



Solution of Second Order Fuzzy Differential Equations using Sumudu Transform under Neutrosophic Environment

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Abstract

Fuzzy differential equations (FDEs) are used to represent dynamical systems under uncertain environments. Finding solutions for fuzzy differential equations (FDEs) is highly challenging. This work employs the neutrosophic version of the Sumudu transform method to determine the solution to fuzzy differential equations (FDEs) that incorporate Neutrosophic Numbers (NNs). By utilising a novel fuzzy arithmetic operations on the parametric representations of NNs, significant theorems are established to demonstrate the characteristics of Neutrosophic Sumudu Transform (NST). The proposed NST approach is efficient in approximating the solutions of FDEs without converting them into their crisp equivalent forms. An illustrative numerical example is provided to demonstrate the efficacy of the proposed methodology.

Keywords: Neutrosophic number; Parametric form; Arithmetic operations; Fuzzy Sumudu transform; Neutrosophic differential equations

1 Introduction

Differential equations can represent various real-world problems. However, the models might not be completely accurate. For instance, the initial value of the models may not be fully understood. Phrases like “less than x_0 ,” “around x_0 ,” or “greater than x_0 ” illustrate some potential uncertainties regarding the starting value. In such cases, it becomes challenging to address the problem using traditional differential equations. Therefore, it is essential to explore alternative hypotheses to tackle this issue. To address this problem, it is crucial to explore alternative hypotheses. Fuzzy set theory, formulated by Zadeh¹ in 1965, is one of the prominent theories that tackle this issue. This approach proves to be more effective than traditional set theory when assessing unknown environmental factors as membership values. Certain real-world scenarios require an evaluation of both membership and non-membership participation values. When defining fuzzy sets, only the degree of membership of the information is considered, while non-membership is disregarded. Atanassov² proposed intuitionistic fuzzy sets to manage hesitation margins.

Smarandache³ in his 1995 paper, he introduced the concept of the neutrosophic set (NS), which incorporates the degree of indeterminacy as a distinct element. This document was subsequently published in 1998. As a result of this, the neutrosophic set may be defined by identifying the three components of the neutrosophic set - membership (T), indeterminacy (I), and nonmembership (F). Neutrosophic set theory, which examines the causes, descriptions, and potentials of neutral ideas, is another topic in the field of uncertainty. Naeem Saleem et al.⁴ introduced the concept of neutrosophic rectangular extended b-metric spaces and obtained fixed point results for contraction mappings based on this notion. In addition, they presented numerous complex illustrations.

Mohamed et al.⁵ studied a real case in Zagazig, Egypt, for evaluating, operating, and determining the most sustainable location for an Electric vehicles (EVs) charging station is presented. Sasikala and Divya⁶ introduced the novel idea of the Fermatean neutrosophic Dombi fuzzy graph. In addition, they discovered several products resulting from the direct, Cartesian composition of Fermatean neutrosophic Dombi fuzzy graphs. The study of fuzzy differential equations (FDEs) provides an appropriate context for the mathematical modelling of real-world issues when uncertainties or ambiguity are pervasive. Studying fuzzy differential equations may be done in a variety of ways. Buckley⁷ and Chalco-Cano and Roman-Flores⁸ discussed fuzzy derivatives, as well as fuzzy differential equations. Dubois et al.⁹ used simply membership values to establish fuzzy differential equations. For the purpose of solving differential equations in control engineering problems, Watugala¹⁰ introduced the Sumudu transform during the 1990s. This transform's primary benefit is its ability to preserve units of measurement and scale. That is why the original differential equation and the converted differential equation are identical. Sumudu transform essential theorems and characteristics are shown in.¹¹ Weerakon¹² employed the Sumudu transform in the future to resolve partial differential equations. This finding inspired several scholars to use the Sumudu transform on various kinds of partial differential equation.

Many scientists and engineers now use neutrosophic sets to represent many real-world phenomena utilising differential equations that have unknown parameters. A fuzzy variation of the traditional Sumudu transform, known as the fuzzy Sumudu transform (FST), was presented by Abdul Rahman and Ahmad.¹³ The traditional Sumudu transform was effectively incorporated into the fuzzy environment by the authors. There were new findings about linearity, fuzzy derivative, preservation, convolution, and shifting. Following that, the strongly generalised differentiability notion was utilised to solve fuzzy differential equations with fuzzy starting values. They have developed a unique technique for using FST to solve fuzzy partial differential equations as well as two new theorems for FST addressing fuzzy partial derivatives. They used FST to solve Caputo fuzzy fractional derivative-based linear fuzzy fractional differential equations. Additionally, they covered a system of fuzzy linear differential equations with fuzzy constant coefficients, as well as some FST applications to fuzzy fractional differential equations (FFDE) and fuzzy Volterra integral equations (FVIE).

Recently, Abdul Rahman and Ahmad¹³ proposed a fuzzy version of classical Sumudu transform called fuzzy Sumudu transform (FST). The traditional Sumudu transform was effectively incorporated by the authors into a fuzzy context. The linearity, fuzzy derivative, preservation, convolution, and shifting were proposed. Following this, the FST was applied to solve strongly generalised differentiability-based fuzzy differential equations with fuzzy starting values. Two new theorems for FST concerning fuzzy partial derivatives have been given, and a unique approach to solving fuzzy partial differential equations has been proposed. Linear fuzzy fractional differential equations with Caputo fuzzy fractional derivatives were solved using FST. Fuzzy set theory (FST) was also explored in relation to various applications to FFDE and FVIE, as well as a system of FDEs.

