Subfamilies of analytic functions associated with Rabotnov function

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Abstract

The aim of this paper is to investigate various subfamilies of analytic functions to find inclusion properties, and necessary and sufficient conditions for the Rabotnov function to be in these subfamilies. Furthermore, several corollaries will be implied from our main results.

Keywords: Analytic; Starlike; Convex; Rabotnov function

1 Preliminaries and Definitions

The Rabotnov function is a mathematical model that describes the time-dependent deformation behavior of materials under continuous load in the field of creep mechanics and for predicting the progressive deformation of materials under continuous stresses.

Let $\mathbf{\Lambda}$ be the family of all analytic functions of the form:

$$q(\varepsilon) = \varepsilon + \sum_{\upsilon=2}^{\infty} C_\upsilon \varepsilon^\upsilon, \quad |\varepsilon| < 1, \tag{1}$$

which are normalized by $q(0) = q'(0) - 1 = 0$. Also, denote by H the subclass of $\mathbf{\Lambda}$ consisting the functions of the form:

$$q(\varepsilon) = \varepsilon - \sum_{\upsilon=2}^{\infty} C_\upsilon \varepsilon^\upsilon, \quad C_\upsilon \geq 0. \tag{2}$$

In 1948,[1] Rabotnov introduced the Rabotnov function (Rabotnov fractional exponential function) as follows:

$$M_{\vartheta,\mu}(\varepsilon) = \varepsilon^\vartheta \sum_{\upsilon=0}^{\infty} \frac{\mu^\upsilon \varepsilon^{\upsilon(1+\vartheta)}}{\Gamma((\upsilon+1)(1+\vartheta))}, \quad (\vartheta, \mu, \varepsilon \in \mathbb{C}). \tag{3}$$

It is clear that $M_{\vartheta,\mu}(\varepsilon) \notin \mathbf{\Lambda}$. Accordingly, we consider the Rabotnov functions as follows:

$$\begin{aligned} \mathcal{R}_{\vartheta,\mu}(\varepsilon) &= \varepsilon^{\frac{1}{1+\vartheta}} \Gamma(1+\vartheta) M_{\vartheta,\mu}(\varepsilon^{\frac{1}{1+\vartheta}}) \\ &= \varepsilon + \sum_{\upsilon=2}^{\infty} \frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)} \varepsilon^\upsilon, \quad |\varepsilon| < 1. \end{aligned} \tag{4}$$

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Also, the linear operator $\mathcal{R}_{\vartheta,\mu}(\varepsilon)q : \mathbf{\Lambda} \to \mathbf{\Lambda}$ as:

$$\mathcal{R}_{\vartheta,\mu}(\varepsilon)q = \mathcal{R}_{\vartheta,\mu}(\varepsilon) * q(\varepsilon) = \varepsilon + \sum_{\upsilon=2}^{\infty} \frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)} C_\upsilon \varepsilon^\upsilon.$$

We consider the following subfamily of analytic functions introduced by Gupta and Jain.[2]

For some $\beta_1 \in [0,1)$, $\beta_2 \in (0,1]$ and $q \in \mathbf{\Lambda}$ in the form (2), then $q \in L^*(\beta_1, \beta_2)$ if satisfying the criteria

$$\left| \frac{q'(\varepsilon) - 1}{1 + q'(\varepsilon) - 2\beta_1} \right| < \beta_2, \quad |\varepsilon| < 1. \tag{5}$$

Also, Gupta and Jain[3] studied the following two subfamilies by replacing $q'(\varepsilon)$ with $\frac{\varepsilon q'(\varepsilon)}{q(\varepsilon)}$ and $1 + \frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}$, respectively, in (5).

