Extended Refined Neutrosophic Vector Spaces, Subspaces and their Application

T. Sathinathan[1], **J. Aldring**[2,*], **S. John Borg**[1], **D. Ajay**[1]
[1]PG and Research Department of Mathematics, Sacred Heart College, Tirupattur - 635601, India
[2]Department of Mathematics, Panimalar Engineering College, Poonamalle, Chennai -600123, India
Emails: sathithoma@gmail.com; jaldring24@gmail.com; sjborg@gmail.com; dajaypravin@gmail.com

Abstract

The objective of this paper is to present a perspective of Refined Neutrosophic Vector Space (r-NVS), Sub spaces and some basic operations on Refined Neutrosophic Sets such as Algebraic sum and Algebraic product. Further some basic propositions, lemma and examples are presented. Finally an application on Refined Neutrosophic Vector Space is presented in the field of e-Commerce buyer oriented product (Smart Phones) ranking to illustrate the advantage of representing r-NVS.

Keywords: Refined Neutrosophic Sets (r-NSs); Refined Neutrosophic Vector Space (r-NVS); Distance Measure; MCDM

1 Introduction

In modern times, a number of uncertainty theories have been put forth to deal with uncertainty and ambiguity. Theory of probability, fuzzy set theory,[1] rough sets[2] and so on have widely been in usage to overcome uncertainty and ambiguity. For instance, the statement 'he has long nails' is ambiguous because its interpretation depends on whether the word "nails" is meant as "a small metal spike with a broadened flat head" or "a hard surface at the tip of human finger". Uncertainty typically arises in situations where multiple analysts or observers have different interpretations of the same statements. However, these theories failed to discuss the indeterminacy which occurs in various uncertain events. A new definition called neutrosophic set (NS) was developed by Smarandache,[3] which generalizes probability set, fuzzy set, and intuitionistic fuzzy set.[4] Neutrosophic set is represented by membership degree, indeterminacy degree and non-membership degree. Later, Smarandache introduced the notion of refined neutrosophic membership values of the form $\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r}), \ldots, \mu_{\tilde{N}_{\mathfrak{f}}^p}(\dot{r})$; $\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r}), \ldots, \sigma_{\tilde{N}_{\mathfrak{f}}^p}(\dot{r})$; $\nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r}), \ldots, \nu_{\tilde{N}_{\mathfrak{f}}^p}(\dot{r})$ of the neutrosophic membership value $\left\langle \mu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) \right\rangle$.[5] The idea of refined neutrosophic algebraic structures was introduced by Agboola and refined neutrosophic groups with their basic and fundamental properties were studied.[6]

Kandasamy and Smarandache have developed the concept of neutrosophic vector space.[7] Agboola and Akinleye performed further studies on neutrosophic vector spaces in which they generalized certain properties of vector spaces and showed that any neutrosophic vector space is a vector space over a neutrosophic area (resp. field). Malath.F.Aswad[8] introduced a novel concept of n-cyclic refined neutrosophic vector spaces with their basic elementary properties such as homomorphisms and subspaces. And also they introduced complex n-refined neutrosophic numbers with their basic properties.[9] Abobala M.[10] proved that every neutrosophic matrix can be represented uniquely by a neutrosophic linear vector space transformation. Then Ali, R.[11] newly developed the concept of n-refined neutrosophic linear Diophantine equations and validated with examples.

DOI: https://doi.org/10.54216/IJNS.260104

In this paper a new perspective of (strong /weak) r-Neutrosophic Vector Space (r-NVS), subspaces different from the one proposed in (on refined neutrosophic vector space 1),[12 13] and some basic operations on Refined Neutrosophic Sets such as Algebraic sum and Algebraic product are studied. Further some basic proposition, lemma and examples are presented. Finally, an application on Refined Neutrosophic Vector Space is presented in the field of e-Commerce for buyer-oriented product ranking to illustrate the advantage of representing r-NVS.

2 Preliminaires

Definition 2.1. [1] A fuzzy set $\widetilde{A}$ in Υ is characterized by a membership value $\mu_{\widetilde{A}}$ which takes the value in the interval $[0,1]$ i.e.

$$\mu_{\widetilde{A}} : \Upsilon \longrightarrow [0,1] \tag{1}$$

The value $\mu_{\widetilde{A}}$ at $\wp \in \Upsilon$, denote by $\mu_{\widetilde{A}}(\wp)$, represents the grade of membership of $\wp$ in $\widetilde{A}$ and is a point of $[0,1]$. Then $\widetilde{A}$ is given by

$$\widetilde{A} = \sum_j \mu_{\widetilde{A}}(\wp_j)/\wp_j, \wp_j \in \Upsilon. \tag{2}$$

Definition 2.2. [3] let $\Re$ be a universe of discourse or a non empty set. Any object in the neutrosophic set $\tilde{N}_{\mathfrak{f}}$ has the form $\tilde{N}_{\mathfrak{f}} = \left\{\left\langle \dot{r}, \mu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) \right\rangle \in \Re\right\}$, where $\mu_{\tilde{N}_{\mathfrak{f}}}(\dot{r})$, $\sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r})$ and $\nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r})$ represent the degree of truth membership, the degree of indeterminacy and the degree of false membership respectively of each element $\dot{r} \in \Re$ to the set $\tilde{N}_{\mathfrak{f}}$ and it is defined as

$$\tilde{N}_{\mathfrak{f}} = \left\{\left\langle \dot{r}, \mu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) \right\rangle \in \Re\right\}$$

where $\mu_{\tilde{N}_f}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) : \Re \rightarrow [0,1]$ such that $0 \leq \mu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) + \sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) + \nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}) \leq 3$.