Jafari et al.¹⁴ developed FDE solutions using the fuzzy Sumudu transform technique. Khan et al.¹⁵ solved linear differential models with uncertainty using Sumudu transform. Georgieva¹⁶ investigated partial Volterra fuzzy integro-differential equations (PVFIDE) with convolution kernel under Hukuhara differentiability applying the double fuzzy Sumudu transform (DFST) technique. Haydar¹⁷ used fuzzy Sumudu transform to solve nth-order FDEs with fuzzy initial values. Jesuraj and Rajkumar¹⁸ studied fuzzy differential equations using a new integral transform called modified Sumudu transform when the initial condition in terms of nonagonal neutrosophic numbers and obtained strong solution. Using the Sumudu transform approach, Parikh and Sahni¹⁹ have studied the spring mass system by solving second order FDE with NNs as the initial conditions. Razvarz et al.²⁰ obtained the solutions of FDEs by applying the fuzzy Sumudu transform (FST) method. A mechanical vibration system's ordinary differential equation with fuzzy starting values and substantially generalised differentiability conditions was studied by using the Sumudu transform method.

1.1 Novelty of the research work

Previous research has not addressed the Sumudu transform under neutrosophic environment and propose a new technique by applying NST to solve FDEs without reducing to equivalent crisp forms. This paper aims to fill this research gap based on the aforementioned motivation and facts. Here are the significant contributions of our work:

- In this paper, we present Sumudu transform under neutrosophic environment and propose a new technique by applying NST to solve FDEs without reducing to equivalent crisp forms to address a gap in the current literature.

- The modelling of a spring mass system in an uncertain environment involves the formulation of a second order FDE, where the initial conditions are defined using neutrosophic numbers. Expressing the neutrosophic numbers in their parametric form and using a new type of arithmetic operations in NST, we obtain the solution of a spring mass system under uncertain environment in the form of neutrosophic numbers.
- At various computational steps, the solution is presented in a table. In addition, the findings for different values of (α, β, γ) and time are visually presented in the form of a table and graphically.

Our work aims to bridge this research gap and offer new insights into the study of Sumudu transform under neutrosophic environment and propose a new technique by applying NST to solve FDEs without reducing to equivalent crisp forms.

1.2 Structure of the research work

The paper is organized as follows. After this introduction, In section 2, we provide the basic notions and operations of neutrosophic set, neutrosophic numbers, triangular neutrosophic numbers, arithmetic operation on triangular neutrosophic numbers, continuity and differentiability. Based on a continuity and differentiability, Section 3 proposes Neutrosophic Sumudu transform and solving FDEs with Sumudu transform under neutrosophic environment. Neutrosophic Sumudu transform related theorems are proven. In Section 4, a spring mass system under uncertain environment is modeled as a second order FDE with the initial conditions involving neutrosophic numbers is solved, the solution is presented in a table. The last section summarizes the conclusion.

2 Preliminaries

We recall neutrosophic numbers, including some necessary definitions, ideas, and arithmetic operations to understand the contribution in this paper.

Definition 2.1. (Fuzzy Number).²¹ A fuzzy number is a fuzzy set $\tilde{a} : R \rightarrow [0, 1]$ which satisfies

- \tilde{a} is upper semicontinuous;
- $\tilde{a}(y) = 0$ outside some interval $[c, d]$;
- There are real numbers a, b such that $c \leq a \leq b \leq d$ for which
 - $\tilde{a}(y)$ is monotonic increasing on $[c, a]$;
 - $\tilde{a}(y)$ is monotonic decreasing on $[b, d]$;
 - $\tilde{a}(y) = 1; a \leq y \leq b$.

Definition 2.2. (Neutrosophic Set(NS)).³ Let X be a universe of discourse. A neutrosophic set (NS) \tilde{a} on X is given by $\tilde{a} = \{(y, \mu_{\tilde{T}}(y), \mu_{\tilde{I}}(y), \mu_{\tilde{F}}(y)) : x \in X\}$, where the functions $\mu_{\tilde{T}}(y), \mu_{\tilde{I}}(y), \mu_{\tilde{F}}(y) : X \rightarrow [0, 1]$ determine the degree of membership, indeterminacy, non-membership of the element $y \in X$ respectively and for every $y \in X$,

$$0 \leq \mu_{\tilde{T}}(y) + \mu_{\tilde{I}}(y) + \mu_{\tilde{F}}(y) \leq 3.$$

Definition 2.3. (Neutrosophic Number(NN)).²² A neutrosophic number (NN) \tilde{a} on R is

- Neutrosophic subset of the real line.
- \tilde{a} is normal. i.e. there exist an $y_0 \in R$, such that $\mu_{\tilde{T}}(y_0) = 1, \mu_{\tilde{I}}(y_0) = 0, \mu_{\tilde{F}}(y_0) = 0$.
- \tilde{a} is convex for the membership function $\mu_{\tilde{T}}$
i.e., $\mu_{\tilde{T}}(\lambda y_1 + (1 - \lambda)y_2) \geq \min(\mu_{\tilde{T}}(y_1), \mu_{\tilde{T}}(y_2)), \forall y_1, y_2 \in R, \lambda \in [0, 1]$.

- \tilde{a} is concave for the indeterminacy and non-membership functions μ_I, μ_F
 i.e., $\mu_{\tilde{I}}(\lambda y_1 + (1 - \lambda)y_2) \geq \max(\mu_{\tilde{I}}(y_1), \mu_{\tilde{I}}(y_2))$,
 $\mu_{\tilde{F}}(\lambda y_1 + (1 - \lambda)y_2) \geq \max(\mu_{\tilde{F}}(y_1), \mu_{\tilde{F}}(y_2))$, for all $y_1, y_2 \in R, \lambda \in [0, 1]$.