For some $\beta_1 \in [0,1)$, $\beta_2 \in (0,1]$ and $q \in \mathbf{\Lambda}$ in the form (2), then $q \in S^*(\beta_1, \beta_2)$ if satisfying the criteria

$$\left| \frac{\frac{\varepsilon q'(\varepsilon)}{q(\varepsilon)} - 1}{1 + \frac{\varepsilon q'(\varepsilon)}{q(\varepsilon)} - 2\beta_1} \right| < \beta_2, \quad |\varepsilon| < 1,$$

and $q \in C^*(\beta_1, \beta_2)$ if satisfying the criteria

$$\left| \frac{\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}}{\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} + 2(1-\beta_1)} \right| < \beta_2, \quad |\varepsilon| < 1.$$

Remark 1.1. For $\beta_2 = 1$, both subfamilies $S^*(\beta_1, \beta_2)$ and $C^*(\beta_1, \beta_2)$ reduce to the well known families of starlike and convex functions of order β_1 , respectively (see[4]).

Now, we provide some sufficient and necessary restrictions for functions to be in the subfamilies $L^*(\beta_1, \beta_2)$, $S^*(\beta_1, \beta_2)$ and $C^*(\beta_1, \beta_2)$.

Lemma 1.2. [3] *A function* $q \in \mathrm{L}^*(\beta_1, \beta_2)$ *if and only if*

$$\sum_{\upsilon=2}^{\infty} \upsilon(\beta_2 + 1) |C_\upsilon| \le 2\beta_2(1 - \beta_1). \tag{6}$$

The result (6) is sharp.

Lemma 1.3. [2] *A function* $q \in \mathrm{S}^*(\beta_1, \beta_2)$ *if and only if*

$$\sum_{\upsilon=2}^{\infty} [(\upsilon(\beta_2 + 1) + \beta_2(1 - 2\beta_1) - 1] |C_\upsilon| \le 2\beta_2(1 - \beta_1), \tag{7}$$

and $q \in \mathrm{C}^*(\beta_1, \beta_2)$ *if and only if*

$$\sum_{\upsilon=2}^{\infty} \upsilon [(\upsilon(\beta_2 + 1) + \beta_2(1 - 2\beta_1) - 1] |C_\upsilon| \le 2\beta_2(1 - \beta_1). \tag{8}$$

The results (7) and (8) are sharp.

In geometric function theory, special functions are generally recognized for their importance after De Branges response to the well-known Bieberbach conjecture. The geometric characteristics of numerous types of special functions are covered in many literature, see for example.[5–7]

Motivated by the work of Al-Hawary et al.,[8] we establish inclusion properties and necessary and sufficient conditions for the Rabotnov function to belong to these subfamilies in this work.

2 Main Results for the Subfamilies $L^*(\beta_1,\beta_2)$, $S^*(\beta_1,\beta_2)$ and $C^*(\beta_1,\beta_2)$

In this section, we will give sufficient conditions for Rabotnov function to be in the subfamilies which are introduced in the previous section.

For our initial findings, we require the subsequent lemma provided by Eker and Ece:[7]

Lemma 2.1. *For $\upsilon \in \mathbb{N}$ and $\vartheta \geq 0$, then*

$$(\vartheta+1)^{\upsilon-1}(\upsilon-1)!\Gamma(\vartheta+1) \leq \Gamma((\vartheta+1)\upsilon).$$

And based on this lemma, we may write:

$$\frac{\Gamma(\vartheta+1)}{\Gamma((\vartheta+1)\upsilon)} \leq \frac{1}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!}. \tag{9}$$

Theorem 2.2. *If $\beta_1 \in [0,1)$, $\beta_2 \in (0,1]$, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon) \in \mathrm{L}^*(\beta_1,\beta_2)$ if the following condition is satisfied:*

$$\frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}} + (\beta_2+1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right) \leq 2\beta_2(1-\beta_1).$$

Proof. By Rabotnov function given by (4) and equation (6), it suffices to show that

$$\wp_\upsilon(\beta_1,\beta_2,\mu,\vartheta,\gamma) = \sum_{\upsilon=2}^{\infty} \upsilon(\beta_2+1)\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)} \leq 2\beta_2(1-\beta_1).$$