Definition 2.3. A two-valued function $t : [0,1] \times [0,1] \longrightarrow [0,1]$ is named as t-norm if it satisfies following properties:

1. **Associativity**
$$t(a, t(b,c)) = t(t(a,b), c)$$
2. **Commutativity**
$$t(a,b) = t(b,a)$$
3. **Monotonicity**
$$If a \leq c \, and \, b \leq d \Rightarrow t(a,b) \leq t(c,d)$$
4. **Boundary Condition**
$$t(0,0) = 0, t(a,1) = a \text{and} t(1,a) = a \forall a,b,c,d \in [0,1].$$

Definition 2.4. A two-valued function $t : [0,1] \times [0,1] \longrightarrow [0,1]$ is named as s-norm if it satisfies following properties:

1. **Associativity**
$$s(a, s(b,c)) = s(s(a,b), c)$$
2. **Commutativity**
$$s(a,b) = s(b,a)$$
3. **Monotonicity**
$$If a \leq c \, and \, b \leq d \Rightarrow s(a,b) \leq s(c,d)$$

4. **Boundary Condition**

$$s(1,1)=1, s(a,0)=a \text{ and } s(0,a)=a \ \forall a,b,c,d \in [0,1].$$

Definition 2.5. [5] Refined Neutrosophic Set (r-NS) $r\tilde{N}_{\mathfrak{f}}$ can be denoted as,

$$r\tilde{N}_{\mathfrak{f}} = \left\{\left\langle \dot{r}, \mu_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}}(\dot{r})\right\rangle \in \Re\right\}$$

$$r\tilde{N}_{\mathfrak{f}} = \left\{\left\langle \dot{r}, \left(\mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right)\right\rangle ; \dot{r} \in \Re\right\}.$$

$$\text{where,} \quad \mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) : \Re \longrightarrow [0,1]$$
$$\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) : \Re \longrightarrow [0,1]$$
$$\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) : \Re \longrightarrow [0,1]$$

such that $0 \le \mu_{\tilde{N_{\mathfrak{f}i}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}i}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}i}}}(\dot{r}) \le 3 \ \forall (i = 1, 2, \ldots, p)$.
And, $\mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}) \le \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}) \le \ldots \le \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) \forall \dot{r} \in \Re$. $\mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})$ is the truth membership sequence $\left(\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right)$ is the indeterminacy membership sequence and $\left(\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right)$ is the false membership sequence, where p is the dimension of r-NS. Here the truth membership sequences are monotonically increasing sequences and the other two are not so.

Definition 2.6. Let $r\tilde{N}_{\mathfrak{f}^A}$, $r\tilde{N}_{\mathfrak{f}^B}$ be a two r-NSs. Then $r\tilde{N}_{\mathfrak{f}^A}$ is said to be r-Neutrosophic subset of $r\tilde{N}_{\mathfrak{f}^B}$ denoted by $r\tilde{N}_{\mathfrak{f}^A} \subseteq r\tilde{N}_{\mathfrak{f}^B}$ if $\mu_{\tilde{N_{\mathfrak{f}A}}}(\dot{r}) \le \mu_{\tilde{N_{\mathfrak{f}B}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}A}}}(\dot{r}) \ge \sigma_{\tilde{N_{\mathfrak{f}B}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}A}}}(\dot{r}) \ge \nu_{\tilde{N_{\mathfrak{f}B}}}(\dot{r}) \forall \dot{r} \in \Re$.

Definition 2.7. $r\tilde{N}_{\mathfrak{f}^A}$ is said to be neutrosophic equal of $r\tilde{N}_{\mathfrak{f}^B}$ and is denoted by $r\tilde{N}_{\mathfrak{f}^A} = r\tilde{N}_{\mathfrak{f}^A}$ if $\mu_{\tilde{N_{\mathfrak{f}A}}}(\dot{r}) = \mu_{\tilde{N_{\mathfrak{f}B}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}A}}}(\dot{r}) = \sigma_{\tilde{N_{\mathfrak{f}B}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}A}}}(\dot{r}) = \nu_{\tilde{N_{\mathfrak{f}B}}}(\dot{r}) \ \forall \dot{r} \in \Re$.

Definition 2.8. The Complement of $r\tilde{N}_{\mathfrak{f}}$ denoted by $r\tilde{N}_{\mathfrak{f}}^C$ is defined by,

$$r\tilde{N}_{\mathfrak{f}}^C = \left\{\left\langle \dot{r}, \left(\mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right), \left(1-\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), 1-\sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, 1-\sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}), \ldots, \nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\right)\right\rangle ; \dot{r} \in \Re\right\}.$$

Definition 2.9. The union of $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ is denoted by $r\tilde{N}_{\mathfrak{f}^A} \cup r\tilde{N}_{\mathfrak{f}^B} = r\tilde{N}_{\mathfrak{f}^D}$ and is defined by,

$$r\tilde{N}_{\mathfrak{f}^D} = \left\{\left\langle \dot{r}, \left(\mu_{\tilde{N_{\mathfrak{f}D1}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}D2}}}(\dot{r}), \ldots, \mu_{\tilde{N_{\mathfrak{f}Dp}}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}D1}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}D2}}}(\dot{r}), \ldots, \sigma_{\tilde{N_{\mathfrak{f}Dp}}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}D1}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}D2}}}(\dot{r}), \ldots, \nu_{\tilde{N_{\mathfrak{f}Dp}}}(\dot{r})\right)\right\rangle ; \dot{r} \in \Re\right\}.$$

$$\text{where} \quad \mu_{\tilde{N_{\mathfrak{f}Di}}}(\dot{r}) = s\left\{\mu_{\tilde{N_{\mathfrak{f}Ai}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}Bi}}}(\dot{r})\right\}$$
$$\sigma_{\tilde{N_{\mathfrak{f}Di}}}(\dot{r}) = t\left\{\sigma_{\tilde{N_{\mathfrak{f}Ai}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}Bi}}}(\dot{r})\right\}$$
$$\nu_{\tilde{N_{\mathfrak{f}Di}}}(\dot{r}) = t\left\{\nu_{\tilde{N_{\mathfrak{f}Ai}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}Bi}}}(\dot{r})\right\} \forall \dot{r} \in \Re (i = 1, 2, \ldots, p)$$

