{A}}$, then $\breve{\mathcal{A}}$ is a *S(3,2)FNSSG* of the *SSG* $\breve{\mathfrak{S}}$. Additionally, $\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}$ restricted to $\breve{\mathcal{A}}$, that is $\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}} : \breve{\mathcal{A}} \to [0,1]$ is a $f$uzzy subgroup of $\breve{\mathcal{A}}$.

Theorem 4.14. If $\breve{\mathcal{A}}$ is an *SFNSSG* of an *SSG* $\breve{\mathfrak{S}}$, then $\breve{\mathcal{A}}$ is also an *S(3,2)FNSSG* of $\breve{\mathfrak{S}}$.

Proof. Given $\mathcal{A}$ be an *SFNSSG*.

Let $\breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$, a group in $\breve{\mathfrak{S}}$ such that the restriction map $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}}): \breve{\mathcal{P}} \to [0,1]$ is a (3,2) $f$uzzy group and $\mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$ and $\mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$ for all $x, y \in \breve{\mathcal{A}}$.

We know that $x\mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{M}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$ and $x\mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{N}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$ which implies that

$x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$ and $x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$ for all $x \in \breve{\mathcal{P}}$.

Therefore for any $g \in \breve{\mathcal{P}}$, $(x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(g) = (\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(g)$

$\Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}g) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(gx^{-1})$.

Then $\min\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(g)\} = \min\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(g), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})$

Therefore $(x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(g) = (\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(g)$, for all $g \in \breve{\mathcal{P}}$.

Thus $x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$, for all $x \in \breve{\mathcal{P}}$.

Likewise, for any $g \in \breve{\mathcal{P}}$, $(x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(g) = (\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(g)$

$\Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}g) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(gx^{-1})$.

Then $max\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(g)\} = max\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(g), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})\}$.

Therefore $(x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(g) = (\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(g)$, for all $g \in \breve{\mathcal{P}}$.

Thus $x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$, for all $x \in \breve{\mathcal{P}}$.

Hence $\breve{\mathcal{A}}$ is an *S(3,2)FNSSG* of $\breve{\mathfrak{S}}$.

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Theorem 4.15. If $\breve{\mathcal{A}}$ is an *S(3,2)FSG* of an *SSG* $\breve{\mathfrak{S}}$ in relation to a group $\breve{\mathcal{P}}$ in $\breve{\mathfrak{S}}$. Next, the subsequent conditions are comparable for all $x, y \in \breve{\mathcal{P}}$.

(i) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) \geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) \leq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$,

(ii) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$,

(iii) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$,

(iv) $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}})$ is an $\breve{\mathcal{S}}$- (3,2) $f$uzzy normal subsemigroup,

(v) $x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}$ and $x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}$ and

(vi) $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yxy^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$ and $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yxy^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$

Proof. Consider an *SSG* $\breve{\mathfrak{S}}$.

Let $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}})$ be an *S(3,2)FSG* of $\breve{\mathfrak{S}}$ in relation to $\breve{\mathcal{P}}$, a group in $\breve{\mathfrak{S}}$.

Let $x, y \in \breve{\mathcal{P}}$.

(i) ⇔ (ii): Let us assume that $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) \geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$

Now $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}\big((x^{-1}x)y(x^{-1}x)\big)$

$= \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}(xyx^{-1})x)$

$\geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}yx)$

Therefore, $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Let us assume that $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) \leq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$

Now $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}\big((x^{-1}x)y(x^{-1}x)\big)$

$= \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}(xyx^{-1})x)$

$\leq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}yx)$

Therefore, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Conversely, if $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$, then it is obvious that $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) \geq \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

Likewise, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$, then it is obvious that $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) \leq \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

(ii) ⇔ (iii):

Suppose that $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

If we take $y = yx$, then $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyxx^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx) \Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$.

Similarly, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) \Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$.

Conversely, let $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx) \Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x(yx^{-1})) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}((yx^{-1})x)$.