Definition 2.4. (Triangular Neutrosophic Number(TNN)).²³ A neutrosophic number (NN) \tilde{a} is a triangular neutrosophic number(TNN) and is denoted by $\tilde{a} = (\langle a_{\tilde{T}_1}, a_{\tilde{T}_2}, a_{\tilde{T}_3} \rangle; \langle a_{\tilde{I}_1}, a_{\tilde{I}_2}, a_{\tilde{I}_3} \rangle; \langle a_{\tilde{F}_1}, a_{\tilde{F}_2}, a_{\tilde{F}_3} \rangle)$ having the membership function, indeterminacy function and non-membership function as will be shown in the following

$$\mu_{\tilde{T}}(y) = \begin{cases} \frac{y - a_{\tilde{T}_1}}{a_{\tilde{T}_2} - a_{\tilde{T}_1}}, & \text{for } a_{\tilde{T}_1} \leq y \leq a_{\tilde{T}_2} \\ \frac{a_{\tilde{T}_3} - y}{a_{\tilde{T}_3} - a_{\tilde{T}_2}}, & \text{for } a_{\tilde{T}_2} \leq y \leq a_{\tilde{T}_3} \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_{\tilde{I}}(y) = \begin{cases} \frac{a_{\tilde{I}_2} - y}{a_{\tilde{I}_2} - a_{\tilde{I}_1}}, & \text{for } a_{\tilde{I}_1} \leq y \leq a_{\tilde{I}_2} \\ \frac{y - a_{\tilde{I}_3}}{a_{\tilde{I}_3} - a_{\tilde{I}_2}}, & \text{for } a_{\tilde{I}_2} \leq y \leq a_{\tilde{I}_3} \\ 1, & \text{otherwise,} \end{cases}$$

and

$$\mu_{\tilde{F}}(y) = \begin{cases} \frac{a_{\tilde{F}_2} - y}{a_{\tilde{F}_2} - a_{\tilde{F}_1}}, & \text{for } a_{\tilde{F}_1} \leq y \leq a_{\tilde{F}_2} \\ \frac{y - a_{\tilde{F}_3}}{a_{\tilde{F}_3} - a_{\tilde{F}_2}}, & \text{for } a_{\tilde{F}_2} \leq y \leq a_{\tilde{F}_3} \\ 0, & \text{otherwise.} \end{cases}$$

We use $F(R)$ to denote the set of all TNNs defined on R .

Definition 2.5. (α, β, γ) -cut.²² The (α, β, γ) -cut of neutrosophic set is represented by $F(\alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in [0, 1]$ and are fixed values, such that $\alpha + \beta + \gamma \leq 3$ and is illustrated as $F(\alpha, \beta, \gamma) = (\mu_{\tilde{T}}(y), \mu_{\tilde{I}}(y), \mu_{\tilde{F}}(y))$, where $y \in X, \mu_{\tilde{T}}(y) \geq \alpha, \mu_{\tilde{I}}(y) \leq \beta, \mu_{\tilde{F}}(y) \leq \gamma$.

Definition 2.6. A triangular neutrosophic number (TNN) $\tilde{a} = (a_{\tilde{T}}, a_{\tilde{I}}, a_{\tilde{F}}) \in F(R)$ can also be represented as pairs $a_{\tilde{T}} = (\underline{a_T}, \overline{a_T}), a_{\tilde{I}} = (\underline{a_I}, \overline{a_I}), a_{\tilde{F}} = (\underline{a_F}, \overline{a_F})$ of functions $\underline{a_T}(r), \overline{a_T}(r), \underline{a_I}(r), \overline{a_I}(r), \underline{a_F}(r), \overline{a_F}(r)$ for $0 \leq r \leq 1$ which satisfies the following requirements:

- For membership function, $\underline{a_T}(r)$ is a bounded monotonic increasing left continuous function.
- For membership function, $\overline{a_T}(r)$ is a bounded monotonic decreasing left continuous function.
- For indeterminacy function, $\underline{a_I}(r)$ is a bounded monotonic increasing left continuous.
- For indeterminacy function, $\overline{a_I}(r)$ is a bounded monotonic decreasing left continuous function.
- For non-membership function, $\underline{a_F}(r)$ is a bounded monotonic increasing left continuous function.
- For non-membership function, $\overline{a_F}(r)$ is a bounded monotonic decreasing left continuous function.
- If $\underline{a_T}(r) \leq \overline{a_T}(r); \underline{a_I}(r) \leq \overline{a_I}(r); \underline{a_F}(r) \leq \overline{a_F}(r), 0 \leq r \leq 1$.

Definition 2.7. For an arbitrary triangular neutrosophic number $\tilde{a} = (\tilde{a}_T, \tilde{a}_I, \tilde{a}_F) = (\underline{a}_T, \overline{a}_T; \underline{a}_I, \overline{a}_I; \underline{a}_F, \overline{a}_F)$, the numbers $a_{T_0} = \left(\frac{\underline{a}_T(1) + \overline{a}_T(1)}{2}\right)$, $a_{I_0} = \left(\frac{\underline{a}_I(1) + \overline{a}_I(1)}{2}\right)$, and $a_{F_0} = \left(\frac{\underline{a}_F(1) + \overline{a}_F(1)}{2}\right)$ are called the location index numbers of membership and indeterminacy and non-membership functions respectively.

