On Energy of Double Dominating Bipolar Single-Valued Neutrosophic Graph

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Abstract

Domination and graph energy are fundamental concepts in graph theory for addressing unpredictable phenomena, and they have attracted considerable interest from researchers. In recent developments, the concept of dominating energy has become increasingly significant in the study of graph energies. While fuzzy graphs (FG) sometimes fall short in delivering optimal results, the neutrosophic set (NS) as well as neutrosophic graphs (NG) offer a robust alternative, effectively managing the uncertainties linked with inconsistent and indeterminate information in real-world scenarios. Most existing research on domination energy in the fuzzy environment focus solely on a single membership function. In contrast, bipolar neutrosophic models, which account for both positive and negative influences, provide a more versatile and applicable approach. This paper focuses on advancements in NG theory to address scenarios where imprecision is represented by both positive and negative membership functions. It introduces a new concept called the double dominating energy graph, relying on the currently developed bipolar single-valued neutrosophic graphs (BSVNG). The study further explores the energy of double domination within the BSVNG framework. Specifically, it develops the adjacency matrix of a dominating BSVNG, analyzes the spectrum of this matrix, and elaborates on the associated theoretical aspects using illustrative examples. Additionally, the double domination energy of BSVNG is calculated to demonstrate its applicability. At the end of this study, conclusions are drawn and avenues for future research are discussed.

Keywords: Bipolar neutrosophic; Domination; Double domination

1. Introduction

Graph energy is the sum of the absolute values of respective eigenvalues, a concept initially introduced by I. Gutman, inspired by its relevance to chemical studies. In the field of chemistry, the molecular energy graph holds significance due to its connection to the total π-electron energy of the molecule that the graph represents. This relationship helps in understanding molecular stability and reactivity, making the graph energy concept specifically valuable in molecular chemistry. A graph having all isolated vertices K_n^c possesses zero energy; while the complete graph K_n having, n vertices possess energy $2(n-1)$. The concept of domination in graphs can be applied to a wide range of problems, which includes facility location, matching theory, social network

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analysis, communication networks, coding theory, security systems, and logistical operations like clutters and block cutters. For instance, in facility location scenarios, such as determining the placement of police stations, hospitals, fire stations, or even supermarkets, domination theory helps optimize the locations by minimizing the distance people must travel to reach the nearest facility. The goal is often to reduce the maximum distance to a facility while minimizing the number of facilities required serving the entire population efficiently. Additionally, the concept of domination is valuable in monitoring communication or electrical networks, selecting representative groups, and conducting land surveys, where the aim is to ensure comprehensive coverage with minimal resources.

In real-life scenarios, conditions are often uncertain and may not be accurately represented by deterministic or crisp models, making it difficult to make precise statements. To address these vague as well as imprecise phenomena, Zadeh [1] established the concept of fuzzy sets, incorporating degrees of membership for elements within these sets. Since then, the idea of fuzzy sets has evolved, leading to the development of more advanced models such as intuitionistic fuzzy sets, hesitant fuzzy sets, orthopair fuzzy sets as well as neutrosophic sets, which give a more flexible framework for dealing with uncertainty in various contexts.

Despite these extensions, fuzzy sets still struggle to address certain types of uncertainties, particularly when dealing with indeterminate or inconsistent information. To overcome this limitation, Smarandache [2] introduced neutrosophic logic and neutrosophic sets. A neutrosophic set consists of three components: the degree of truth (T), the degree of indeterminacy (I), and the degree of falsity (F). These components are defined within the non-standard unit interval $]^-0, 1^+[$, which allows for a more comprehensive representation of uncertainty, accommodating values beyond traditional boundaries.

In a neutrosophic set, the membership function is defined by the truth subset, the non-membership function by the falsity subset, and the hesitancy or uncertainty by the indeterminacy subset. These unique characteristics give neutrosophic sets an advantage over other extensions of fuzzy sets, as they provide a more nuanced way to handle varying degrees of uncertainty. We employ neutrosophic sets primarily because they excel in distinguishing between the relative and absolute preferences of decision-makers, offering a more precise representation of ambiguity and uncertainty in decision-making processes.

Bipolarity is the human mind tendency to evaluate situations and make decisions relying on both positive as well as negative outcomes. Positive analysis involves reasoning that aligns with what is acceptable, appropriate, or permissible, while negative analysis focuses on what is impossible, forbidden, or rejected. In this context, positive thoughts relate to preferences, as they determine which options are more desirable without necessarily dismissing those that fall outside these preferences. On the other hand, negative thoughts act as constraints, clearly indicating which values or options should be rejected or avoided. Building on these concepts, Deli et al. [3] proposed extending bipolar fuzzy sets and neutrosophic sets into the framework of bipolar neutrosophic sets, where they analysed positive and negative membership degrees along with their respective operations. Bipolar fuzzy sets are particularly valuable for handling uncertainty in real-life scenarios as they effectively address both positive and negative membership values, providing a more comprehensive approach to evaluating situations with inherent dualities.

2. Literature review

In the year 1978, I. Gutman introduced the concept of "graph energy," expressed as the sum of the absolute values of the eigenvalues of a graph's adjacency matrix. This idea connects the structure of a graph to the electron energy of a molecule, making graph energy a useful tool in quantum theory and various energy-related

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applications. Later, Gutman and Zhou [5] expanded this concept by defining the Laplacian energy of a graph as the sum of the absolute differences between the average vertex degree of the graph and its Laplacian eigenvalues.

Following Zadeh's expansion of fuzzy sets, Kauffman [6] as well as Rosenfeld [7] established the concept of fuzzy graphs to handle uncertainty within graph structures. Fuzzy graphs effectively model the relationships between objects when the existence of the objects or the connections between them are ambiguous or uncertain. Later, Anjali and Mathew [8] pioneered the idea of graph energy in the context of fuzzy sets. Building on this foundation, Praba et al. [9] extended the concept further by developing the notion of energy for intuitionistic fuzzy graphs, broadening its application to scenarios with even higher levels of uncertainty.

Basha and Kartheek [10] later introduced the innovative concept of Laplacian energy graphs within the intuitionistic fuzzy framework, exploring various properties associated with these graphs. Moreover, Akram et al. [11] also developed the concept of energy in bipolar fuzzy graphs (BFGs), demonstrating its application in multi-criteria decision-making for commercial partnerships and resolving communication challenges. More recently, Patra et al. [12] proposed new methodologies for calculating graph energy in interval-valued fuzzy graphs, utilizing max–min operators to compute the eigenvalues.

The concept of domination is a fundamental topic in graph theory. Somasundaram [13] further explored domination in the context of products of fuzzy graphs, examining operations such as join, union, composition, Cartesian product, and their domination parameters. Furthermore, Ghobadi et al. [14] later expanded the field by presenting the concept of an inverse dominating set in fuzzy graphs. Additionally, Natarajan and Ayyaswamy [15] developed the idea of strong and weak domination within fuzzy graphs, broadening the scope of domination theory in fuzzy environments.