Definition 2.10. The Intersection of $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ is denoted by $r\tilde{N}_{\mathfrak{f}^A} \cup r\tilde{N}_{\mathfrak{f}^B} = r\tilde{N}_{\mathfrak{f}^E}$ and is defined by,

$$r\tilde{N}_{\mathfrak{f}^E} = \left\{\left\langle \dot{r}, \left(\mu_{\tilde{N_{\mathfrak{f}E1}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}E2}}}(\dot{r}), \ldots, \mu_{\tilde{N_{\mathfrak{f}Ep}}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}E1}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}E2}}}(\dot{r}), \ldots, \sigma_{\tilde{N_{\mathfrak{f}Ep}}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}E1}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}E2}}}(\dot{r}), \ldots, \nu_{\tilde{N_{\mathfrak{f}Ep}}}(\dot{r})\right)\right\rangle ; \dot{r} \in \Re\right\}.$$

$$\text{where} \quad \mu_{\tilde{N_{\mathfrak{f}Ei}}}(\dot{r}) = t\left\{\mu_{\tilde{N_{\mathfrak{f}Ai}}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}Bi}}}(\dot{r})\right\}$$
$$\sigma_{\tilde{N_{\mathfrak{f}Ei}}}(\dot{r}) = s\left\{\sigma_{\tilde{N_{\mathfrak{f}Ai}}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}Bi}}}(\dot{r})\right\}$$
$$\nu_{\tilde{N_{\mathfrak{f}Ei}}}(\dot{r}) = s\left\{\nu_{\tilde{N_{\mathfrak{f}Ai}}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}Bi}}}(\dot{r})\right\} \forall \dot{r} \in \Re (i = 1, 2, \ldots, p)$$

Definition 2.11. Let $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ be two r-NSs. The algebraic sum of $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ is denoted by $r\tilde{N}_{\mathfrak{f}^A} \oplus r\tilde{N}_{\mathfrak{f}^B} = r\tilde{N}_{\mathfrak{f}^D}$ and is defined by,

$$r\tilde{N}_{\mathfrak{f}^D} = \Big\{ \Big\langle \dot{r}, \Big(\mu_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r}) + \mu_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) - \mu_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r})\mu_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) \Big), \Big(\sigma_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r})\sigma_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) \Big), \Big(\nu_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r})\nu_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) \Big) \Big\rangle : \dot{r} \in \Re \Big\}$$

Definition 2.12. Let $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ be two r-NSs. The algebraic product of $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ is denoted by $r\tilde{N}_{\mathfrak{f}^A} \otimes r\tilde{N}_{\mathfrak{f}^B} = r\tilde{N}_{\mathfrak{f}^E}$ and is defined by,

$$r\tilde{N}_{\mathfrak{f}^E} = \Big\{ \Big\langle \dot{r}, \Big(\mu_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r})\mu_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) \Big), \Big(\sigma_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r}) + \sigma_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) - \sigma_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r})\sigma_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) \Big), \Big(\nu_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r}) + \nu_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) - \nu_{N_{\mathfrak{f}Ai}^{\sim}}(\dot{r})\nu_{N_{\mathfrak{f}Bi}^{\sim}}(\dot{r}) \Big) \Big\rangle : \dot{r} \in \Re \Big\}$$

Definition 2.13. For $\omega > 0$, the scalar product is defined by,

$$\omega.r\tilde{N}_{\mathfrak{f}} = \Big\{ \Big\langle \dot{r}, \Big(\Big[1 - \Big(1 - \mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big)\Big]^{\omega}, \Big[1 - \Big(1 - \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big)\Big]^{\omega}, \ldots, \Big[1 - \Big(1 - \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\Big)\Big]^{\omega} \Big), \Big(\Big[\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big]^{\omega}, \Big[\sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big]^{\omega}, \ldots, \Big[\sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\Big]^{\omega} \Big), \Big(\Big[\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big]^{\omega}, \Big[\nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big]^{\omega}, \ldots, \Big[\nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\Big]^{\omega} \Big) \Big\rangle ; \dot{r} \in \Re \Big\}.$$

Definition 2.14. For $\omega > 0$, the power of fuzzy number is defined by,

$$\Big[r\tilde{N}_{\mathfrak{f}}\Big]^{\omega} = \Big\{ \Big\langle \dot{r}, \Big(\Big[\mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big]^{\omega}, \Big[\mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big]^{\omega}, \ldots, \Big[\mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\Big]^{\omega} \Big), \Big(\Big[1 - \Big(1 - \sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big)\Big]^{\omega}, \Big[1 - \Big(1 - \sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big)\Big]^{\omega}, \ldots, \Big[1 - \Big(1 - \sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\Big)\Big]^{\omega} \Big), \Big(\Big[1 - \Big(1 - \nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big)\Big]^{\omega}, \Big[1 - \Big(1 - \nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big)\Big]^{\omega}, \ldots, \Big[1 - \Big(1 - \nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r})\Big)\Big]^{\omega} \Big) \Big\rangle ; \dot{r} \in \Re \Big\}.$$

Definition 2.15. Let $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ be two r-NSs. Then

1. Hamming Distance $d_H\left(r\tilde{N}_{\mathfrak{f}^A}, r\tilde{N}_{\mathfrak{f}^B}\right)$ between $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ defined by,

$$d_H\left(r\tilde{N}_{\mathfrak{f}^A}, r\tilde{N}_{\mathfrak{f}^B}\right) = \sum_{j=1}^{n}\sum_{i=1}^{n} \Big[\Big|\mu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r_i}) - \mu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r_i})\Big| + \Big|\sigma_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r_i}) - \sigma_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r_i})\Big| + \Big|\nu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r_i}) - \nu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r_i})\Big| \Big] \tag{3}$$

2. Normalized Hamming Distance $d_H\left(r\tilde{N}_{\mathfrak{f}^A}, r\tilde{N}_{\mathfrak{f}^B}\right)$ between $r\tilde{N}_{\mathfrak{f}^A}$ and $r\tilde{N}_{\mathfrak{f}^B}$ is defined by,

$$d_H\left(r\tilde{N}_{\mathfrak{f}^A}, r\tilde{N}_{\mathfrak{f}^B}\right) = \frac{1}{3np}\sum_{j=1}^{n}\sum_{i=1}^{n} \Big[\Big|\mu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r_i}) - \mu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r_i})\Big| + \Big|\sigma_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r_i}) - \sigma_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r_i})\Big| + \Big|\nu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r_i}) - \nu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r_i})\Big| \Big] \tag{4}$$

3 r-Neutrosophic Vector Space And Subspaces

In this section for convenience the dimension of membership sequence is taken as 2 i.e. $p = 2$. Also, we use the following logical operators:

- $\mu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r}) \otimes \mu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) = \mu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) \otimes \mu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r})$
- $\sigma_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r}) \otimes \sigma_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) = \sigma_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) \otimes \sigma_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r})$
- $\nu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r}) \otimes \nu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) = \nu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) \otimes \nu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r})$
- $\mu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r}) \oplus \mu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) = \mu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) \oplus \mu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r})$
- $\sigma_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r}) \oplus \sigma_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) = \sigma_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) \oplus \sigma_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r})$
- $\nu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r}) \oplus \nu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) = \nu_{N_{\mathfrak{f}Bj}^{\sim}}(\dot{r}) \oplus \nu_{N_{\mathfrak{f}Aj}^{\sim}}(\dot{r})$

Definition 3.1.

$$\begin{aligned} \text{If} \quad * : \Upsilon &\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right) \times \\ \Upsilon &\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right) \\ &\longrightarrow \Upsilon \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right) \end{aligned}$$

is a binary operation defined on Υ.