The non-decreasing left continuous functions $a_{T*} = (a_{T_0} - \underline{a}_T)$, $a_{T}^* = (\overline{a}_T - a_{T_0})$ are called the left and right fuzziness index functions for the membership function, the non-decreasing left continuous functions $a_{I*} = (a_{I_0} - \underline{a}_I)$, $a_I^* = (\overline{a}_I - a_{I_0})$ are called the left fuzziness index function and right fuzziness index function for the indeterminacy function and the non-decreasing left continuous functions $a_{F*} = (a_{F_0} - \underline{a}_F)$, $a_F^* = (\overline{a}_F - a_{F_0})$ are called the left and right fuzziness index functions for the non-membership function respectively. Hence every TNN $\tilde{a} = (\langle a_{\tilde{T}_1}, a_{\tilde{T}_2}, a_{\tilde{T}_3} \rangle; \langle a_{\tilde{I}_1}, a_{\tilde{I}_2}, a_{\tilde{I}_3} \rangle; \langle a_{\tilde{F}_1}, a_{\tilde{F}_2}, a_{\tilde{F}_3} \rangle)$ can also be interpreted in terms of $\tilde{a} = (\langle a_{T_0}, a_{T*}, a_T^* \rangle; \langle a_{I_0}, a_{I*}, a_I^* \rangle; \langle a_{F_0}, a_{F*}, a_F^* \rangle)$.

2.1 Arithmetic operations on triangular Neutrosophic numbers

Ming Ma et al.²¹ have proposed a new type of arithmetic operations for fuzzy numbers by expressing them in the form of location index and fuzziness index functions. We extend this new fuzzy arithmetic on triangular neutrosophic numbers based upon both location index and fuzziness index functions for membership, indeterminacy, non-membership functions. The location index functions (LIF) follows the usual arithmetic operations while the fuzziness index functions (FIF) follows the lattice rule which is least upper bound (LUB) in the lattice L , i.e. $a \vee b = \max\{a, b\}$, for $a, b \in L$. Specifically for every pair of triangular neutrosophic numbers

$$\tilde{a} = (\langle a_{T_0}, a_{T*}, a_T^* \rangle; \langle a_{I_0}, a_{I*}, a_I^* \rangle; \langle a_{F_0}, a_{F*}, a_F^* \rangle), \tilde{b} = (\langle b_{T_0}, b_{T*}, b_T^* \rangle; \langle b_{I_0}, b_{I*}, b_I^* \rangle; \langle b_{F_0}, b_{F*}, b_F^* \rangle).$$

We define,

(i) Addition

$$\begin{aligned} \tilde{a} + \tilde{b} &= (\langle a_{T_0}, a_{T*}, a_T^* \rangle; \langle a_{I_0}, a_{I*}, a_I^* \rangle; \langle a_{F_0}, a_{F*}, a_F^* \rangle) + (\langle b_{T_0}, b_{T*}, b_T^* \rangle; \langle b_{I_0}, b_{I*}, b_I^* \rangle; \langle b_{F_0}, b_{F*}, b_F^* \rangle) \\ &= (\langle a_{T_0} + b_{T_0}, \max\{a_{T*}, b_{T*}\}, \max\{a_T^*, b_T^*\} \rangle, \langle a_{I_0} + b_{I_0}, \max\{a_{I*}, b_{I*}\}, \max\{a_I^*, b_I^*\} \rangle, \\ &\quad \langle a_{F_0} + b_{F_0}, \max\{a_{F*}, b_{F*}\}, \max\{a_F^*, b_F^*\} \rangle). \end{aligned}$$

(ii) Subtraction

$$\begin{aligned} \tilde{a} - \tilde{b} &= (\langle a_{T_0}, a_{T*}, a_T^* \rangle; \langle a_{I_0}, a_{I*}, a_I^* \rangle; \langle a_{F_0}, a_{F*}, a_F^* \rangle) - (\langle b_{T_0}, b_{T*}, b_T^* \rangle; \langle b_{I_0}, b_{I*}, b_I^* \rangle; \langle b_{F_0}, b_{F*}, b_F^* \rangle) \\ &= (\langle a_{T_0} - b_{T_0}, \max\{a_{T*}, b_{T*}\}, \max\{a_T^*, b_T^*\} \rangle, \langle a_{I_0} - b_{I_0}, \max\{a_{I*}, b_{I*}\}, \max\{a_I^*, b_I^*\} \rangle, \\ &\quad \langle a_{F_0} - b_{F_0}, \max\{a_{F*}, b_{F*}\}, \max\{a_F^*, b_F^*\} \rangle). \end{aligned}$$

(iii) Multiplication

$$\begin{aligned} \tilde{a} \times \tilde{b} &= (\langle a_{T_0}, a_{T*}, a_T^* \rangle; \langle a_{I_0}, a_{I*}, a_I^* \rangle; \langle a_{F_0}, a_{F*}, a_F^* \rangle) \times (\langle b_{T_0}, b_{T*}, b_T^* \rangle; \langle b_{I_0}, b_{I*}, b_I^* \rangle; \langle b_{F_0}, b_{F*}, b_F^* \rangle) \\ &= (\langle a_{T_0} \times b_{T_0}, \max\{a_{T*}, b_{T*}\}, \max\{a_T^*, b_T^*\} \rangle, \langle a_{I_0} \times b_{I_0}, \max\{a_{I*}, b_{I*}\}, \max\{a_I^*, b_I^*\} \rangle, \\ &\quad \langle a_{F_0} \times b_{F_0}, \max\{a_{F*}, b_{F*}\}, \max\{a_F^*, b_F^*\} \rangle). \end{aligned}$$