Karunambigai et al. [16] explored various aspects of BFGs focusing on the concepts of cardinality, dominating set, independent set, total dominating number, and independent dominating number. Building on these foundations, Umamageswari et al. [17] introduced the idea of multiple dominating sets within BFGs, where they defined the k-dominating set and its domination number, further expanding the study of domination parameters in this context. Akram et al. [18] expanded the discussion by examining various domination concepts, including dominating sets, total dominating sets, equitable dominating sets, total equitable dominating sets, and independent and equitable independent sets within BFGs.

The study of domination has recently expanded into the neutrosophic environment, drawing considerable attention. Banitalebi and Borzooei [19] later extended this work by exploring the notion of a neutrosophic special dominating set, defining related parameters such as neutrosophic special domination numbers, neutrosophic special cobondage sets, and neutrosophic special cobondage numbers. Additionally, Ramya et al. [20] introduced the concept of complementary domination in single-valued neutrosophic graphs (SVNG) and investigated the bounds and characteristics of the inverse domination number (IDN) within various SVNG scenarios.

Additionally, Kartheek and Basha [21] extended the concept of minimum dominating energy to fuzzy graphs, defining its properties and establishing various upper and lower bounds for this energy in fuzzy graph contexts. Apart from that, Kalimulla et al. [22] explored the transition of energy from IFG to dominating energy through operations on IFGs. Their research-utilized operations like complement, union, join, Cartesian product, and composition to calculate the value of dominating energy, offering deeper insights into the dynamics of graph operations.

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Vijayaragavan et al. [23] investigated the dominating Laplacian energy in fuzzy graphs, specifically focusing on the Cartesian product and tensor product. They explored the relationship between these products concerning the dominating Laplacian energy in IFGs. Sarwar et al. [24] introduced new concepts of dominating and double dominating energy in *m*-polar fuzzy graphs and developed a decision-making model relying on *m*-polar fuzzy preference relations to tackle multi-criteria decision-making problems. In addition, Akram et al. [25] suggested novel ideas related to the energy of dominating BFGs and the energy of double dominating BFGs, further advancing the study of energy in fuzzy graph frameworks.

Since the introduction of neutrosophic sets, numerous researchers have sought to combine the study of energy graphs, dominating sets, and neutrosophic sets. For example, in 2020, Mullai and Broumi [26] proposed the concept of dominating energy within SVNG. More recently, Mohamad et al. [27] extended this idea to include dominating energy in BSVNG. Building on these foundations motivated by [25, 26, 27], we propose a new approach by introducing the concept of double dominating energy in BSVNG. Our analysis focuses on the double dominating energy within the BSVNG framework, where we develop the adjacency matrix, explore the spectrum of the adjacency matrix, and elaborate on the associated theoretical aspects using illustrative examples. Furthermore, we compute the double domination energy of BSVNG, providing a deeper understanding of its properties and applications.

The structure of this study is organized as follows: The first section provides an overview of the historical background of domination and energy graphs, as well as an introduction to the concepts of fuzzy and neutrosophic sets. The second section reviews existing literature on domination and energy graphs within both fuzzy and neutrosophic environments. In the third section, a concise introduction to graphs and neutrosophic sets is presented, setting the stage for their use in the study. The fourth section explores double dominating energy in a variety of operations within BSVNG and introduces new theorems related to these concepts. Finally, the fifth section summarizes the research findings, discusses its limitations, and provides recommendations for future research directions.

3. Preliminaries

This section recalls several fundamental definitions corrsponding to BFG, BSVNG as well as domination essential for this research.

Definition 3.1 [28]

A BFG with a finite set *X* as the underlying set refers to a pair of $G=(V,E)$, in which $V=(\mu_V^+,\mu_V^-)$ denotes a bipolar fuzzy set on *X*, while $E=(\mu_E^+,\mu_E^-)$ denotes a bipolar fuzzy relation on *X* provided that $\mu_E^+(v_m v_n)\le \mu_V^+(v_m)\wedge\mu_V^+(v_n)$ and $\mu_E^-(v_m v_n)\ge \mu_V^+(v_m)\vee\mu_V^+(v_n)$ where $v_m v_n\in E$. We call $V=(\mu_V^+,\mu_V^-)$ the bipolar fuzzy vertex set of *X*, while $E=(\mu_E^+,\mu_E^-)$ denotes the bipolar fuzzy edge set of *X*.

Definition 3.2 [3]

A bipolar neutrosophic set *A* on a non-empty set *X* refers to an object of the form

$A=\left\{\left\langle x,T_A^+(x),I_A^+(x),F_A^+(x),T_A^-(x),I_A^-(x),F_A^-(x)\right\rangle : x\in X\right\}$,

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where $T_A^+, I_A^+, F_A^+ : X \to [0,1]$ and $T_A^-, I_A^-, F_A^- : X \to [-1,0]$. Here, the positive values $T_A^+(x), I_A^+(x), F_A^+(x)$ express the truth, indeterminacy as well as false-memberships degrees of an element $x \in X$, respectively, while $T_A^-(x), I_A^-(x), F_A^-(x)$ express the implicit counter property of the truth, indeterminacy and false-memberships degrees of the element $x \in X$, accordingly, which corresponds to the bipolar neutrosophic set *A*.

Definition 3.3 [29]

Let $A = (T_A^+, I_A^+, F_A^+, T_A^-, I_A^-, F_A^-)$ and $B = (T_B^+, I_B^+, F_B^+, T_B^-, I_B^-, F_B^-)$ be bipolar NG on a set X. If $B = (T_B^+, I_B^+, F_B^+, T_B^-, I_B^-, F_B^-)$ denotes a bipolar neutrosophic relation on $A = (T_A^+, I_A^+, F_A^+, T_A^-, I_A^-, F_A^-)$, then

$$T_B^+(xy) \le \min(T_A^+(x), T_A^+(y)), I_B^+(xy) \ge \max(I_A^+(x), I_A^+(y)), F_B^+(xy) \ge \max(F_A^+(x), F_A^+(y)),$$
$$T_B^-(xy) \ge \max(T_A^-(x), T_A^-(y)), I_B^-(xy) \le \min(I_A^-(x), I_A^-(y)), F_B^-(xy) \le \min(F_A^-(x), F_A^-(y)), \forall x, y \in X.$$

Definition 3.4 [30]

Suppose $G = (V, E)$ denote a BSVNG while $x, y \in V$ in G. We then state that *x* dominates *y* if

$$T_E^+(xy) = T_V^+(x) \wedge T_V^+(y), I_E^+(xy) = I_V^+(x) \vee I_V^+(y), F_E^+(xy) = F_V^+(x) \vee F_V^+(y),$$
$$T_E^-(xy) = T_V^-(x) \vee T_V^-(y), I_E^-(xy) = I_V^-(x) \wedge I_V^-(y), F_E^-(xy) = F_V^-(x) \wedge F_V^-(y).$$