Then the doublet $\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right)$ is called r-Neutrospohic algebraic structure if it satisfies following conditions.

$$\begin{aligned} u &= \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle \\ v &= \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle \end{aligned}$$

If $u, v \in \Upsilon \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)$
then, $u * v \in \Upsilon \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)$.

Definition 3.2. Let $\left(\Upsilon \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right), \oplus, \otimes\right)$ be a **r- Neutrosophic algebraic structure** where $\oplus$ and $\otimes$ are algebraic sum and algebraic product respectively. For any two elements,

$$\begin{aligned} u &= \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle \\ v &= \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle \end{aligned}$$

where $u, v \in \Upsilon \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right)$ and $\wp_1, \wp_2 \in \Upsilon$. We define,

$$\begin{aligned} u \oplus v = &\left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle \oplus \\ &\left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle \\ = &\left\langle \dot{r_1} + \dot{r_2}, \left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \oplus \mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \oplus \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \right. \\ &\left. \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \oplus \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \oplus \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \oplus \nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \oplus \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right\rangle \\ u \otimes v = &\left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle \otimes \\ &\left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle \\ = &\left\langle \dot{r_1} \bullet \dot{r_2}, \left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \otimes \mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \otimes \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \right. \\ &\left. \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \otimes \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \otimes \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \otimes \nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \otimes \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right\rangle \end{aligned}$$

Definition 3.3. Let $(G,*)$ be any group.
The doublet $\left(G\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),*\right)$ is called **r- Neutrosophic group** generated by
$\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)$.
$\left(G\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),*\right)$ is said to be **commutative** if $\forall u,v\in(G,*)$ then $u*v=v*u$.

Otherwise we say $\left(G\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),*\right)$ a **non-commutative r-Neutrosophic group**.

Example 3.4. Let Z_2 be integer modulo group defined as $Z_2=\{0,1\}$.

$$
\begin{aligned}
&Z_2\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)=\{(0,(0,0),(0,0),(0,0)),\\
&\left(0,\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),(0,0),(0,0)\right),\left(0,(0,0),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),(0,0)\right),\left(0,(0,0),(0,0),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\\
&\left(0,\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),(0,0)\right),\left(0,\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),(0,0),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\\
&\left(0,(0,0),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\left(0,\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right),\\
&(1,(0,0),(0,0),(0,0)),\left(1,\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),(0,0),(0,0)\right),\left(1,(0,0),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),(0,0)\right),\\
&\left(1,(0,0),(0,0),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),(1,(\mathcal{T}(\wp),\mathcal{T}(\wp)),(\mathcal{I}(\wp),\mathcal{I}(\wp)),(0,0)),(1,(\mathcal{T}(\wp),\mathcal{T}(\wp)),(0,0),(\mathcal{F}(\wp),\mathcal{F}(\wp))),\\
&\left(1,(0,0),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\left(0,\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)\}
\end{aligned}
$$

Then $\left(Z_2\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus\right)$ is a commutative r- Neutrosophic group of integer modulo 2.

In General, $\left(Z_n\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus\right)$ is a commutative r-Neutrosophic group of integer modulo **n**

Definition 3.5. Let $(V,+,\bullet)$ be any vector space over a Field F.
Let $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ is called **r-Neutrosophic set** generated by $\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)$.
We call the triplet $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ a **weak r-Neutrosophic vector space** over the field K.

If $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ a r- Neutrosophic vector space over the r-Neutrosophic field
$K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)$ then $(V,\oplus,\otimes)$ is called a **strong r-Neutrosophic vector space**.

The elements of $(V,\oplus,\otimes)$ are called **r-neutrosophic elements**.
The elements of $K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)$ are called **r-Neutrosophic scalars** if and only if,

1. $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus\right)$ is an r-Neutrosophic abelian group under algebraic sum.
2. $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\otimes\right)$ satisfies scalar multiplication (weak r-Neutrosophic vector space) and algebraic product (strong r-neutrosophic vector space).

3.1 Illustration

$$\text{Let}\quad u = \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right) \right) \right\rangle$$
$$v = \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right) \right) \right\rangle$$

where $u, v \in \left(V \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r}) \right) \right) \right)$ and $\dot{r_1}, \dot{r_2}$ be vectors in V.

$$\alpha = \left\langle \ell, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right) \right) \right\rangle \in K \left(\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right) \right) \right)$$

we define

1. Algebraic sum

$$u \oplus v = \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right) \right) \right\rangle \oplus \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right) \right) \right\rangle$$
$$= \left\langle \dot{r_1} + \dot{r_2}, \left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right) \right\rangle$$

2. Scalar multiplication, For $\omega > 0$ (weak r-Neutrosophic vector Space)

$$\omega \bullet r\tilde{N}_{\mathfrak{f}} = \left\{ \left\langle \dot{r}, \left(\left[1 - \left(1 - \mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}) \right) \right]^{\omega}, \left[1 - \left(1 - \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}) \right) \right]^{\omega}, \ldots, \left[1 - \left(1 - \mu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) \right) \right]^{\omega} \right), \left(\left[\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}) \right]^{\omega}, \left[\sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}) \right]^{\omega}, \ldots, \left[\sigma_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) \right]^{\omega} \right), \left(\left[\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r}) \right]^{\omega}, \left[\nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r}) \right]^{\omega}, \ldots, \left[\nu_{\tilde{N_{\mathfrak{f}p}}}(\dot{r}) \right]^{\omega} \right) \right\rangle ; \dot{r} \in \Re \right\}.$$