By writing $\upsilon = (\upsilon-1)+1$, we have

$$\begin{aligned}&\wp_\upsilon(\beta_1,\beta_2,\mu,\vartheta,\gamma)\\ =\ &(\beta_2+1)\sum_{\upsilon=2}^{\infty}(\upsilon-1)\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)} + (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}.\end{aligned}$$

Using the inequality *(9)*, we get

$$\begin{aligned}&\wp_\upsilon(\beta_1,\beta_2,\mu,\vartheta,\gamma)\\ &\leq (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{(\upsilon-1)\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!} + (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!}\\ &= (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-2)!} + (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!}\\ &= \frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}} + (\beta_2+1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right).\end{aligned} \tag{10}$$

But the inequality (10) is bounded above by $2\beta_2(1-\beta_1)$, thus

$$\frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}} + (\beta_2+1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right) \leq 2\beta_2(1-\beta_1).$$

□

Theorem 2.3. *If $\beta_1 \in [0,1)$, $\beta_2 \in (0,1]$, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon) \in \mathrm{S}^*(\beta_1,\beta_2)$ if the following condition is satisfied:*

$$\frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}} + 2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right) \leq 2\beta_2(1-\beta_1). \tag{11}$$

Proof. By Rabotnov function given by (4) and equation (7), it suffices to show that

$$\begin{aligned}
& {}_{\upsilon}(\beta_1,\beta_2,\mu,\vartheta,\gamma) \\
= & \sum_{\upsilon=2}^{\infty}\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)}\gamma(\upsilon)\le 2\beta_2(1-\beta_1).
\end{aligned}$$

By writing $\upsilon=(\upsilon-1)+1$, we have

$$\begin{aligned}
& {}_{\upsilon}(\beta_1,\beta_2,\mu,\vartheta,\gamma) = \\
& (\beta_2+1)\sum_{\upsilon=2}^{\infty}(\upsilon-1)\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)}+2\beta_2(1-\beta_1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)}.
\end{aligned}$$

Using the inequality (9), we get

$$\begin{aligned}
& {}_{\upsilon}(\beta_1,\beta_2,\mu,\vartheta,\gamma) \\
& \le (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{(\upsilon-1)\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!}+2\beta_2(1-\beta_1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!} \\
& = (\beta_2+1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-2)!}+2\beta_2(1-\beta_1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!} \\
& = \frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right). \qquad (12)
\end{aligned}$$

But the inequality (12) is bounded above by $2\beta_2(1-\beta_1)$, thus

$$\frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\le 2\beta_2(1-\beta_1).$$

□

Theorem 2.4. *If $\beta_1\in[0,1)$, $\beta_2\in(0,1]$, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon)\in \mathrm{C}^*(\beta_1,\beta_2)$ if the following condition is satisfied:*

$$\begin{aligned}
& \frac{\mu^2(\beta_2+1)}{(\vartheta+1)^2}e^{\frac{\mu}{\vartheta+1}}+\frac{2\mu\left(\beta_2(2-\beta_1)+1\right)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right) \\
\le\ & 2\beta_2(1-\beta_1).
\end{aligned}$$

Proof. By Rabotnov function given by (4) and equation (8), it suffices to show that

$$\begin{aligned}
& {}_{\upsilon}(\beta_1,\beta_2,\mu,\vartheta,\gamma) \\
= & \sum_{\upsilon=2}^{\infty}\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)}\le 2\beta_2(1-\beta_1).
\end{aligned}$$

By writing $\upsilon=(\upsilon-1)+1$ and $\upsilon^2=(\upsilon-1)(\upsilon-2)+3(\upsilon-1)+1$, we have

$$\begin{aligned}
& {}_{\upsilon}(\beta_1,\beta_2,\mu,\vartheta,\gamma) \\
= & (\beta_2+1)\sum_{\upsilon=2}^{\infty}(\upsilon-1)(\upsilon-2)\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)} \\
& +2\left(\beta_2(2-\beta_1)+1\right)\sum_{\upsilon=2}^{\infty}(\upsilon-1)\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)} \\
& +2\beta_2(1-\beta_1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}\Gamma\left(1+\vartheta\right)}{\Gamma\left((1+\vartheta)\,\upsilon\right)}.
\end{aligned}$$