4. Double dominating energy in BSVNG

Definition 4.1

A graph of the form $\tilde{G} = (V, E, \mu^+, \gamma^+, \sigma^+, \mu^-, \gamma^-, \sigma^-, \mu_1^+, \gamma_1^+, \sigma_1^+, \mu_1^-, \gamma_1^-, \sigma_1^-)$ is called double dominating BSVNG, in which $\mu_1^+ : V \to [0,1]$, refers to a positive degree of membership, $\gamma_1^+ : V \to [0,1]$, refers to a positive degree of indeterminacy, $\sigma_1^+ : V \to [0,1]$, refers to a positive degree of non-membership, $\mu_1^- : V \to [-1,0]$, refers to a negative degree of membership, $\gamma_1^- : V \to [-1,0]$, refers to a negative degree of indeterminacy, while $\sigma_1^- : V \to [-1,0]$, refers to a negative degree of non-membership defined provided that

$$\mu_1^+(v_m) = \min_{v_n}[\mu^+(v_m, v_n)], \gamma_1^+(v_m) = \max_{v_n}[\gamma^+(v_m, v_n)], \sigma_1^+(v_m) = \max_{v_n}[\sigma^+(v_m, v_n)],$$
$$\mu_1^-(v_m) = \max_{v_n}[\mu^-(v_m, v_n)], \gamma_1^-(v_m) = \min_{v_n}[\gamma^-(v_m, v_n)], \sigma_1^-(v_m) = \min_{v_n}[\sigma^+(v_m, v_n)].$$

Definition 4.2

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For any BSVNG, $\tilde{G}=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$, a subset $\tilde{D}_d \subseteq V$ is known as double dominating set (DDS) in $\tilde{G}$ provided that every vertex in $V-\tilde{D}_d$ is dominated by at least two vertices in $\tilde{D}_d$. The minimum cardinality of all DDS of $\tilde{G}$ is known as the double domination number of $\tilde{G}$, represented by $\gamma_{D_d}^{N}\left(\tilde{G}\right)$.

Definition 4.3

Let $\tilde{G}=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ denote a double dominating BSVNG. Suppose $u,v\in V$. Here, u dominates v in $\tilde{G}$ provided that strong edges exists from u to v.

Definition 4.4

Let $\tilde{G}=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ denote a double dominating BSVNG. Thus, the adjacency matrix with respect to $\tilde{G}$ is expressed as $A_{\tilde{D}_d}\left(\tilde{G}\right)=\left[d_{mn}\right]$, in which

$$d_{mn}=\begin{cases}\begin{pmatrix}\mu_{mn}^{+},\gamma_{mn}^{+},\sigma_{mn}^{+}\\ \mu_{mn}^{-},\gamma_{mn}^{-},\sigma_{mn}^{-}\end{pmatrix} & if\ \left(v_m,v_n\right)\in E\\ \begin{pmatrix}1,1,1,\\ -1,-1,-1\end{pmatrix} & if\ m=n\ and\ v_m\in D_{\tilde{D}_d}\\ \begin{pmatrix}0,0,0,\\ 0,0,0\end{pmatrix} & otherwise\end{cases}$$

This dominating bipolar single-valued neutrosophic adjacency matrix, $A_{\tilde{D}_d}\left(\tilde{G}\right)$ may be expressed as $A_{\tilde{D}_d}\left(\tilde{G}\right)=\left(\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right),\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right),\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right),\mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right),\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right),\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)$ in which

$$\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right)=\begin{cases}\mu_{mn}^{+} & if\ \left(v_m,v_n\right)\in E\\ 1 & if\ m=n\ and\ v_j\in\tilde{D}_d\\ 0 & otherwise\end{cases},\qquad \gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)=\begin{cases}\gamma_{mn}^{+} & if\ \left(v_m,v_n\right)\in E\\ 1 & if\ m=n\ and\ v_m\in\tilde{D}_d\\ 0 & otherwise\end{cases},$$

$$\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)=\begin{cases}\sigma_{mn}^{+} & if\ \left(v_m,v_n\right)\in E\\ 1 & if\ m=n\ and\ v_m\in\tilde{D}_d\\ 0 & otherwise\end{cases},\qquad \mu_{\tilde{D}_d}^{-}\left(G\right)=\begin{cases}\mu_{mn}^{-} & if\ \left(v_m,v_n\right)\in E\\ -1 & if\ m=n\ and\ v_m\in\tilde{D}_d\\ 0 & otherwise\end{cases},$$

$$\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)=\begin{cases}\gamma_{mn}^{-} & if\ \left(v_m,v_n\right)\in E\\ -1 & if\ m=n\ and\ v_m\in\tilde{D}_d\\ 0 & otherwise\end{cases},\qquad \sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)=\begin{cases}\sigma_{mn}^{-} & if\ \left(v_m,v_n\right)\in E\\ -1 & if\ m=n\ and\ v_m\in\tilde{D}_d\\ 0 & otherwise\end{cases}.$$

Definition 4.5

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The spectrum with respect to the adjacency matrix of a double dominating BSVNG $\tilde{G}$ is denoted as $\left(S_{\tilde{D}_d}^{\mu^+}, S_{\tilde{D}_d}^{\gamma^+}, S_{\tilde{D}_d}^{\sigma^+}, S_{\tilde{D}_d}^{\mu^-}, S_{\tilde{D}_d}^{\gamma^-}, S_{\tilde{D}_d}^{\sigma^-}\right)$, where $S_{\tilde{D}_d}^{\mu^+}, S_{\tilde{D}_d}^{\gamma^+}, S_{\tilde{D}_d}^{\sigma^+}, S_{\tilde{D}_d}^{\mu^-}, S_{\tilde{D}_d}^{\gamma^-}, S_{\tilde{D}_d}^{\sigma^-}$ are the sets of eigenvalues of $\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right), \gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right), \sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right), \mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right), \gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right),$ and $\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)$, respectively.

Definition 4.6

The energy of a double dominating BSVNG $\tilde{G} = \left(V, E, \mu^+, \gamma^+, \sigma^+, \mu^-, \gamma^-, \sigma^-, \mu_1^+, \gamma_1^+, \sigma_1^+, \mu_1^-, \gamma_1^-, \sigma_1^-\right)$ is defined as

$$
\begin{aligned}
E_{\tilde{D}_d}\left(\tilde{G}\right) &= \left(E\left(\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right), E\left(\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right), E\left(\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right), E\left(\mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right), E\left(\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right), E\left(\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\right) \\
&= \left(\sum_{p=1}^{n}\left|\zeta_p\right|, \sum_{p=1}^{n}\left|\tau_p\right|, \sum_{p=1}^{n}\left|\upsilon_p\right|, \sum_{p=1}^{n}\left|\vartheta_p\right|, \sum_{p=1}^{n}\left|\xi_p\right|, \sum_{p=1}^{n}\left|\varepsilon_p\right|\right)
\end{aligned}
$$

where $S_{\tilde{D}_d}^{\mu^+} = \left\{\zeta_p\right\}_{p=1}^{n}, S_{\tilde{D}_d}^{\gamma^+} = \left\{\tau_p\right\}_{p=1}^{n}, S_{\tilde{D}_d}^{\sigma^+} = \left\{\upsilon_p\right\}_{p=1}^{n}, S_{\tilde{D}_d}^{\mu^-} = \left\{\vartheta_p\right\}_{p=1}^{n}, S_{\tilde{D}_d}^{\gamma^-} = \left\{\xi_p\right\}_{p=1}^{n}$ and $S_{\tilde{D}_d}^{\sigma^-} = \left\{\varepsilon_p\right\}_{p=1}^{n}$.