(iv) Division

$$\begin{aligned} \tilde{a} \div \tilde{b} &= (\langle a_{T_0}, a_{T*}, a_T^* \rangle; \langle a_{I_0}, a_{I*}, a_I^* \rangle; \langle a_{F_0}, a_{F*}, a_F^* \rangle) \div (\langle b_{T_0}, b_{T*}, b_T^* \rangle; \langle b_{I_0}, b_{I*}, b_I^* \rangle; \langle b_{F_0}, b_{F*}, b_F^* \rangle) \\ &= (\langle a_{T_0} \div b_{T_0}, \max\{a_{T*}, b_{T*}\}, \max\{a_T^*, b_T^*\} \rangle, \langle a_{I_0} \div b_{I_0}, \max\{a_{I*}, b_{I*}\}, \max\{a_I^*, b_I^*\} \rangle, \\ &\quad \langle a_{F_0} \div b_{F_0}, \max\{a_{F*}, b_{F*}\}, \max\{a_F^*, b_F^*\} \rangle), \text{ provided } b_{T_0}, b_{I_0}, b_{F_0} \neq 0. \end{aligned}$$

Definition 2.8. (Continuity)²². A neutrosophic-valued function (NVF) $\tilde{f} : R \rightarrow F(R)$ is said to be continuous on R if for every $y_0 \in R$ and every $\epsilon > 0$, there exists $\delta > 0$ such that if $|y - y_0| < \delta$, then $D(\tilde{f}(y), \tilde{f}(y_0)) < \epsilon$.

Definition 2.9. (Differentiability)²². Let $\tilde{f} : R \rightarrow F(R)$ be a neutrosophic-valued function (NVF). We say f is differentiable at $y_0 \in R$, if there exist an element $\tilde{f}'(y_0)$ of \tilde{f} at the point y_0 such that limits

$$\tilde{f}'(y_0) = \lim_{\Delta x \rightarrow 0} \frac{\tilde{f}(y_0 + \Delta x) - \tilde{f}(y_0)}{\Delta x}$$

exist and are equal to $f'(y_0)$. Hence the limits are taken in the metric space D .

3 Neutrosophic Sumudu Transformation

Let $\tilde{f}(t)$ be a continuous neutrosophic valued function. Assume that $\tilde{f}(ut)e^{-t}$ be an improper neutrosophic Riemann integrable on $[0, \infty)$, then $\int_0^\infty \tilde{f}(ut)e^{-t}dt$ is called neutrosophic Sumudu transform of $\tilde{f}(t)$ and it is represented by,

$$G(u) = S[\tilde{f}(t)] = \left(\int_0^\infty \tilde{f}_T(ut)e^{-t}dt, \max \tilde{f}_I(ut)e^{-t}, \max \tilde{f}_F(ut)e^{-t} \right), \quad (u \in [-\tau, \tau]). \quad (1)$$

In the neutrosophic valued function argument, the variable u is used to factor the variable t . We have, $\tilde{f}(t, r) = \{\tilde{f}_T(t, r), \tilde{f}_I(t, r), \tilde{f}_F(t, r)\}$, which are represented in neutrosophic parametric form as

$$\tilde{f}_T(t, r) = (f_{T_0}, f_{T_*}, f_T^*), \tilde{f}_I(t, r) = (f_{I_0}, f_{I_*}, f_I^*), \tilde{f}_F(t, r) = (f_{F_0}, f_{F_*}, f_F^*).$$

$$\begin{aligned} \text{Then } \int_0^\infty \tilde{f}_T(ut)e^{-t}dt &= \left(\int_0^\infty f_{T_0}e^{-t}dt, \max f_{T_*}e^{-t}, \max f_T^*e^{-t} \right), \\ \int_0^\infty \tilde{f}_I(ut)e^{-t}dt &= \left(\int_0^\infty f_{I_0}e^{-t}dt, \max f_{I_*}e^{-t}, \max f_I^*e^{-t} \right), \\ \int_0^\infty \tilde{f}_F(ut)e^{-t}dt &= \left(\int_0^\infty f_{F_0}e^{-t}dt, \max f_{F_*}e^{-t}, \max f_F^*e^{-t} \right). \end{aligned}$$

Also applying the NST on the components, we have

$$\begin{aligned} S[f_{T_0}(t, r)] &= \int_0^\infty f_{T_0}e^{-t}dt, \quad S[f_{T_*}(t, r)] = \max f_{T_*}e^{-t}, \quad S[f_T^*(t, r)] = \max f_T^*e^{-t}, \\ S[f_{I_0}(t, r)] &= \int_0^\infty f_{I_0}e^{-t}dt, \quad S[f_{I_*}(t, r)] = \max f_{I_*}e^{-t}, \quad S[f_I^*(t, r)] = \max f_I^*e^{-t}, \\ S[f_{F_0}(t, r)] &= \int_0^\infty f_{F_0}e^{-t}dt, \quad S[f_{F_*}(t, r)] = \max f_{F_*}e^{-t}, \quad S[f_F^*(t, r)] = \max f_F^*e^{-t} \end{aligned}$$

and also

$$\begin{aligned} S[\tilde{f}(t)] &= (\langle S[f_{T_0}(t, r)], S[f_{T_*}(t, r)], S[f_T^*(t, r)] \rangle; \\ &\quad \langle S[f_{I_0}(t, r)], S[f_{I_*}(t, r)], S[f_I^*(t, r)] \rangle; \\ &\quad \langle S[f_{F_0}(t, r)], S[f_{F_*}(t, r)], S[f_F^*(t, r)] \rangle). \end{aligned}$$

Theorem 3.1. If $\tilde{G}_N(u)$ is NST of $\tilde{f}_N(t) = (f_{N_0}(t), (f_N(t))_*, (f_N(t))^*)$, then the NST of the derivatives with integer order is given by

$$S \left[\frac{d^n \tilde{f}_N(t)}{dt^n} \right] = u^{-n} \left[\tilde{G}_N(u) - \sum_{k=1}^{n-1} u^k \frac{d^k \tilde{f}_N(t)}{dt^k} \Big|_{t=0} \right]. \quad (2)$$