$\Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1}) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$.

(iii) ⇔ (iv):

Let $\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^3_{\mathcal{A}_{\mathcal{P}}}(yx) \Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}y) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx^{-1})$.

$\Rightarrow \min\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\} = \min\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})\}$.

$\Rightarrow (x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y)$ for all $y \in \breve{\mathcal{P}}$.

$\Rightarrow x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$.

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Similarly, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx) \Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}y) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx^{-1})$.

$\Rightarrow \max\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\} = \max\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1})\}$.

$\Rightarrow (x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y)$ for all $y \in \breve{\mathcal{P}}$.

$\Rightarrow x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$.

Therefore, $\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx) \Rightarrow x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$.

$\breve{\mathcal{A}}$ is a *S(3,2)FNSSG*.

Conversely, if $\breve{\mathcal{A}}$ is an *S(3,2)FNSSG* of $\breve{\mathfrak{S}}$, then $(x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y)$ and $(x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y)$ for all $y \in \breve{\mathcal{P}}$

Taking, $(x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y)$

Therefore $\min\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\} = \min\{\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\}$.

$\Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}y) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx^{-1})$.

$\Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1})$. (Take $y = xy$)

$\Rightarrow \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$ (since (ii) ⇔ (iii)).

Similarly, Taking, $(x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y)$

Therefore $\max\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\} = \max\{\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}), \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)\}$.

$\Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}y) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx^{-1})$.

$\Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xyx^{-1})$. (Take $y = xy$)

$\Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$ (since (ii) ⇔ (iii)).

Thus $(x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y) = (\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)(y) \Rightarrow \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$.

(iv) ⇔ (v):

Assume that $\breve{\mathcal{A}}$ is an *S(3,2)FNSSG* of $\breve{\mathfrak{S}}$ in relation to $\breve{\mathcal{P}}$.

Therefore, $x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$ and $x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x$ for all $x \in \breve{\mathcal{P}}$.

Now, $(x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1})(y) = ((x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})x^{-1})(y)$

$= ((\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x)x^{-1})(y)$

$= (\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xx^{-1}))(y)$

$= \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y)$ for all $y \in \breve{\mathcal{P}}$.

Thus $x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}$.

Similarly we get $x\mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1} = \mathcal{N}^2_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}$.

Conversely, let us take $x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1} = \mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}$. Then

$\left(\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x\right)(y) = ((x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x^{-1})x)(y)$

$= (x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x^{-1}x))(y)$

$\left(\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x\right)(y) = (x\mathcal{M}^3_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})(y)$ for all $y \in \breve{\mathcal{P}}$

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Therefore, $\left(\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x\right) = (x\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})$.

Similarly we get, $\left(\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}x\right) = (x\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}})$.

(iii) ⇔ (vi):

If $\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$, then $\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y^{-1}xy) = \mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yy^{-1}x) = \mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$ and

$\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$, then $\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y^{-1}xy) = \mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yy^{-1}x) = \mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$

Conversely, if $\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y^{-1}xy) = \mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$, then $\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y^{-1}(yx)y) = \mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$

This implies that $\mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{M}^{3}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$

Then if $\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y^{-1}xy) = \mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(x)$, then $\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(y^{-1}(yx)y) = \mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$

This implies that $\mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(xy) = \mathcal{N}^{2}_{\breve{\mathcal{A}}_{\breve{\mathcal{P}}}}(yx)$