Example 4.7

Consider a double dominating BSVNG $\tilde{G} = \left(V, E, \mu^+, \gamma^+, \sigma^+, \mu^-, \gamma^-, \sigma^-, \mu_1^+, \gamma_1^+, \sigma_1^+, \mu_1^-, \gamma_1^-, \sigma_1^-\right)$ on $V = \left\{v_1, v_2, v_3, v_4, v_5, v_6\right\}$ and $\mu_1^+, \gamma_1^+, \sigma_1^+, \mu_1^-, \gamma_1^-, \sigma_1^-$ are defined by $\mu_1^+ : V \to \left[0,1\right], \gamma_1^+ : V \to \left[0,1\right]$, $\sigma_1^+ : V \to \left[0,1\right], \mu_1^- : V \to \left[-1,0\right], \gamma_1^- : V \to \left[-1,0\right], \sigma_1^- : V \to \left[-1,0\right]$, as shown in Figure 1, where

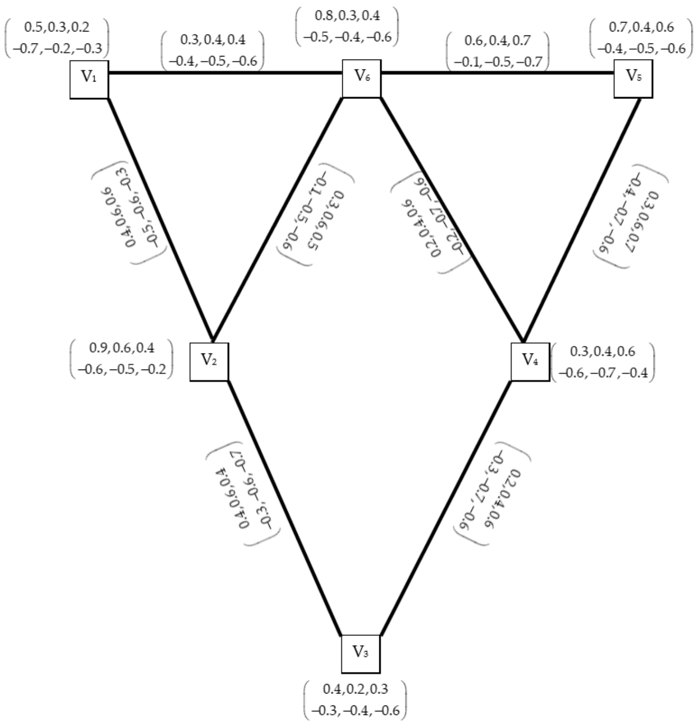

Figure 1. Double dominating BSVNG

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$$\begin{aligned}\mu_1^+(v_1) &= \min_{v_n}\left[\mu^+(v_m,v_n)\right] \\ &= \min\left[\mu^+(v_1,v_2),\mu^+(v_1,v_6)\right] \\ &= \min\left[0.4,0.3\right]=0.3\end{aligned}$$

$$\begin{aligned}\gamma_1^+(v_1) &= \max_{v_n}\left[\gamma^+(v_m,v_n)\right] \\ &= \max\left[\gamma^+(v_1,v_2),\gamma^+(v_1,v_6)\right] \\ &= \max\left[0.4,0.6\right]=0.6\end{aligned}$$

$$\begin{aligned}\sigma_1^+(v_1) &= \max_{v_n}\left[\sigma^+(v_m,v_n)\right] \\ &= \max\left[\sigma^+(v_1,v_2),\sigma^+(v_1,v_6)\right] \\ &= \max\left[0.6,0.4\right]=0.6\end{aligned}$$

$$\begin{aligned}\mu_1^-(v_1) &= \max_{v_n}\left[\mu^-(v_m,v_n)\right] \\ &= \max\left[\mu^-(v_1,v_2),\mu^-(v_1,v_6)\right] \\ &= \max\left[-0.5,-0.4\right]=-0.4\end{aligned}$$

$$\begin{aligned}\gamma_1^-(v_1) &= \min_{v_n}\left[\gamma^-(v_m,v_n)\right] \\ &= \min\left[\gamma^-(v_1,v_2),\gamma^-(v_1,v_6)\right] \\ &= \min\left[-0.6,-0.5\right]=-0.6\end{aligned}$$

$$\begin{aligned}\sigma_1^-(v_1) &= \min_{v_n}\left[\sigma^-(v_m,v_n)\right] \\ &= \min\left[\sigma^-(v_1,v_2),\sigma^-(v_1,v_6)\right] \\ &= \min\left[-0.3,-0.6\right]=-0.6\end{aligned}$$

Similarly,

$\mu_1^+(v_2)=0.3,\mu_1^+(v_3)=0.2,\mu_1^+(v_4)=0.2,\mu_1^+(v_5)=0.3,\mu_1^+(v_6)=0.2$,

$\gamma_1^+(v_2)=0.6,\gamma_1^+(v_3)=0.6,\gamma_1^+(v_4)=0.6,\gamma_1^+(v_5)=0.6,\gamma_1^+(v_6)=0.6$,

$\sigma_1^+(v_2)=0.6,\sigma_1^+(v_3)=0.6,\sigma_1^+(v_4)=0.7,\sigma_1^+(v_5)=0.7,\sigma_1^+(v_6)=0.7$,

$\mu_1^-(v_2)=-0.1,\mu_1^-(v_3)=-0.3,\mu_1^-(v_4)=-0.2,\mu_1^-(v_5)=-0.1,\mu_1^-(v_6)=-0.1$,

$\gamma_1^-(v_2)=-0.6,\gamma_1^-(v_3)=-0.7,\gamma_1^-(v_4)=-0.7,\gamma_1^-(v_5)=-0.7,\gamma_1^-(v_6)=-0.7$,

$\sigma_1^-(v_2)=-0.7,\sigma_1^-(v_3)=-0.7,\sigma_1^-(v_4)=-0.6,\sigma_1^-(v_5)=-0.7,\sigma_1^-(v_6)=-0.7$.

Thus, $\tilde{D}_d^1=\{V_2,V_4,V_6\}$ and $\tilde{D}_d^2=\{V_1,V_3,V_5\}$ are DDS since every vertex in $V-\tilde{D}_d^1$ as well as $V-\tilde{D}_d^2$, is dominated by at least two vertices in $\tilde{D}_d$.