Proof: The NST of $\tilde{f}'_N(t) = \frac{d\tilde{f}_N(t)}{dt}$ is denoted by

$$\begin{aligned} S\left[\frac{d\tilde{f}_N(t)}{dt}\right] &= S[\tilde{f}'_N(t)] = \int_0^\infty \tilde{f}'_N(ut)e^{-t} dt = \frac{1}{u} \int_0^\infty \tilde{f}'_N(x)e^{-x/u} dx, \\ &= \frac{1}{u} \int_0^\infty \tilde{f}'_N(t)e^{-t/u} dt = \frac{1}{u} \lim_{p \rightarrow \infty} \int_0^p \tilde{f}'_N(t)e^{-t/u} dt, \\ &= \frac{1}{u} \lim_{p \rightarrow \infty} \int_0^p \frac{d\tilde{f}_N(t)}{dt} e^{-t/u} dt, \\ &= \frac{1}{u} \lim_{p \rightarrow \infty} \left\{ \left[e^{-t/u} \tilde{f}_N(t) \right]_{t=0}^p + \frac{1}{u} \int_0^p \tilde{f}_N(t)e^{-t/u} dt \right\}, \\ &= \frac{1}{u} \left\{ -\tilde{f}_N(0) + \frac{1}{u} \lim_{p \rightarrow \infty} \int_0^p \tilde{f}_N(t)e^{-t/u} dt \right\}, \\ &= \frac{1}{u} \left\{ -\tilde{f}_N(0) + \frac{1}{u} \int_0^\infty \tilde{f}_N(t)e^{-t/u} dt \right\}, \\ &= -\frac{1}{u} \tilde{f}_N(0) + \frac{1}{u^2} \tilde{G}_N(u). \end{aligned}$$

That is $S[\tilde{f}'_N(t)] = -\frac{1}{u} \tilde{f}_N(0) + \frac{1}{u} \tilde{G}_N(u).$ (3)

Continuing using the same approach, we obtain the NST of the nth order derivative as

$$S\left[\frac{d^n \tilde{f}_N(t)}{dt^n}\right] = u^{-n} \left[\tilde{G}_N(u) - \sum_{k=1}^{n-1} u^k \frac{d^k \tilde{f}_N(t)}{dt^k} \Big|_{t=0} \right].$$

3.1 Solving second order fuzzy differential equations (FDEs) with Neutrosophic Sumudu Transform (NST)

Consider the following second-order general ordinary differential equation:

$$x''(t) = f(t, x(t), x'(t)). \tag{4}$$

Considering the initial states $x(t_0) = x_0, x'(t_0) = z_0$, where $f : [t_0, P] \times R \rightarrow R$. Suppose that initial states x and z are uncertain and are defined in terms of location index and fuzziness index functions of membership, indeterminacy and non-membership, i.e., neutrosophic number. Then the second-order ordinary differential equation (ODE) (4) becomes the following FDE under fuzzy initial value:

$$\tilde{x}''(t) = f(t, \tilde{x}(t), \tilde{x}'(t)), 0 \leq t \leq P, \tag{5}$$

$$\tilde{x}_T(t_0) = \tilde{x}_0 = (x_{T_0}(0), x_{T_*}(0), x_{T^*}^*(0)), \tilde{x}'_T(t_0) = \tilde{z}_0 = (z_{T_0}(0), z_{T_*}(0), z_{T^*}^*(0)), 0 < \alpha \leq 1, \tag{6}$$

$$\tilde{x}_I(t_0) = \tilde{y}_0 = (x_{I_0}(0), x_{I_*}(0), x_{I^*}^*(0)), \tilde{x}'_I(t_0) = \tilde{z}_0 = (z_{I_0}(0), z_{I_*}(0), z_{I^*}^*(0)), 0 < \beta \leq 1, \tag{7}$$

$$\tilde{x}_F(t_0) = \tilde{x}_0 = (x_{F_0}(0), x_{F_*}(0), x_{F^*}^*(0)), \tilde{x}'_F(t_0) = \tilde{z}_0 = (z_{F_0}(0), z_{F_*}(0), z_{F^*}^*(0)), 0 < \gamma \leq 1, \tag{8}$$

where $f : [t_0, P] \times F(R) \rightarrow F(R)$ is a measure of continuity. Making use of NST on the given second order differential equation (5), we have

$$S[\tilde{x}''(t)] = s[f(t, \tilde{x}(t), \tilde{x}'(t))] .$$

Let $\tilde{x}'(t)$ is differentiable, and making use of parametric form, we have $\tilde{x}'(t) = [x'_0(t), x'_*(t), x'^*(t)]$. the differential equations can be expressed as:

$$x'_T(t) = f(t, x_T(t), x'_T(t)), x_T(t) = x_T(0),$$

$$x'_I(t) = f(t, x_I(t), x'_I(t)), x_I(t) = x_I(0),$$

$$x'_F(t) = f(t, x_F(t), x'_F(t)), x_F(t) = x_F(0).$$

After implementing the ST , we have

$$S[\tilde{x}''(t)] = \frac{S[\tilde{x}(t)] - \tilde{x}(t_0) - u\tilde{x}'(t_0)}{u^2}.$$