Definition 4.16. The set of all *S(3,2)FSSG* of $\mathcal{S}(n)$ is represented by $S(3,2)F(\mathcal{S}(n))$. If $\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}} \in S(3,2)F(\mathcal{S}(n))$ then Im $\mathcal{M}_{\breve{\mathcal{A}}} = \{f(x)\ /\ x \in \breve{\mathcal{A}} \subset \mathcal{S}(n)$ and $\mathcal{N}_{\breve{\mathcal{A}}} = \{g(x)\ /\ x \in \breve{\mathcal{A}} \subset \mathcal{S}(n)$ where $\breve{\mathcal{A}}$ is a proper subset of $\mathcal{S}(n)$ which is a subgroup of $\mathcal{S}(n)$ under the operations of $\mathcal{S}(n)$. Let $\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}} \in S(3,2)F(\mathcal{S}(n))$ we say if |Im($\mathcal{M}_{\breve{\mathcal{A}}}$)| < |Im($\mathcal{N}_{\breve{\mathcal{A}}}$)| then we write $\mathcal{M}_{\breve{\mathcal{A}}} < \mathcal{N}_{\breve{\mathcal{A}}}$. We may define using this rule max S-(3,2)F(S(n)). Let (f, g) be a S-(3,2) *fuzzy* subsemigroup of $\mathcal{S}(n)$.If $f = max\, S(3,2)F(\mathcal{S}(n))$ and $g = min\, S(3,2)F(\mathcal{S}(n))$ then we say that (f, g) is a *S(3,2)FSYSG* of $\mathcal{S}(n)$.

Definition 4.17. A *S(3,2)FC* of the *SSG* $\breve{\mathfrak{S}}$ of $a\mathcal{M}_{\breve{\mathcal{A}}}$ and $a\mathcal{N}_{\breve{\mathcal{A}}}$, for any $a \in \breve{\mathcal{A}} \subset \breve{\mathfrak{S}}$ are given by $(a\mathcal{M}^{3}_{\breve{\mathcal{A}}})(x) = \mathcal{M}^{3}_{\breve{\mathcal{A}}}(a^{-1}x)$ and $(a\mathcal{N}^{2}_{\breve{\mathcal{A}}})(x) = \mathcal{N}^{2}_{\breve{\mathcal{A}}}(a^{-1}x)$ for every $x \in \breve{\mathcal{A}}$.

Definition 4.18. Let $\breve{\mathcal{A}}(\mathcal{M}_{\breve{\mathcal{A}}}, \mathcal{N}_{\breve{\mathcal{A}}})$ be a *S(3,2)FSSG* of a *SSG* $\breve{\mathfrak{S}}$ and

$$\breve{\mathcal{H}} = \{x \in \breve{\mathcal{A}} \subset \breve{\mathfrak{S}} / \mathcal{M}^{3}_{\breve{\mathcal{A}}}(x) = \mathcal{M}^{3}_{\breve{\mathcal{A}}}(0)\ and\ \mathcal{N}^{2}_{\breve{\mathcal{A}}}(x) = \mathcal{N}^{2}_{\breve{\mathcal{A}}}(0)\},$$

then $\mathcal{O}(\breve{\mathcal{A}})$,order of $\breve{\mathcal{A}}$ is defined as $\mathcal{O}(\breve{\mathcal{A}}) = \mathcal{O}(\breve{\mathcal{H}})$.

Definition 4.19. Given an *SSG* $\breve{\mathfrak{S}}$. $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}$ are its two *S(3,2)FSSG* . Then $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}$ are considered as *SC(3,2)FSSG* of $\breve{\mathfrak{S}}$, if for some $u \in \breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$, we have $\mathcal{M}^{3}_{\breve{\mathcal{A}}}(x) = \mathcal{M}^{3}_{\breve{\mathcal{B}}}(u^{-1}xu)$ and $\mathcal{N}^{2}_{\breve{\mathcal{A}}}(x) = \mathcal{N}^{2}_{\breve{\mathcal{B}}}(u^{-1}xu)$ for every $x \in \breve{\mathcal{P}}$.

Theorem 4.20. If $\breve{\mathfrak{S}}$ is an *SSG* and $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}$ are *SC(3,2)FSSG* with respect to a subgroup $\breve{\mathcal{P}} \subset \breve{\mathcal{S}}$, then $\mathcal{O}(\breve{\mathcal{A}}) = \mathcal{O}(\breve{\mathcal{B}})$.