1. For $\tilde{D}_d^1=\{V_2,V_4,V_6\}$, the adjacency matrix of $\tilde{G}$ is as follows:

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$$A_{\tilde{D}_d^1}\left(\tilde{G}\right)=$$

$$\begin{bmatrix} \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.4,0.6,0.6\\-0.5,-0.6,-0.3\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.3,0.4,0.4\\-0.4,-0.5,-0.6\end{pmatrix} \\ \begin{pmatrix}0.4,0.6,0.6\\-0.5,-0.6,-0.3\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.4,0.6,0.4\\-0.3,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.5\\-0.1,-0.5,-0.6\end{pmatrix} \\ \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.4,0.6,0.4\\-0.3,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.2,0.4,0.6\\-0.3,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} \\ \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.2,0.4,0.6\\-0.3,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.7\\-0.4,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}0.2,0.4,0.6\\-0.2,-0.7,-0.6\end{pmatrix} \\ \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.7\\-0.4,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.6,0.4,0.7\\-0.1,-0.5,-0.7\end{pmatrix} \\ \begin{pmatrix}0.3,0.4,0.4\\-0.4,-0.5,-0.6\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.5\\-0.1,-0.5,-0.6\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.2,0.4,0.6\\-0.2,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}0.6,0.4,0.7\\-0.1,-0.5,-0.7\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} \end{bmatrix}$$

This can be written in six different matrices as:

$$A\left(\mu^{+}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)=\begin{bmatrix} 0 & 0.4 & 0 & 0 & 0 & 0.3 \\ 0.4 & 1 & 0.4 & 0 & 0 & 0.3 \\ 0 & 0.4 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 1 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0.3 & 0 & 0.6 \\ 0.3 & 0.3 & 0 & 0.2 & 0.6 & 1 \end{bmatrix} \quad A\left(\gamma^{+}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)=\begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0.6 & 1 & 0.6 & 0 & 0 & 0.6 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 1 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 & 0.4 & 0.4 & 1 \end{bmatrix}$$

$$A\left(\sigma^{+}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)=\begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0.6 & 1 & 0.4 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 1 & 0.7 & 0.6 \\ 0 & 0 & 0 & 0.7 & 0 & 0.7 \\ 0.4 & 0.5 & 0 & 0.6 & 0.7 & 1 \end{bmatrix} \quad A\left(\mu^{-}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)=\begin{bmatrix} 0 & -0.5 & 0 & 0 & 0 & -0.4 \\ -0.5 & -1 & -0.3 & 0 & 0 & -0.1 \\ 0 & -0.3 & 0 & -0.3 & 0 & 0 \\ 0 & 0 & -0.3 & -1 & -0.4 & -0.2 \\ 0 & 0 & 0 & -0.4 & 0 & -0.1 \\ -0.4 & -0.1 & 0 & -0.2 & -0.1 & -1 \end{bmatrix}$$

$$A\left(\gamma^{-}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)=\begin{bmatrix} 0 & -0.6 & 0 & 0 & 0 & -0.5 \\ -0.6 & -1 & -0.6 & 0 & 0 & -0.5 \\ 0 & -0.6 & 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.7 & -1 & -0.7 & -0.7 \\ 0 & 0 & 0 & -0.7 & 0 & -0.5 \\ -0.5 & -0.5 & 0 & -0.7 & -0.5 & -1 \end{bmatrix} \quad A\left(\sigma^{-}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)=\begin{bmatrix} 0 & -0.3 & 0 & 0 & 0 & -0.6 \\ -0.3 & -1 & -0.7 & 0 & 0 & -0.6 \\ 0 & -0.7 & 0 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & -1 & -0.6 & -0.6 \\ 0 & 0 & 0 & -0.6 & 0 & -0.7 \\ -0.6 & -0.6 & 0 & -0.6 & -0.7 & -1 \end{bmatrix}$$

Since,

$$Spec\left(A\left(\mu^{+}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)\right)=\{-0.361,-0.248,-0.050,0.825,1.125,1.709\},$$

$$Spec\left(A\left(\gamma^{+}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)\right)=\{-0.568,-0.307,-0.192,0.622,1.282,2.162\},$$

$$Spec\left(A\left(\sigma^{+}_{\tilde{D}_d^1}\left(\tilde{G}\right)\right)\right)=\{-0.615,-0.395,-0.281,0.630,1.309,2.353\},$$

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$$Spec\left(A\left(\mu^{-}_{\tilde{D}^1_d}(\tilde{G})\right)\right)=\{-1.516,-1.170,-0.891,0.037,0.197,0.343\},$$

$$Spec\left(A\left(\gamma^{-}_{\tilde{D}^1_d}(\tilde{G})\right)\right)=\{-2.430,-1.313,-0.676,0.299,0.360,0.759\},$$

$$Spec\left(A\left(\sigma^{-}_{\tilde{D}^1_d}(\tilde{G})\right)\right)=\{-2.424,-1.190,-0.797,0.161,0.566,0.684\}.$$

Therefore,

$$Spec\left(A_{\tilde{D}^1_d}(\tilde{G})\right)=\left\{\begin{array}{c}(-0.361,-0.568,-0.615,-1.516,-2.430,2.424),\\(-0.248,-0.307,-0.395,-1.170,-1.313,-1.190),\\(-0.050,-0.192,-0.281,-0.891,-0.676,-0.797),\\(0.825,0.622,0.630,0.037,0.299,0.161),\\(1.125,1.282,1.309,0.197,0.360,0.566),\\(1.709,2.162,2.353,0.343,0.759,0.684)\end{array}\right\}.$$

Hence, the energy of double dominating BSVNG $\tilde{G}$ is

$$\begin{aligned}E_{\tilde{D}^1_d}(\tilde{G})&=\left(E\left(\mu^{+}_{\tilde{D}^1_d}(\tilde{G})\right),E\left(\gamma^{+}_{\tilde{D}^1_d}(\tilde{G})\right),E\left(\sigma^{+}_{\tilde{D}^1_d}(\tilde{G})\right),E\left(\mu^{-}_{\tilde{D}^1_d}(\tilde{G})\right),E\left(\gamma^{-}_{\tilde{D}^1_d}(\tilde{G})\right),E\left(\sigma^{-}_{\tilde{D}^1_d}(\tilde{G})\right)\right)\\&=\left(\sum_{p=1}^{n}|\zeta_p|,\sum_{p=1}^{n}|\tau_p|,\sum_{p=1}^{n}|\upsilon_p|,\sum_{p=1}^{n}|\vartheta_p|,\sum_{p=1}^{n}|\xi_p|,\sum_{p=1}^{n}|\varepsilon_p|\right)\\&=(4.318,5.133,5.583,4.154,5.837,5.822).\end{aligned}$$

2. For $\tilde{D}^2_d=\{V_1,V_3,V_5\}$, the adjacency matrix of $\tilde{G}$ is as follows:

$$A_{\tilde{D}^2_d}(\tilde{G})=$$

$$\begin{bmatrix}\begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix}&\begin{pmatrix}0.4,0.6,0.6\\-0.5,-0.6,-0.3\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.3,0.4,0.4\\-0.4,-0.5,-0.6\end{pmatrix}\\\begin{pmatrix}0.4,0.6,0.6\\-0.5,-0.6,-0.3\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.4,0.6,0.4\\-0.3,-0.6,-0.7\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.3,0.6,0.5\\-0.1,-0.5,-0.6\end{pmatrix}\\\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.4,0.6,0.4\\-0.3,-0.6,-0.7\end{pmatrix}&\begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix}&\begin{pmatrix}0.2,0.4,0.6\\-0.3,-0.7,-0.6\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\\\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.2,0.4,0.6\\-0.3,-0.7,-0.6\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.3,0.6,0.7\\-0.4,-0.7,-0.6\end{pmatrix}&\begin{pmatrix}0.2,0.4,0.6\\-0.2,-0.7,-0.6\end{pmatrix}\\\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.3,0.6,0.7\\-0.4,-0.7,-0.6\end{pmatrix}&\begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix}&\begin{pmatrix}0.6,0.4,0.7\\-0.1,-0.5,-0.7\end{pmatrix}\\\begin{pmatrix}0.3,0.4,0.4\\-0.4,-0.5,-0.6\end{pmatrix}&\begin{pmatrix}0.3,0.6,0.5\\-0.1,-0.5,-0.6\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}&\begin{pmatrix}0.2,0.4,0.6\\-0.2,-0.7,-0.6\end{pmatrix}&\begin{pmatrix}0.6,0.4,0.7\\-0.1,-0.5,-0.7\end{pmatrix}&\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\end{bmatrix}$$