3. Algebraic product (Strong r-Neutrosophic vector Space)

$$\alpha \otimes u = \left\langle \ell, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \right) \right) \right\rangle \otimes \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right) \right) \right\rangle$$
$$= \left\langle \ell \bullet \dot{r_1}, \left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell) \otimes \mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \otimes \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell) \otimes \sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \otimes \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell) \otimes \nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \otimes \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right) \right\rangle$$

Example 3.6. Let $\left(R \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r}) \right) \right), \oplus, \otimes \right)$ denote the r- Neutrosophic set where u, v are r-Neutrosophic Fuzzy numbers over the field R given by,

$$u = \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right) \right) \right\rangle$$
$$v = \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right) \right) \right\rangle$$
$$u \oplus v = \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \right) \right) \right\rangle \oplus \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right) \right) \right\rangle$$
$$= \left\langle \dot{r_1} + \dot{r_2}, \left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2}) \right) \right\rangle$$

where $\dot{r_1}, \dot{r_2} \in R$. It is obvious that $\forall \dot{r_1} + \dot{r_2} \in R$ and also every membership value's algebraic sum lies in $[0, 1]$.
$u \oplus v \in \left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right)\right), \oplus, \otimes\right)$
For $\omega > 0$,

$$\omega \bullet u = \Big\{\Big\langle \dot{r}, \Big(\Big[1 - \Big(1 - \mu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big)\Big]^{\omega}, \Big[1 - \Big(1 - \mu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big)\Big]^{\omega}\Big), \Big(\Big[\sigma_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big]^{\omega}, \Big[\sigma_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big]^{\omega}\Big), \Big(\Big[\nu_{\tilde{N_{\mathfrak{f}1}}}(\dot{r})\Big]^{\omega}, \Big[\nu_{\tilde{N_{\mathfrak{f}2}}}(\dot{r})\Big]^{\omega}\Big)\Big\rangle ; \dot{r} \in \Re\Big\}.$$

Hence the scalar product.

Then $\left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right)\right), \oplus, \otimes\right)$ is a weak r-Neutrosophic vector space over R.

Example 3.7. Let $\left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right)\right), \oplus, \otimes\right)$ denote a r-Neutrosophic set where u, v are r-Neutrosophic Fuzzy numbers over the r-Neutrosophic field $\left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right)\right)\right)$ given by

$$u = \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right)\right)\right\rangle$$
$$v = \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right)\right)\right\rangle$$

$$\alpha = \left\langle \ell, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right)\right)\right\rangle \in K\left(\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right)\right)\right)$$

$$\begin{aligned} u \oplus v &= \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right)\right)\right\rangle \oplus \\ &\quad \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right)\right)\right\rangle \\ &= \Big\langle \dot{r_1} + \dot{r_2}, \left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right), \\ &\quad \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}) \oplus \nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_2}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1}) \oplus \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_2})\right)\Big\rangle \end{aligned}$$

where $\dot{r_1}, \dot{r_2} \in R$. It is obvious that $\forall \dot{r_1} + \dot{r_2} \in R$ and also every membership value's algebraic sum lies in $[0, 1]$.

$u \oplus v \in \left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right)\right), \oplus, \otimes\right)$

$$\begin{aligned} \alpha \otimes u &= \left\langle \ell, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right)\right)\right\rangle \otimes \\ &\quad \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right)\right)\right\rangle \end{aligned}$$

$$\begin{aligned} &= \Big\langle \ell \bullet \dot{r_1}, \left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell) \otimes \mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \otimes \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \\ &\quad \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell) \otimes \sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \otimes \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell) \otimes \nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r_1}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell) \otimes \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r_1})\right)\Big\rangle \end{aligned}$$

Hence, algebraic product holds.

Then $\left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\dot{r}), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\dot{r})\right)\right), \oplus, \otimes\right)$ is a strong r-Neutrosophic Vector space over
$\left(R\left(\left(\mu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \mu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\sigma_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \sigma_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right), \left(\nu_{\tilde{N_{\mathfrak{f}}^1}}(\ell), \nu_{\tilde{N_{\mathfrak{f}}^2}}(\ell)\right)\right)\right)$

Proposition 3.8. *Every Strong r-Neutrosophic Vector Space is a Weak r- Neutrosophic Vector Space.*

Proof. Suppose that
$\left(V\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ is a strong r-Neutrosophic Vector Space over strong r-Neutrosophic
Field $\left(K\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right)\right)\right)$.
Since, K$\subseteq$ $\left(K\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right)\right)\right)$
$\left(V\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ is also a Weak r- Neutrosophic Vector Space over K. □

Proposition 3.9. *Every weak (strong) r-Neutrosophic vector space is a vector space.*

Proof. Suppose that
$\left(V\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ is a strong r-Neutrosophic Vector Space over strong r-Neutrosophic
Field $\left(K\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right)\right)\right)$.
$\left(V\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right)\right),\oplus\right)$ is an Abelian r-Neutrosophoic Group can be established easily. Consider

$$u=\left\langle\dot{r_1},\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\right)\right)\right\rangle$$
$$v=\left\langle\dot{r_2},\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right)\right)\right\rangle$$

where $u,v\in\left(V\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r})\right)\right),\oplus,\otimes\right)$

$$\alpha=\left\langle\ell_1,\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right)\right)\right\rangle$$
$$\beta=\left\langle\ell_2,\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell_2),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell_2)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell_2),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell_2)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell_2),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell_2)\right)\right)\right\rangle$$

where $\alpha,\beta\in\left(K\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell)\right)\right)\right)$. $\forall\wp_1,\wp_2,\ell_1,\ell_2\in\Re$.