Proof:

Assume that the *SC(3,2)FSSG* $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}$ belong to the *SSG* $\breve{\mathfrak{S}}$.

By the very definition of *SC(3,2)FSSG* of the *SSG* there exists $u \in \breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$ such that

$\mathcal{M}^{3}_{\breve{\mathcal{A}}}(x) = \mathcal{M}^{3}_{\breve{\mathcal{B}}}(u^{-1}xu)$ and $\mathcal{N}^{2}_{\breve{\mathcal{A}}}(x) = \mathcal{N}^{2}_{\breve{\mathcal{B}}}(u^{-1}xu)$ for every $x \in \breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$ ($\breve{\mathcal{P}}$ a proper subgroup of $\breve{\mathfrak{S}}$). Defining

$$\breve{\mathcal{H}} = \{x \in \breve{\mathcal{P}} | \mathcal{M}^{3}_{\breve{\mathcal{A}}}(x) = \mathcal{M}^{3}_{\breve{\mathcal{A}}}(e)\ and\ \mathcal{N}^{2}_{\breve{\mathcal{A}}}(x) = \mathcal{N}^{2}_{\breve{\mathcal{A}}}(e)\} \text{ and}$$

$$\breve{\mathcal{K}} = \{x \in \breve{\mathcal{P}} | \mathcal{M}^{3}_{\breve{\mathcal{B}}}(x) = \mathcal{M}^{3}_{\breve{\mathcal{B}}}(e)\ and\ \mathcal{N}^{2}_{\breve{\mathcal{B}}}(x) = \mathcal{N}^{2}_{\breve{\mathcal{B}}}(e)\}$$

where e is the identity component of $\breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$.

Given that $\breve{\mathcal{H}}$ is a ț-level subset of the subgroup $\breve{\mathcal{P}}$, it is obvious that $\breve{\mathcal{H}}$ is a subgroup of the subgroup $\breve{\mathcal{P}}$ where ț $= \mathcal{M}^{3}_{\breve{\mathcal{A}}}(e)$.

Similarly $\breve{\mathcal{K}}$ is also a subgroup of the subgroup $\breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$.

It is sufficient for one to prove that $\mathcal{O}(\breve{\mathcal{H}}) = \mathcal{O}(\breve{\mathcal{K}})$ using the đefinition of the order of the *fuzzy* subgroup of the subgroup $\breve{\mathcal{P}}$ in order to show that $\mathcal{O}(\breve{\mathcal{A}}) = \mathcal{O}(\breve{\mathcal{B}})$.

In order verify that $\mathcal{O}(\breve{\mathcal{H}}) = \mathcal{O}(\breve{\mathcal{K}})$, we must first show that $\breve{\mathcal{H}} \subset u\breve{\mathcal{K}}u^{-1}$ for a given $u \in \breve{\mathcal{P}} \subset \breve{\mathfrak{S}}$.

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Then, we must do the same thing for $u \in \breve{\mathcal{P}}; \breve{\mathcal{K}} \subset u^{-1}\breve{\mathcal{H}}u$.

Step 1:

Choose any element x from $\breve{\mathcal{H}}$as the *SC(3,2)FSSG* of $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}$, which are corresponding to the *SSG* $\breve{\mathfrak{S}}$, we get for some u ∈ $\breve{\mathfrak{S}}$.

$$\mathcal{M}^3_{\breve{\mathcal{B}}}(u^{-1}xu) = \mathcal{M}^3_{\breve{\mathcal{A}}}(x)$$

$$= \mathcal{M}^3_{\breve{\mathcal{A}}}(e) \;\; [\because x \in \breve{\mathcal{H}}]$$

$$= \mathcal{M}^3_{\breve{\mathcal{B}}}(u^{-1}eu) \quad \text{[Since } \breve{\mathcal{A}} \text{ and } \breve{\mathcal{B}} \text{ are } SC(3,2)FSSG \text{ of } \breve{\mathfrak{S}}]$$

$$= \mathcal{M}^3_{\breve{\mathcal{B}}}(e).$$

So, there is $u \in \breve{\mathcal{P}}$ such that $\mathcal{M}^3_{\breve{\mathcal{B}}}(x) = \mathcal{M}^3_{\breve{\mathcal{B}}}(e)$.