This can be written in six different matrices as:

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$$A\left(\mu^{+}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\begin{bmatrix} 1 & 0.4 & 0 & 0 & 0 & 0.3 \\ 0.4 & 0 & 0.4 & 0 & 0 & 0.3 \\ 0 & 0.4 & 1 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0.3 & 1 & 0.6 \\ 0.3 & 0.3 & 0 & 0.2 & 0.6 & 0 \end{bmatrix} \quad A\left(\gamma^{+}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0.6 & 0 & 0.6 & 0 & 0 & 0.6 \\ 0 & 0.6 & 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.6 & 1 & 0.4 \\ 0.4 & 0.6 & 0 & 0.4 & 0.4 & 0 \end{bmatrix}$$

$$A\left(\sigma^{+}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0.5 \\ 0 & 0.4 & 1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.7 & 0.6 \\ 0 & 0 & 0 & 0.7 & 1 & 0.7 \\ 0.4 & 0.5 & 0 & 0.6 & 0.7 & 0 \end{bmatrix} \quad A\left(\mu^{-}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\begin{bmatrix} -1 & -0.5 & 0 & 0 & 0 & -0.4 \\ -0.5 & 0 & -0.3 & 0 & 0 & -0.1 \\ 0 & -0.3 & -1 & -0.3 & 0 & 0 \\ 0 & 0 & -0.3 & 0 & -0.4 & -0.2 \\ 0 & 0 & 0 & -0.4 & -1 & -0.1 \\ -0.4 & -0.1 & 0 & -0.2 & -0.1 & 0 \end{bmatrix}$$

$$A\left(\gamma^{-}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\begin{bmatrix} -1 & -0.6 & 0 & 0 & 0 & -0.5 \\ -0.6 & 0 & -0.6 & 0 & 0 & -0.5 \\ 0 & -0.6 & -1 & -0.7 & 0 & 0 \\ 0 & 0 & -0.7 & 0 & -0.7 & -0.7 \\ 0 & 0 & 0 & -0.7 & -1 & -0.5 \\ -0.5 & -0.5 & 0 & -0.7 & -0.5 & 0 \end{bmatrix} \quad A\left(\sigma^{-}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\begin{bmatrix} -1 & -0.3 & 0 & 0 & 0 & -0.6 \\ -0.3 & 0 & -0.7 & 0 & 0 & -0.6 \\ 0 & -0.7 & -1 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0 & -0.6 & -0.6 \\ 0 & 0 & 0 & -0.6 & -1 & -0.7 \\ -0.6 & -0.6 & 0 & -0.6 & -0.7 & 0 \end{bmatrix}$$

Since,

$$Spec\left(A\left(\mu^{+}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)\right)=\left\{-0.503,-0.149,-0.087,1.040,1.181,1.517\right\},$$

$$Spec\left(A\left(\gamma^{+}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)\right)=\left\{-0.884,-0.285,-0.066,1.091,1.286,1.858\right\},$$

$$Spec\left(A\left(\sigma^{+}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)\right)=\left\{-0.927,-0.380,-0.187,1.131,1.303,2.060\right\},$$

$$Spec\left(A\left(\mu^{-}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)\right)=\left\{-1.408,-1.173,-1.033,-0.006,0.284,0.336\right\},$$

$$Spec\left(A\left(\gamma^{-}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)\right)=\left\{-2.119,-1.314,-1.155,0.176,0.316,1.095\right\},$$

$$Spec\left(A\left(\sigma^{-}_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)\right)=\left\{-2.111,-1.273,-1.153,0.113,0.317,1.106\right\}.$$

Therefore,

$$Spec\left(A_{\tilde{D}^2_d}\left(\tilde{G}\right)\right)=\left\{\begin{array}{c} \left(-0.503,-0.884,-0.927,1.408,-2.119,-2.111\right), \\ \left(-0.149,-0.285,-0.380,-1.173,-1.314,-1.273\right), \\ \left(-0.087,-0.066,-0.187,-1.033,-1.155,-1.153\right), \\ \left(1.040,1.091,1.131,-0.006,0.176,0.113\right), \\ \left(1.181,1.286,1.303,0.284,0.316,0.317\right), \\ \left(1.517,1.858,2.060,0.336,1.095,1.106\right) \end{array}\right\}.$$

Hence, the energy of double dominating BSVNG $\tilde{G}$ is

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$$E_{\tilde{D}_d^2}\left(\tilde{G}\right)=\left(E\left(\mu^{+}_{\tilde{D}_d^2}\left(\tilde{G}\right)\right),E\left(\gamma^{+}_{\tilde{D}_d^2}\left(\tilde{G}\right)\right),E\left(\sigma^{+}_{\tilde{D}_d^2}\left(\tilde{G}\right)\right),E\left(\mu^{-}_{\tilde{D}_d^2}\left(\tilde{G}\right)\right),E\left(\gamma^{-}_{\tilde{D}_d^2}\left(\tilde{G}\right)\right),E\left(\sigma^{-}_{\tilde{D}_d^2}\left(\tilde{G}\right)\right)\right)$$
$$=\left(\sum_{p=1}^{n}\left|\zeta_p\right|,\sum_{p=1}^{n}\left|\tau_p\right|,\sum_{p=1}^{n}\left|\upsilon_p\right|,\sum_{p=1}^{n}\left|\vartheta_p\right|,\sum_{p=1}^{n}\left|\xi_p\right|,\sum_{p=1}^{n}\left|\varepsilon_p\right|\right)$$
$$=\left(4.477,5.470,5.988,4.240,6.175,6.073\right).$$

Theorem 4.8

Suppose $\tilde{G}=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ refers to a double dominating BSVNG having n vertices as well as m edges. Suppose also that $\tilde{D}_d=\left\{\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_k\right\}$ is a DDS. Provided that $\zeta_1,\zeta_2,\ldots\zeta_n,\tau_1,\tau_2,\ldots\tau_n,\upsilon_1,\upsilon_2,\ldots\upsilon_n$, $\vartheta_1,\vartheta_2,\ldots,\vartheta_n,\xi_1,\xi_2,\ldots,\xi_n$, and $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ denote the eigen values with respect to adjacency matrix $A\left(\mu^{+}_{\tilde{D}_d}\left(\tilde{G}\right)\right)$, $A\left(\gamma^{+}_{\tilde{D}_d}\left(\tilde{G}\right)\right),A\left(\sigma^{+}_{\tilde{D}_d}\left(\tilde{G}\right)\right),A\left(\mu^{-}_{\tilde{D}_d}\left(\tilde{G}\right)\right),A\left(\gamma^{-}_{\tilde{D}_d}\left(\tilde{G}\right)\right),A\left(\sigma^{-}_{\tilde{D}_d}\left(\tilde{G}\right)\right)$, respectively, then