1.

$$\begin{aligned}\alpha(u+v)=&\left\langle\ell_1,\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right)\right)\right\rangle\\
&\left[\left\langle\dot{r_1},\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\right)\right)\right\rangle\oplus\right.\\
&\left.\left\langle\dot{r_2},\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right)\right)\right\rangle\right]\\
=&\left\langle\ell_1,\left(\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\mu_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right),\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\sigma_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\ell_1),\nu_{\tilde{N_\mathfrak{f}^2}}(\ell_1)\right)\right)\right\rangle\\
&\left\langle\dot{r_1}+\dot{r_2},\left(\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1})\oplus\mu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\oplus\mu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right),\right.\\
&\left.\left(\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1})\oplus\sigma_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\oplus\sigma_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right),\left(\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_1})\oplus\nu_{\tilde{N_\mathfrak{f}^1}}(\dot{r_2}),\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_1})\oplus\nu_{\tilde{N_\mathfrak{f}^2}}(\dot{r_2})\right)\right\rangle\end{aligned}$$

$$= \left\langle \ell_1 \left(\dot{r_1} + \dot{r_2}\right), \left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \oplus \mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \oplus \mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \oplus \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \oplus \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \oplus \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \oplus \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \oplus \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \oplus \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \oplus \nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}) \oplus \nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \oplus \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1}) \oplus \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right\rangle$$

$$= \left\langle \ell_1 \bullet \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \otimes \mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1})\right), \left(\mu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1) \otimes \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right), \left(\left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \otimes \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1) \otimes \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right), \left(\left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \otimes \nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1) \otimes \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle \oplus \left\langle \ell_1 \bullet \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \otimes \mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2})\right), \left(\mu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1) \otimes \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right), \left(\left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \otimes \sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1) \otimes \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right), \left(\left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1) \otimes \nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1) \otimes \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle$$

$$= \left\langle \ell_1, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1)\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1)\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1)\right)\right)\right\rangle \otimes \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle \oplus \left\langle \ell_1, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1)\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1)\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell_1), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\ell_1)\right)\right)\right\rangle \otimes \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle = \alpha u + \alpha v$$

2. Similarly, $(\alpha + \beta) \bullet u = \alpha \bullet u + \beta \bullet u$
3. $(\alpha \bullet \beta) \bullet u = (\alpha \bullet u) \bullet (\beta \bullet u)$
4. For $1 = \langle 1, (0,0), (0,0), (0,0)\rangle \Rightarrow 1 \bullet u = u$

Accordingly, $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right), \oplus, \otimes\right)$ is a Vector Space. □

Lemma 3.10. *Let* $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right), \oplus, \otimes\right)$ *be a strong r-Neutrosophic Vector Space over strong r-Neutrosophic Field* $\left(K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right)\right)$. *Consider*

$$u = \left\langle \dot{r_1}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_1}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_1})\right)\right)\right\rangle$$
$$v = \left\langle \dot{r_2}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_2}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_2})\right)\right)\right\rangle$$
$$w = \left\langle \dot{r_3}, \left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_3}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_3})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_3}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_3})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r_3}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r_3})\right)\right)\right\rangle$$

where $u, v, w \in \left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right), \oplus, \otimes\right)$ *and* $\beta \in \left(K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\ell), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\ell)\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\ell), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\ell)\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\ell), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\ell)\right)\right)\right)$ *then,*

1. $u+w=v+w \Longrightarrow u=v$
2. $0 \bullet u=0$
3. $\beta \bullet 0=0$
4. $(-\beta)\dot{r}=\beta(-\dot{r})=-(\beta\dot{r})$

Proof. We prove the above lemma in the same way that we proved Proposition3.9. □

Definition 3.11. Let $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ be a strong r-Neutrosophic vector space over strong r-Neutrosophic field $\left(K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right)\right)\right)$.
A non-empty subset $\left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ of $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ is called a strong r-Neutrosophic subspace of V if W itself is a strong r- Neutrosophic subspace over Field $\left(K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right)\right)\right)$. Also $W \subset V$.

Definition 3.12. Let $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ be a strong r-Neutrosophic vector space over K
and $\left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ be a non-empty subset of $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$.
Then W is called a weak r-Neutrosophic subspace of V.

Proposition 3.13. *Let* $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ *be a strong r-Neutrosophic vector space over strong r-Neutrosophic field* $\left(K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right)\right)\right)$.
A non-empty subset $\left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ *of* $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right),\oplus,\otimes\right)$ *is called a strong r -Neutrosophic subspace of V iff the following conditions hold:*

1. $u,v \in \left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$
$\rightarrow u\oplus v \in \left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$
2. $u \in \left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$
$\rightarrow \alpha u \in \left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$
$\forall \alpha \in \left(K\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\ell),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\ell)\right)\right)\right)$.
3. *W contains a proper subset of V*

Example 3.14. Let $R^2\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)$ be a r-Neutrosophic vector space over r-Neutrosophic field $R\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)$.
Let $\left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ be a set of vectors in R^2.
Then $\left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$ is a strong r-neutrosophic subspace of $\left(V\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\mu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\sigma_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\sigma_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right),\left(\nu_{\tilde{N}_{\mathfrak{f}}^{1}}(\dot{r}),\nu_{\tilde{N}_{\mathfrak{f}}^{2}}(\dot{r})\right)\right)\right)$.

It is obivious that $1, 2$ from previous proposition holds good.

Since $W \subset \left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right)\right)$ is a proper subset which is also a r-Neutrosophic Vector Space. Thus 3 holds good.

$\Longrightarrow \left(W\left(\left(\mu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \mu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\sigma_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \sigma_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right), \left(\nu_{\tilde{N}_{\mathfrak{f}}^1}(\dot{r}), \nu_{\tilde{N}_{\mathfrak{f}}^2}(\dot{r})\right)\right)\right)$ is a strong r-Neutrosophic subspace.