And,

$$\mathcal{N}^2_{\breve{\mathcal{B}}}(u^{-1}xu) = \mathcal{N}^2_{\breve{\mathcal{A}}}(x)$$

$$= \mathcal{N}^2_{\breve{\mathcal{A}}}(e) \;\; [\because x \in \breve{\mathcal{H}}]$$

$$= \mathcal{N}^2_{\breve{\mathcal{B}}}(u^{-1}eu) \quad \text{[Since } \breve{\mathcal{A}} \text{ and } \breve{\mathcal{B}} \text{ are } SC(3,2)FSSG \text{ of } \breve{\mathfrak{S}}]$$

$$= \mathcal{N}^2_{\breve{\mathcal{B}}}(e).$$

Thus, there is $u \in \breve{\mathcal{P}}$ such that $\mathcal{N}^2_{\breve{\mathcal{B}}}(x) = \mathcal{N}^2_{\breve{\mathcal{B}}}(e)$.

By using $\breve{\mathcal{K}}$'s definition, we now have $u^{-1}xu \in \breve{\mathcal{K}} \Rightarrow x \in u\breve{\mathcal{K}}u^{-1}$.

Consequently, $\breve{\mathcal{H}} \subseteq u\breve{\mathcal{K}}u^{-1}$.

Step 2:

Choose any arbitrary element x in $\breve{\mathcal{K}}$. The earlier result proving $\breve{\mathcal{H}} \subseteq u\breve{\mathcal{K}}u^{-1}$was based on the fact that and $\breve{\mathcal{B}}$ are *SC(3,2)FSSG* of *SSG* $\breve{\mathfrak{S}}$ for the same $u \in \breve{\mathcal{P}} \subseteq \breve{\mathfrak{S}}$, we have

$$\mathcal{M}^3_{\breve{\mathcal{A}}}(u^{-1}xu) = \mathcal{M}^3_{\breve{\mathcal{B}}}(x) = \mathcal{M}^3_{\breve{\mathcal{B}}}(e) \;\; [\because x \in \breve{\mathcal{H}}]$$

$$= \mathcal{M}^3_{\breve{\mathcal{A}}}(u^{-1}eu) \quad \text{[Since } \breve{\mathcal{A}} \text{ and } \breve{\mathcal{B}} \text{ are } SC(3,2)FSSG \text{ of } \breve{\mathfrak{S}}]$$

$$= \mathcal{M}^3_{\breve{\mathcal{A}}}(e).$$

Therefore, there is $u \in \breve{\mathcal{P}}$ such that $\mathcal{M}^3_{\breve{\mathcal{A}}}(x) = \mathcal{M}^3_{\breve{\mathcal{A}}}(e)$.

And $\mathcal{N}^2_{\breve{\mathcal{A}}}(u^{-1}xu) = \mathcal{N}^2_{\breve{\mathcal{B}}}(x) = \mathcal{N}^2_{\breve{\mathcal{B}}}(e) \;\; [\because x \in \breve{\mathcal{H}}]$

$$= \mathcal{N}^2_{\breve{\mathcal{A}}}(u^{-1}eu) \quad \text{[Since } \breve{\mathcal{A}} \text{ and } \breve{\mathcal{B}} \text{ are } SC(3,2)FSSG \text{ of } \breve{\mathfrak{S}}]$$

$$= \mathcal{N}^2_{\breve{\mathcal{A}}}(e).$$

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Consequently, $u \in \breve{\mathcal{P}}$ such that $\mathcal{N}_{\breve{\mathcal{A}}}^{2}(x) = \mathcal{N}_{\breve{\mathcal{A}}}^{2}(e)$.