1. $\sum_{p=1}^{n}\zeta_p=\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\tau_p=\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\upsilon_p=\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\vartheta_p=-\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\xi_p=-\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\varepsilon_p=-\left|\tilde{D}_d\right|$.
2. $\sum_{p=1}^{n}\left(\zeta_p\right)^2=\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn},\sum_{p=1}^{n}\left(\tau_p\right)^2=\sum_{p=1}^{n}\left(\gamma^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\gamma^{+}_{pq}\gamma^{+}_{qn},\sum_{p=1}^{n}\left(\upsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\sigma^{+}_{pq}\sigma^{+}_{qn},$
$\sum_{p=1}^{n}\left(\vartheta_p\right)^2=\sum_{p=1}^{n}\left(\mu^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{-}_{pq}\mu^{-}_{qn},\sum_{p=1}^{n}\left(\xi_p\right)^2=\sum_{p=1}^{n}\left(\gamma^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\gamma^{-}_{pq}\gamma^{-}_{qn},\sum_{p=1}^{n}\left(\varepsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\sigma^{-}_{pq}\sigma^{-}_{qn}.$

Proof.

1. According to the trace property of matrices, where the sum of the eigenvalues equals the trace of the matrix, we can express the following:

$$\sum_{p=1}^{n}\zeta_p=\sum\mu^{+}_{pp}=\left|\tilde{D}_d\right|.$$

Analogously, we can show that

$$\sum_{p=1}^{n}\tau_p=\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\upsilon_p=\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\vartheta_p=-\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\xi_p=-\left|\tilde{D}_d\right|,\sum_{p=1}^{n}\varepsilon_p=-\left|\tilde{D}_d\right|.$$

2. Equivalently, the sum of square of eigenvalues of $\left(\mu^{+}_{\tilde{D}_d}\left(\tilde{G}\right)\right)$ is equal to the trace of $\left(\mu^{+}_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2$

$$tr\left(A\left(\mu^{+}_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2\right)=\sum_{p=1}^{n}\left(\zeta_p\right)^2$$
$$=\mu^{+}_{11}\mu^{+}_{11}+\mu^{+}_{12}\mu^{+}_{21}+\ldots+\mu^{+}_{1n}\mu^{+}_{n1}+\mu^{+}_{21}\mu^{+}_{12}+\mu^{+}_{22}\mu^{+}_{22}+\ldots+\mu^{+}_{2n}\mu^{+}_{n2}$$
$$+\ldots+\mu^{+}_{n1}\mu^{+}_{1n}+\mu^{+}_{n2}\mu^{+}_{2n}+\ldots+\mu^{+}_{nn}\mu^{+}_{nn}$$
$$=\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn}$$

Analogously, we can show that,

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$\sum_{p=1}^{n}\left(\tau_p\right)^2=\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+},\sum_{p=1}^{n}\left(\upsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+},\sum_{p=1}^{n}\left(\vartheta_p\right)^2=\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-},$

$\sum_{p=1}^{n}\left(\xi_p\right)^2=\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-},\sum_{p=1}^{n}\left(\varepsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}.$

This completes the proof.

Theorem 4.9

Suppose $\tilde{G}=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ denote a double dominating BSVNG having n vertices and m edges. If $\tilde{D}_d=\left\{\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_k\right\}$ is the DDS, then

i. $$\sqrt{\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}+n(n-1)\left|\tilde{A}\right|^{\frac{2}{n}}}\le E\left(\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\right)}$$

where $\left|\tilde{A}\right|$ refers to the determinant of $\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right)$.

ii. $$\sqrt{\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+}+n(n-1)\left|\tilde{B}\right|^{\frac{2}{n}}}\le E\left(\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+}\right)}$$

where $\left|\tilde{B}\right|$ refers to the determinant of $\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)$.

iii. $$\sqrt{\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+}+n(n-1)\left|\tilde{C}\right|^{\frac{2}{n}}}\le E\left(\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+}\right)}$$

where $\left|\tilde{C}\right|$ refers to the determinant of $\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)$.

iv. $$\sqrt{\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-}+n(n-1)\left|\tilde{D}\right|^{\frac{2}{n}}}\le E\left(\mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-}\right)}$$

where $\left|\tilde{D}\right|$ refers to the determinant of $\mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right)$.

v. $$\sqrt{\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-}+n(n-1)\left|\tilde{F}\right|^{\frac{2}{n}}}\le E\left(\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-}\right)}$$

where $\left|\tilde{F}\right|$ refers to the determinant of $\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)$.

vi. $$\sqrt{\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}+n(n-1)\left|\tilde{H}\right|^{\frac{2}{n}}}\le E\left(\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}\right)}$$

where $\left|\tilde{H}\right|$ refers to the determinant of $\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)$.

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Proof.

By Cauchy Schwarz inequality, $\left(\sum_{p=1}^{n} a_p b_p\right)^2 \le \left(\sum_{p=1}^{n} a_p{}^2\right)\left(\sum_{p=1}^{n} b_p{}^2\right)$. Therefore,

Upper bound

If $a_p = 1$ and $b_p = |\zeta_p|$, then $\left(\sum_{p=1}^{n} |\zeta_p|\right)^2 \le \left(\sum_{p=1}^{n} 1\right)\left(\sum_{p=1}^{n} \zeta_p{}^2\right)$. Thus,

$$\left(E\left(\mu^{+}_{\tilde{D}_d}(\tilde{G})\right)\right)^2 \le n\left(\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n} \mu^{+}_{pq}\mu^{+}_{qn}\right)$$

$$\left(E\left(\mu^{+}_{\tilde{D}_d}(\tilde{G})\right)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n} \mu^{+}_{pq}\mu^{+}_{qn}\right)}.$$

Lower bound

$$\begin{aligned}\left(E\left(\mu^{+}_{\tilde{D}_d}(\tilde{G})\right)\right)^2 &= \left(\sum_{p=1}^{n}|\zeta_p|\right)^2\\ &= \left(\sum_{p=1}^{n}\left|\mu^{+}_{pp}\right|^2 + 2\sum_{1\le p<q\le n}\left|\mu^{+}_{pq}\right|\left|\mu^{+}_{qn}\right|\right)\\ &= \left(\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn}\right) + \frac{2n(n-1)}{2}\underset{1\le p<q\le n}{AM}\left\{|\zeta_p||\zeta_q|\right\}.\end{aligned}$$