4 An MCDM model

Finding a better solution over the huge data of uncertainty is a difficult task. It is evident from the literature that most uncertain situations/problems can be found only with magnitude information. Some cases the uncertain information can be handled with magnitude and direction. So these type of information related into vector form. So we have proposed a novel multi criteria decision making method using r-NVS and it involves the following steps:

Step 1. Frame the relation matrix (M°) between criteria and rating/evaluation vectors

Step 2. Determine the decision matrix $(M^{'})$ between alternatives and criteria by r-NS

Step 3. Find the normalized hamming distance of M° and $M^{'}$ using Eq.4

Step 4. Rank/Rate the alternatives based on the distance values

4.1 Application in e-Commerce

In this section, a product rating scheme in online shopping using r-NS is proposed. Let M= $\{M_1, M_2, M_3, M_4\}$ denote the set of Smart Phones (Alternatives). 5star= R_1 ,4star= R_2, 3star= R_3, 2star= R_4, 1star= R_5 denote the set of Rating. Battery life= C_1, Refresh rate= C_2, RAM/ROM= C_3, Camera Quality= C_4,

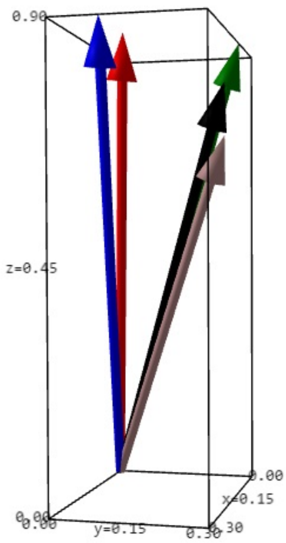

Figure 1: R_1- Rating vector

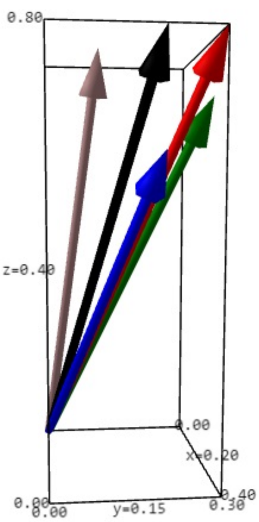

Figure 2: R_2- Rating vector

Price= C_5 are fixed as the Criteria. The relation between criteria and rating vectors are given in matrix M^{o}. Also, the vector representation of R_1, R_2, R_3, R_4 & R_5 over criteria are shown in the following figures 1-5.

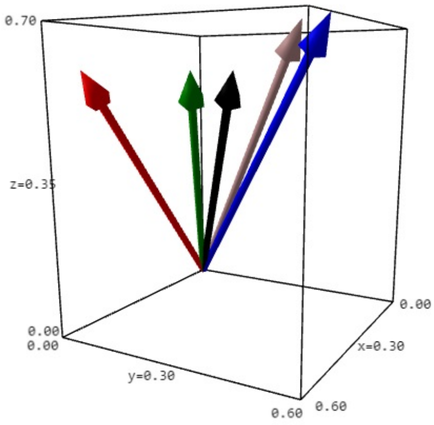

Figure 3: R_1- Rating vector

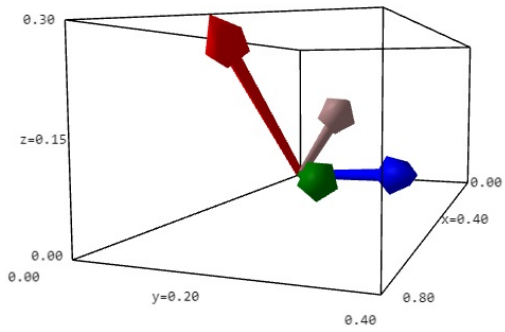

Figure 4: R_2- Rating vector

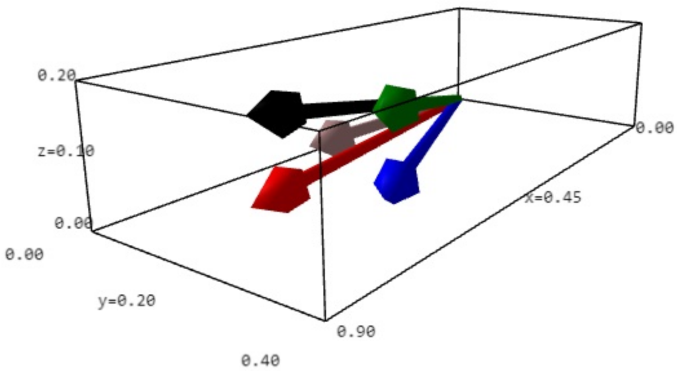

Figure 5: R_3- Rating vector

Step 1 Relation between criteria and rating vectors

$$M^o = \left\{ \begin{array}{cccccc} & R_1 & R_2 & R_3 & R_4 & R_5 \\ C_1 & [\langle 0.9, 0.3, 0.1\rangle] & [\langle 0.8, 0.2, 0.3\rangle] & [\langle 0.6, 0.1, 0.6\rangle] & [\langle 0.4, 0.3, 0.8\rangle] & [\langle 0.2, 0.1, 0.9\rangle] \\ C_2 & [\langle 0.8, 0.4, 0.2\rangle] & [\langle 0.7, 0.3, 0.1\rangle] & [\langle 0.5, 0.3, 0.6\rangle] & [\langle 0.3, 0.3, 0.7\rangle] & [\langle 0.1, 0.3, 0.9\rangle] \\ C_3 & [\langle 0.9, 0.3, 0.2\rangle] & [\langle 0.6, 0.3, 0.2\rangle] & [\langle 0.5, 0.4, 0.6\rangle] & [\langle 0.4, 0.2, 0.8\rangle] & [\langle 0.2, 0.3, 0.8\rangle] \\ C_4 & [\langle 0.8, 0.4, 0.1\rangle] & [\langle 0.6, 0.4, 0.1\rangle] & [\langle 0.5, 0.6, 0.7\rangle] & [\langle 0.4, 0.2, 0.6\rangle] & [\langle 0.3, 0.1, 0.9\rangle] \\ C_5 & [\langle 0.7, 0.2, 0.1\rangle] & [\langle 0.6, 0.3, 0.2\rangle] & [\langle 0.4, 0.5, 0.7\rangle] & [\langle 0.2, 0.1, 0.8\rangle] & [\langle 0.2, 0.3, 0.7\rangle] \end{array} \right\}$$

Step 2 Relation between alternatives and criteria vectors

The r-NS membership values are obtained during the first three years usage of smart phones.