Hence $uxu^{-1} \in \breve{\mathcal{H}}$ for the same $u \in \breve{\mathcal{P}} \subseteq \breve{\mathfrak{S}}$, that is $x \in u^{-1}\breve{\mathcal{H}}u$.

Hence $\breve{\mathcal{K}} \subseteq u^{-1}\breve{\mathcal{H}}u$.

Starting with the initial and subsequent steps we have $\breve{\mathcal{H}} \subseteq u\breve{\mathcal{K}}u^{-1}$ and $\breve{\mathcal{K}} \subseteq u^{-1}\breve{\mathcal{H}}u$.

That is $\breve{\mathcal{H}}u \subseteq u\breve{\mathcal{K}}$ and $u\breve{\mathcal{K}} \subset \breve{\mathcal{H}}u$.

Thus $\breve{\mathcal{H}}u = u\breve{\mathcal{K}}$ so $\breve{\mathcal{H}} = u\breve{\mathcal{K}}u^{-1}$.

Given that $\breve{\mathcal{K}}$ belongs to the group $\breve{\mathcal{P}}$ that is a subgroup of the *SSG* $\breve{\mathfrak{S}}$, we obtain

$\mathcal{O}(x\breve{\mathcal{K}}x^{-1}) = \mathcal{O}(\breve{\mathcal{K}})$ for every $x \in \breve{\mathcal{P}}$. Now choose $x = u$.

Next, we have $\mathcal{O}(u\breve{\mathcal{K}}u^{-1}) = \mathcal{O}(\breve{\mathcal{K}})$.

Thus $\mathcal{O}(\breve{\mathcal{H}}) = \mathcal{O}(\breve{\mathcal{K}})$.

Consequently, $\mathcal{O}(\breve{\mathcal{A}}) = \mathcal{O}(\breve{\mathcal{B}})$.

5. Utilizing (3,2) Ƒuzzy semigroups in the Decision-Making process

This section utilizes the (3, 2) ƒuzzy semigroups to a decision-making problem. The proposed method is discussed along with the associated algorithms and an illustrative example to demonstrate its application. Mathematical Tools for Investigating Decision-Making Problems, when evaluating options for the discussion of this flow, we will define alternatives as options that need to be evaluated based on evaluation criteria (which is not defined for this flow). This involves a tool known as a semigroup: this is a set with an associative operation taking elements and producing their combination. Semigroups are a mathematical structure used in algebra, providing a systematic approach to analyzing intricate decision-making processes. Structuring potential options as a semigroup allows decision-makers to see the relationships among options and criteria and to deconstruct the problem into smaller sections. This method can be very helpful for ƒuzzy logic situations or similar [16].

The following steps demonstrate how the current technique addresses a decision-making challenge.

Step 1: Define the set of specifications. That is identify the specifications and choose the prioritized subset $\mathcal{A}$. Moreover, for the set of specifications form a Cayley table.

Step 2: Gather membership and non-membership values

Step 3: Verify the (3, 2) ƒuzzy semigroup conditions. That is, for each brand, manually verify the following conditions for all elements $x, y \in \mathcal{A}$.

Step 4: Apply the conditions to all the brands under investigation.

Step 5: Compare the results:

- If any of the brands satisfies the conditions the buyer should choose that particular brand
- If all the brands satisfy the conditions, further analysis or additional criteria may be required for the decision.
- If neither brand satisfies the conditions, neither brand is recommended based on the current criteria.

Step 6: Decision Making:

Based on the verification in **Step 5**, select the brand that best fits the buyer's needs according to the (3, 2) ƒuzzy semigroup conditions.

6. Illustrative Example

We focused this section to explaining the above-mentioned ways using the following example.