Since $\underset{1\le p<q\le n}{AM}\left\{|\zeta_p||\zeta_q|\right\} \ge \underset{1\le p<q\le n}{GM}\left\{|\zeta_p||\zeta_q|\right\}$, hence

$$\left(E\left(\mu^{+}_{\tilde{D}_d}(\tilde{G})\right)\right) \ge \sqrt{\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn} + n(n-1)\underset{1\le p<q\le n}{GM}\left\{|\zeta_p||\zeta_q|\right\}}$$

where

$$\begin{aligned}\underset{1\le p<q\le n}{GM}\left\{|\zeta_p||\zeta_q|\right\} &= \left(\prod_{1\le p<q\le n}|\zeta_p||\zeta_q|\right)^{\frac{2}{n(n-1)}}\\ &= \left(\prod_{p=1}|\zeta_p|^{n-1}\right)^{\frac{2}{n(n-1)}}\\ &= \left(\prod_{p=1}|\zeta_p|\right)^{\frac{2}{n}}\\ &= |\tilde{A}|^{\frac{2}{n}}.\end{aligned}$$

Therefore,

$$\left(E\left(\mu^{+}_{\tilde{D}_d}(\tilde{G})\right)\right) \ge \sqrt{\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn} + n(n-1)|\tilde{A}|^{\frac{2}{n}}}.$$

Combining these bounds yields

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i. $$\sqrt{\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^{2}+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}+n(n-1)\left|\tilde{A}\right|^{\frac{2}{n}}}\le E\left(\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^{2}+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\right)}.$$

Analogously, we can show that

ii. $$\sqrt{\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^{2}+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+}+n(n-1)\left|\tilde{B}\right|^{\frac{2}{n}}}\le E\left(\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^{2}+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+}\right)},$$

iii. $$\sqrt{\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^{2}+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+}+n(n-1)\left|\tilde{C}\right|^{\frac{2}{n}}}\le E\left(\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^{2}+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+}\right)},$$

iv. $$\sqrt{\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^{2}+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-}+n(n-1)\left|\tilde{D}\right|^{\frac{2}{n}}}\le E\left(\mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^{2}+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-}\right)},$$

v. $$\sqrt{\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^{2}+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-}+n(n-1)\left|\tilde{F}\right|^{\frac{2}{n}}}\le E\left(\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^{2}+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-}\right)},$$

vi. $$\sqrt{\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^{2}+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}+n(n-1)\left|\tilde{H}\right|^{\frac{2}{n}}}\le E\left(\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^{2}+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}\right)}.$$

This completes the proof.

Theorem 4.10

Suppose $\tilde{G}=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-}\right)$ denote a BSVNG while $A\left(\tilde{G}\right)=(\mu^{+}\left(\tilde{G}\right),\gamma^{+}\left(\tilde{G}\right),\sigma^{+}\left(\tilde{G}\right),\mu^{-}\left(\tilde{G}\right),\gamma^{-}\left(\tilde{G}\right),\ \sigma^{-}\left(\tilde{G}\right)$ express the adjacency matrix of $\tilde{G}$. Suppose $\tilde{G}_1=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ be a double dominating BSVNG of $\tilde{G}$ and $A_{\tilde{D}_d}\left(\tilde{G}\right)=\left(\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right),\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right),\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right),\mu_{\tilde{D}_d}^{-}\left(\tilde{G}\right),\gamma_{\tilde{D}_d}^{-}\left(\tilde{G}\right),\sigma_{\tilde{D}_d}^{-}\left(\tilde{G}\right)\right)$ denote the adjacency matrix with respect to a double dominating BSVNG $\tilde{G}_1$. Thus

i. $$E\left(\mu_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)^{2}\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^{2}+E\left(\mu^{+}\left(\tilde{G}\right)\right)^{2}\right),$$

ii. $$E\left(\gamma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)^{2}\le n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^{2}+E\left(\gamma^{+}\left(\tilde{G}\right)\right)^{2}\right),$$

iii. $$E\left(\sigma_{\tilde{D}_d}^{+}\left(\tilde{G}\right)\right)^{2}\le n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^{2}+E\left(\sigma^{+}\left(\tilde{G}\right)\right)^{2}\right),$$

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iv. $E\left(\mu^-_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\mu^-_{pp}\right)^2 + E\left(\mu^-\left(\tilde{G}\right)\right)^2\right),$

v. $E\left(\gamma^-_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\gamma^-_{pp}\right)^2 + E\left(\gamma^-\left(\tilde{G}\right)\right)^2\right),$

vi. $E\left(\sigma^-_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\sigma^-_{pp}\right)^2 + E\left(\sigma^-\left(\tilde{G}\right)\right)^2\right).$

Proof.

$$E\left(\mu^+\left(\tilde{G}\right)\right)^2 \geq 2\sum_{1\leq p<q\leq n}\mu^+_{pq}\mu^+_{qn} + n\left(n-1\right)\left|\tilde{A}\right|^{\frac{2}{n}} \geq 2\sum_{1\leq p<q\leq n}\mu^+_{pq}\mu^+_{qn}$$

(i.e.) $2\sum_{1\leq p<q\leq n}\mu^+_{pq}\mu^+_{qn} \leq E\left(\mu^+\left(\tilde{G}\right)\right)^2$ (1)

Now,

$$E\left(\mu^+_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\sum_{p=1}^{n}\left(\mu^+_{pp}\right)^2 + 2n\sum_{1\leq p<q\leq n}\mu^+_{pq}\mu^+_{qn}$$

$$\begin{aligned} E\left(\mu^+_{\tilde{D}_d}\left(G\right)\right)^2 &\leq n\left(\sum_{p=1}^{n}\left(\mu^+_{pp}\right)^2 + 2\sum_{1\leq p<q\leq n}\mu^+_{pq}\mu^+_{qn}\right) \\ &\leq n\left(\sum_{p=1}^{n}\left(\mu^+_{pp}\right)^2 + E\left(\mu^+\left(\tilde{G}\right)\right)^2\right) \qquad \left(\text{by Eq (1)}\right) \end{aligned}$$

Analogously, we can show that

ii. $E\left(\gamma^+_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\gamma^+_{pp}\right)^2 + E\left(\gamma^+\left(\tilde{G}\right)\right)^2\right),$

iii. $E\left(\sigma^+_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\sigma^+_{pp}\right)^2 + E\left(\sigma^+\left(\tilde{G}\right)\right)^2\right),$

iv. $E\left(\mu^-_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\mu^-_{pp}\right)^2 + E\left(\mu^-\left(\tilde{G}\right)\right)^2\right),$

v. $E\left(\gamma^-_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\gamma^-_{pp}\right)^2 + E\left(\gamma^-\left(\tilde{G}\right)\right)^2\right),$

vi. $E\left(\sigma^-_{\tilde{D}_d}\left(\tilde{G}\right)\right)^2 \leq n\left(\sum_{p=1}^{n}\left(\sigma^-_{pp}\right)^2 + E\left(\sigma^-\left(\tilde{G}\right)\right)^2\right).$

5. Conclusion

In this study, we investigated several graph-theoretic concepts and introduced the hypothesis of double dominating energy within the framework of BSVNG. We developed a new concept of the adjacency matrix, as well as the spectrum of this matrix for the dominating structures in BSVNG. Subsequently, we calculated the

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energy of double dominating BSVNG and established several properties related to its double dominating energy. This work extends the theoretical understanding of domination in neutrosophic environments, offering insights into its applications and characteristics.

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