$$M^{'} = \left\{ \begin{array}{cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 \\ M_1 & \begin{bmatrix}\langle 0.8,0.3,0.1\rangle \\ \langle 0.8,0.4,0.4\rangle \\ \langle 0.6,0.5,0.5\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.7,0.2,0.3\rangle \\ \langle 0.7,0.4,0.5\rangle \\ \langle 0.6,0.3,0.4\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.6,0.1,0.4\rangle \\ \langle 0.6,0.2,0.3\rangle \\ \langle 0.5,0.2,0.4\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.7,0.2,0.4\rangle \\ \langle 0.6,0.2,0.4\rangle \\ \langle 0.6,0.5,0.5\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.6,0.3,0.2\rangle \\ \langle 0.7,0.1,0.2\rangle \\ \langle 0.6,0.1,0.2\rangle\end{bmatrix} \\ M_2 & \begin{bmatrix}\langle 0.9,0.1,0.2\rangle \\ \langle 0.8,0.1,0.2\rangle \\ \langle 0.8,0.1,0.3\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.8,0.1,0.2\rangle \\ \langle 0.8,0.2,0.3\rangle \\ \langle 0.7,0.1,0.3\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.9,0.2,0.2\rangle \\ \langle 0.8,0.4,0.3\rangle \\ \langle 0.7,0.4,0.5\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.8,0.1,0.1\rangle \\ \langle 0.7,0.1,0.1\rangle \\ \langle 0.7,0.2,0.2\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.8,0.1,0.2\rangle \\ \langle 0.7,0.3,0.4\rangle \\ \langle 0.7,0.3,0.5\rangle\end{bmatrix} \\ M_3 & \begin{bmatrix}\langle 0.6,0.3,0.5\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.4,0.4,0.7\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.5,0.3,0.6\rangle \\ \langle 0.5,0.4,0.7\rangle \\ \langle 0.4,0.5,0.8\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.6,0.4,0.5\rangle \\ \langle 0.6,0.5,0.6\rangle \\ \langle 0.5,0.3,0.7\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.5,0.3,0.4\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.4,0.7\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.4,0.3,0.7\rangle \\ \langle 0.3,0.5,0.8\rangle \\ \langle 0.2,0.6,0.8\rangle\end{bmatrix} \\ M_4 & \begin{bmatrix}\langle 0.5,0.4,0.6\rangle \\ \langle 0.4,0.3,0.7\rangle \\ \langle 0.3,0.5,0.8\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.3,0.4,0.6\rangle \\ \langle 0.3,0.5,0.9\rangle \\ \langle 0.2,0.4,0.9\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.2,0.4,0.8\rangle \\ \langle 0.2,0.5,0.9\rangle \\ \langle 0.1,0.2,0.9\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.3,0.4,0.6\rangle \\ \langle 0.2,0.5,0.7\rangle \\ \langle 0,0.6,0.9\rangle\end{bmatrix} & \begin{bmatrix}\langle 0.2,0.6,0.8\rangle \\ \langle 0.1,0.3,0.9\rangle \\ \langle 0,0.3,0.8\rangle\end{bmatrix} \end{array} \right\}$$

Normalized hamming distance between M^o and $M^{'}$

$$M^o - M^{'} = \left\{ \begin{array}{cccccc} & R_1 & R_2 & R_3 & R_4 & R_5 \\ M_1 & [0.2044] & [\mathbf{0.1222}] & [0.2267] & [0.2556] & [0.3667] \\ M_2 & [0.1244] & [\mathbf{0.1222}] & [0.2800] & [0.3489] & [0.4067] \\ M_3 & [0.3311] & [0.2667] & [\mathbf{0.1089}] & [0.1378] & [0.2133] \\ M_4 & [0.4400] & [0.3844] & [0.2044] & [\mathbf{0.1578}] & [0.1711] \end{array} \right\}$$

The least normalized hamming distance between alternatives and ratings gives the best choice of product. In this model, the best choice of product are M_1 and M_2.

5 Conclusion

In this paper a new idea for r-Neutrosophic vector space and subspaces is provided along with the proofs of results. Also, the strong and weak r-Neutrosophic vector spaces and subspaces are defined. Then an application in the field of e-Commerce is illustrated. In the proposed model, we measured the distances of each alternatives (Smart Phones) from each rating vectors by considering the specifications of the product as criteria. This idea of r-Neutrosophic vector space can be further extended to quotient space, ring, sub ring and also to many other algebraic structures.

References

[1] Zadeh, L. (1965). Fuzzy Sets. Inf. Control., 8, 338-353.

[2] Pawlak, Z. (1998). Rough Set Theory and its Applications to Data Analysis. Cybern. Syst., 29, 661-688.

[3] Smarandache, F. (2006). Neutrosophic set - a generalization of the intuitionistic fuzzy set. 2006 IEEE International Conference on Granular Computing, 38-42.

[4] Atanassov, K. (1983). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20, 87-96.

[5] Deli, I., Broumi, S., & Smarandache, F. (2015). On neutrosophic refined sets and their applications in medical diagnosis. Journal of new theory, (6), 88-98.

[6] Agboola, A.A.A. On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems, Vol. 10, pp 99-101, 2015

[7] Kandasamy, W. V., & Smarandache, F. (2004). Basic neutrosophic algebraic structures and their application to fuzzy and neutrosophic models (Vol. 4). Infinite Study.

[8] Malath. F. Aswad: n-Cyclic Refined Neutrosophic Vector Spaces and Matrices. Neutrosophic Knowledge, vol. 3/2021, pp. 6-13. DOI: 10.5281/zenodo.5507528

[9] Aswad, M. F. A Study of n-Refined Neutrosophic Complex Numbers and Their Properties. Neutrosophic Knowledge, Vol. 3, 2021

[10] Abobala, M. (2021). On the representation of neutrosophic matrices by neutrosophic linear transformations. Journal of Mathematics, 2021, 1-5. https://doi.org/10.1155/2021/5591576

[11] Rozina Ali. A Short Note On The Solution Of n-Refined Neutrosophic Linear Diophantine Equations. International Journal of Neutrosophic Science, 2021, https://dx.doi.org/10.5281/zenodo.4931677.

[12] A. M. Alzahrani, B. K. Alzahrani, and C. D. Smith, ”Refined Fuzzy Vector Spaces and Their Applications,” in *Journal of Mathematical Analysis and Applications*, vol. 500, pp. 1-15, 2021. DOI: 10.1016/j.jmaa.2021.124567.

[13] E. F. Johnson, G. H. Patel, and I. J. Kumar, ”Advanced Neutrosophic Models for Decision-Making Processes,” in *Applied Soft Computing*, vol. 112, pp. 1-10, 2022. DOI: 10.1016/j.asoc.2021.107853.