Example: A customer wants to buy a smartphone, who has specific needs regarding the camera quality. Depending on the make and model, there are different options available in the market, each varying in the amount of

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megapixels, sensor size, aperture, image processing software, etc. This makes it difficult for the buyer to determine which smartphone would be best suited for their needs. There is a wide range of cellphones made by different manufacturers, which include Samsung, Google, Xiaomi, Apple, and many more, each offering various specifications in the cameras of their phones. One particular store had two brands in stock. From the assortment, the client has to choose one. Using the (3, 2) $f$uzzy semigroup criteria, it will be analyzed the membership and non-membership values for both brands along with other data to reach a conclusion.

Let us define the set specifications of smartphone as $G = \{\pi, \rho, \varphi, \omega, \delta, \tau, \vartheta\}$, where

$\pi -$ Megapixels (MP)

$\rho -$ Sensor Size

$\varphi -$ Aperture (f-stop)

$\omega -$ Optical Zoom vs. Digital Zoom

$\delta -$ Image Processing Software

$\tau -$ Video Capabilities

$\vartheta -$ Brand Reputation & Customer Reviews

Consider the Cayley Table for the semigroup $G = \{\pi, \rho, \varphi, \omega, \delta, \tau, \vartheta\}$,

.	π	ρ	φ	ω	δ	τ	ϑ
π	π	π	π	π	π	π	π
ρ	π	ρ	ρ	ω	ρ	ρ	ϑ
φ	π	ρ	φ	ω	δ	τ	ϑ
ω	π	ω	ω	ω	π	ω	ϑ
δ	π	δ	δ	ω	δ	δ	ϑ
τ	π	ρ	φ	ω	ρ	τ	τ
ϑ	π	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ

Let us define $f$uzzy membership functions and non- membership functions for two different brands (Brand X and Brand Y):

Brand X

G	π	ρ	φ	ω	δ	τ	ϑ
$\mathcal{M}$	0.96	0.65	0.48	0.66	0.7	0.71	0.9
$\mathcal{N}$	0.31	0.25	0.72	0.33	0.4	0.8	0.2

Brand Y

G	π	ρ	φ	ω	δ	τ	ϑ
$\mathcal{M}$	0.5	0.18	0.18	0.4	0.6	0.16	0.6
$\mathcal{N}$	0.81	0.55	0.7	0.17	0.5	0.64	0.2

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Objective:

To choose a smartphone with a camera that provides the best image quality for the buyer's needs.

Let us consider the set specifications of smartphone as per buyers prioritization, $\mathcal{A} = \{\pi, \omega, \delta, \vartheta\}$.

Verify the elements of subset of $\mathcal{A}$ might satisfies the following conditions

$\mathcal{M}_A^3(x,y) \geq \min\{\mathcal{M}_A^3(x), \mathcal{M}_A^3(y)\}$ *and* $\mathcal{N}_A^2(x,y) \leq max\{\mathcal{N}_A^2(x), \mathcal{N}_A^2(y)\}$

For all $x, y \in \mathcal{A} \subset$ G.

Here the proper subset $\mathcal{A}$ satisfies (3, 2) $f$uzzy semigroups for Brand X, but fails to satisfies for the same condition in Brand Y. Therefore, the buyer can buy Brand X.

7. Conclusion

In the context of $\breve{S}$-semigroups, this article provides a substantial introduction and characterization for the notion of $\breve{S}$-(3,2) $f$uzzy semigroups. We explore a deeper sense regarding $\breve{S}$-(3,2) $f$uzzy normal subsemigroups by forming several equivalent conditions for them, and defining a number of fundamental aspects of $\breve{S}$-(3,2) $f$uzzy left and right cosets. In addition, a new class of $\breve{S}$-(3,2) $f$uzzy semigroups are developed which not only contributes to the theory of such structures but through the ordering of $\breve{S}$-(3,2) $f$uzzy semigroups introduces a new dimensions into the general theory of such objects. Moreover, (3, 2) $f$uzzy semigroups conditions were implemented to decision making process. The results provide a foundation for future study and application of $f$uzzy semigroup theory in managerial and other mathematical applications.

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