The following nine first-order ordinary differential equations are derived, three for each of the three membership categories (Membership, Indeterminacy, and Non-membership):

$$\begin{aligned} S[f(t, x_T(t), x'_T(t))] &= \frac{S[x_T(t)] - x(t) - ux'_T(t)}{u^2}, \\ S[f(t, x_I(t), x'_I(t))] &= \frac{S[x_I(t)] - x(t) - ux'_I(t)}{u^2}, \\ S[f(t, x_F(t), x'_F(t))] &= \frac{S[x_F(t)] - x(t) - ux'_F(t)}{u^2}, \\ S[x_T(t)] &= G_T(u, \alpha), \\ S[x_I(t)] &= G_T(u, \beta), \\ S[x_F(t)] &= G_T(u, \gamma), \end{aligned}$$

where,

$$G_{T_0}(u, \alpha), G_{T_*}(u, \alpha), G_T^*(u, \alpha), G_{I_0}(u, \beta), G_{I_*}(u, \beta), G_I^*(u, \beta), G_{F_0}(u, \gamma), G_{F_*}(u, \gamma), G_F^*(u, \gamma)$$

are solutions to fuzzy differential equations. By use of neutrosophic sumudu transform, we have

$$(x_{T_0}(t), x_{T_*}(t), x_T^*(t)), (x_{I_0}(t), x_{I_*}(t), x_I^*(t)), (x_{F_0}(t), x_{F_*}(t), x_F^*(t)))$$

and it follows that,

$$\begin{aligned} x_{T_0}(t) &= S^{-1}[G_{T_0}(u, \alpha)], x_{T_*}(t) = S^{-1}[G_{T_*}(u, \alpha)], x_T^*(t) = S^{-1}[G_T^*(u, \alpha)], \\ x_{I_0}(t) &= S^{-1}[G_{I_0}(u, \beta)], x_{I_*}(t) = S^{-1}[G_{I_*}(u, \beta)], x_I^*(t) = S^{-1}[G_I^*(u, \beta)], \\ x_{F_0}(t) &= S^{-1}[G_{F_0}(u, \gamma)], x_{F_*}(t) = S^{-1}[G_{F_*}(u, \gamma)], x_F^*(t) = S^{-1}[G_F^*(u, \gamma)]. \end{aligned}$$

4 An application to Spring Mass system

Imagine a 4-ft elastic string connected to a hard framework. A 8lb load is on that spring. In equilibrium, the spring is 6 ft. If $FR = 2x$ is the damping force and the body is released from equilibrium with a downward beginning velocity of 1 feet per second, compute $x(t)$ analytically and using the fuzzy Sumudu transform for any time t .²⁴

Solution. This system of springs and masses in vibration is mathematically modelled by the second order ordinary differential equation as $px'' + qx' + kx = 0$, where the body's weight is located $p = 8/32 = 1/4$ slug, the spring constant with $k = 4lb/ft$. The force of resistance $F_R = qdx/dt = 2dx/dt$ with the initial conditions with $x(0) = 1, x'(0) = 1$. Hence, we obtain the ordinary differential equation of the second order $x''(t) + 8x'(t) + 16x(t) = 0$ with the initial conditions $x(0) = 1, x'(0) = 1$.

Crisp solution

As analytic results, we have $x(t) = c_1e^{-4t}$. Using initial values $x(0) = 0, x'(0) = 1$, in the solution, we get $x(t) = te^{-4t}$. At $t = 0.1$, we get $x(0.1) = 0.067$.

Neutrosophic solution

In case of imprecision in string length measurement or haziness in the beginning circumstances, both can be represented using Neutrosophic numbers. Thus, the above mechanical spring mass system's ordinary differential equation is transformed into a Neutrosophic initial value problem $\tilde{x}'' + 8\tilde{x}'(t) + 16\tilde{x}(t) = \tilde{0}$ with

$$\tilde{x}_T(0) = (0, 1 - \alpha, 1 - \alpha), \tilde{x}'_T(0) = (0.5, 0.5 - \alpha, 1.5 - \alpha),$$

$$\begin{aligned} \tilde{x}_I(0) &= (0, 0.5\beta, 0.5\beta), \tilde{x}'_I(0) = (1, 0.5\beta, 0.5\beta), \\ \tilde{x}_F(0) &= (0, 0.2\gamma, 0.2\gamma), \tilde{x}'_F(0) = (1, 0.2\gamma, 0.2\gamma). \end{aligned}$$

If we apply the Neutrosophic Sumudu Transform on the expressions for Membership, Indeterminacy, and Non-membership, we obtain,

$$\begin{aligned} \tilde{x}_T(u, \alpha) &= \left(\frac{1 + 8u}{(1 + 4u)^2} (0, 1 - \alpha, 1 - \alpha) + \frac{u}{(1 + 4u)^2} (0.5, 0.5 - \alpha, 1.5 - \alpha) \right), \\ \tilde{x}_I(u, \beta) &= \left(\frac{1 + 8u}{(1 + 4u)^2} (0, 0.5\beta, 0.5\beta) + \frac{u}{(1 + 4u)^2} (1, 0.5\beta, 0.5\beta) \right), \\ \tilde{x}_F(u, \gamma) &= \left(\frac{1 + 8u}{(1 + 4u)^2} (0, 0.2\gamma, 0.2\gamma) + \frac{u}{(1 + 4u)^2} (1, 0.2\gamma, 0.2\gamma) \right). \end{aligned}$$

Membership, indeterminacy, and non-membership functions are determined using the inverse sumudu transform for the location index and fuzziness index functions :

$$\begin{aligned} \tilde{x}_T(t, \alpha) &= (e^{-4t} (0, 1 - \alpha, 1 - \alpha) + 4te^{-4t} (0, 1 - \alpha, 1 - \alpha)), \\ \tilde{x}_I(t, \beta) &= (e^{-4t} (0, 0.5\beta, 0.5\beta) + te^{-4t} (1, 0.5\beta, 0.5\beta)), \\ \tilde{x}_F(t, \gamma) &= (e^{-4t} (0, 0.2\gamma, 0.2\gamma) + te^{-4t} (1, 0.2\gamma, 0.2\gamma)). \end{aligned}$$

4.1 Results and Discussions

While solving the fuzzy differential equation, alternative values of α, β and γ are considered by expressing the initial condition as location index and fuzziness index functions. The solution at $t = 0.1$ is covered in table-1.

From table 1, it is observed that for truth membership value of $x_T(t, \alpha)$, both the left and right fuzziness index functions are shows an inverse behaviour, i.e one is increasing and the other is decreasing. When the value of α approaches to 1 the crisp solution matches with this fuzzy solution.

It is also observed that for indeterminacy membership value $x_I(t, \beta)$ and for False membership value $x_F(t, \gamma)$, left fuzziness index function is decreasing and the right fuzziness index function is increasing. When the values of β, γ approaches to 0 the crisp solution matches with this fuzzy solution.

In Figure 1, the graph is constructed for various values of α, β and γ at $t = 0.1$. It is shown that the answer becomes closer to the precise solution as α (the membership value) increases and as β and γ (the indeterminacy and non-membership values, respectively) decrease.

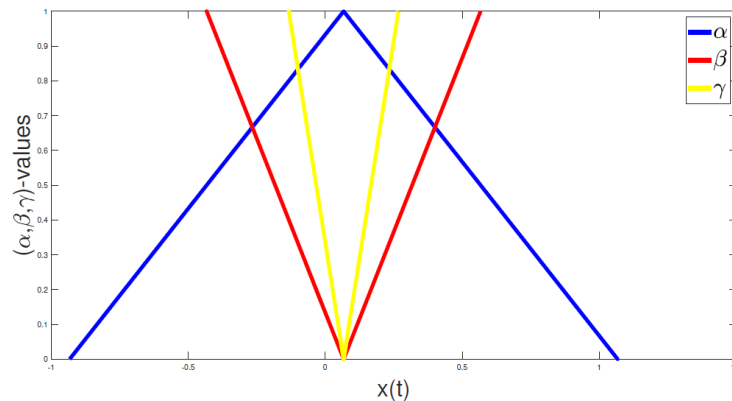


Figure 1: The graph represents the variation for different values of (α, β, γ) -cut and at time $t = 0.1$.

Table 1: The solution in terms of the location index and fuzziness index functions at $t = 0.1$, compared with the crisp solution.

Values of α, β, γ	$x_T(t, \alpha)$	$x_I(t, \beta)$	$x_F(t, \gamma)$	Exact solution
0	(-0.93296, 0.067032, 1.067032)	(0.067032, 0.067032, 0.067032)	(0.0670300, 0.067032, 0.0670320)	0.067032
0.1	(-0.8329, 0.067032, 0.967032)	(0.01703, 0.067032, 0.117032)	(0.047032, 0.067032, 0.087032)	...
0.2	(-0.73296, 0.067032, 0.867032)	(-0.033516, 0.067032, 0.167032)	(0.027032, 0.067032, 0.107032)	...
0.3	(-0.632968, 0.067032, 0.767032)	(-0.082968, 0.067032, 0.217032)	(0.007032, 0.067032, 0.127032)	...
0.4	(-0.532968, 0.067032, 0.667032)	(-0.132968, 0.067032, 0.267032)	(-0.012968, 0.067032, 0.147032)	...
0.5	(-0.432968, 0.067032, 0.567032)	(-0.182968, 0.067032, 0.317032)	(-0.032968, 0.067032, 0.167032)	...
0.6	(-0.332968, 0.067032, 0.467032)	(-0.23296, 0.067032, 0.367032)	(-0.052968, 0.067032, 0.187032)	...
0.7	(-0.232968, 0.067032, 0.367032)	(-0.282968, 0.067032, 0.417032)	(-0.072968, 0.067032, 0.207032)	...
0.8	(-0.132968, 0.067032, 0.267032)	(-0.332968, 0.067032, 0.467032)	(-0.09268, 0.067032, 0.227032)	...
0.9	(-0.032968, 0.067032, 0.167032)	(-0.382968, 0.067032, 0.517032)	(-0.112968, 0.067032, 0.247032)	...
1	(0.067032, 0.067032, 0.067032)	(-0.432980, 0.067032, 0.567032)	(-0.132968, 0.067032, 0.267032)	0.067032

5 Conclusion

Any dynamical systems under uncertainty and imprecision can be modelled in a proper way by using fuzzy differential equations. In this paper, we have presented a new technique by applying NST to solve fuzzy differential equations without reducing to equivalent crisp forms. Neutrosophic Sumudu transform is useful to solve problems without generating a new frequency domain due to the scale and unit-preserving properties of the Sumudu transform. The uncertain parameters of the problem are treated as neutrosophic numbers and are expressed in terms of their parametric forms, i.e. in the form of location index, left and right fuzziness index functions. Using a novel type of arithmetic operations on the parametric forms of neutrosophic numbers, significant theorems and systematic techniques are outlined for the neutrosophic solution of fuzzy differential equations without changing to crisp forms. A second-order ordinary fuzzy differential equation involving neutrosophic numbers based on a vibrating spring mass system is investigated and solved utilising NST without converting to crisp form. The solution is displayed in the form of a table at various computational points. Additionally, the results at various (α, β, γ) -cut and time values are graphically represented and are analytically verified by defuzzifying the data. The proposed methodology can be applied for solving real world problems arising in dynamical systems.

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