Under the inequality (9), we get

$$\begin{aligned}
&{}_{\upsilon}(\beta_1,\beta_2,\mu,\vartheta,\gamma)\\
&\leq (\beta_2+1)\sum_{\upsilon=3}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-3)!}\\
&+2\left(\beta_2(2-\beta_1)+1\right)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-2)!}\\
&+2\beta_2(1-\beta_1)\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}}{(\vartheta+1)^{\upsilon-1}(\upsilon-1)!}\\
&=\frac{\mu^2(\beta_2+1)}{(\vartheta+1)^2}e^{\frac{\mu}{\vartheta+1}}+\frac{2\mu\left(\beta_2(2-\beta_1)+1\right)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right). \qquad (13)
\end{aligned}$$

But the inequality (13) is bounded above by $2\beta_2(1-\beta_1)$, thus

$$\begin{aligned}
&\frac{\mu^2(\beta_2+1)}{(\vartheta+1)^2}e^{\frac{\mu}{\vartheta+1}}+\frac{\mu\left(2+2\beta_2(1-\beta_1)\right)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\\
\leq\;& 2\beta_2(1-\beta_1).
\end{aligned}$$

□

Corollary 2.5. *If $\beta_1\in[0,1)$ and $\beta_2=1$, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon)\in \mathrm{L}^*(\beta_1,1)$ if satisfied*

$$\frac{\mu}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\leq\beta_2(1-\beta_1).$$

Corollary 2.6. *If $\beta_1\in[0,1)$ and $\beta_2=1$, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon)\in \mathrm{S}^*(\beta_1,1)$ if satisfied*

$$\frac{\mu}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\leq\beta_2(1-\beta_1). \qquad (14)$$

Corollary 2.7. *If $\beta_1\in[0,1)$, $\beta_2=1$, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon)\in \mathrm{C}^*(\beta_1,1)$ if satisfied*

$$\frac{\mu^2}{(\vartheta+1)^2}e^{\frac{\mu}{\vartheta+1}}+\frac{\mu\left((3-\beta_1)\right.}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\leq\beta_2(1-\beta_1).$$

3 Inclusion Properties for the Subfamily $C^*(\beta_1,\beta_2)$

For $\gamma_1\in(0,1]$, $\gamma_2<1$, $\gamma_3\in\mathbb{C}\backslash\{0\}$, a function $q\in\mathbf{\Lambda}$ is said to be in the subfamily $\mathcal{ML}^{\gamma_3}(\gamma_1,\gamma_2)$ if it satisfies the following inequality:

$$\left|\frac{(1-\gamma_1)\frac{q(\varepsilon)}{\varepsilon}+\gamma_1 q'(\varepsilon)-1}{2\gamma_3(1-\gamma_2)+(1-\gamma_1)\frac{q(\varepsilon)}{\varepsilon}+\gamma_1 q'(\varepsilon)-1}\right|<1,\ \varepsilon\in\mathbf{U}.$$

The subfamily $\mathcal{ML}^{\gamma_3}(\gamma_1,\gamma_2)$ was introduced by Swaminathan.[9]

Lemma 3.1. [9] *If $q\in\mathcal{ML}^{\gamma_3}(\gamma_1,\gamma_2)$ is in the form (2), then*

$$|C_\upsilon|\leq\frac{2\left|\gamma_3\right|(1-\gamma_2)}{\gamma_1(\upsilon-1)+1},\quad \upsilon\in\mathbb{N}-\{1\}.$$

Using Lemma 3.1, we will study the influence of the Rabotnov function $\mathcal{R}_{\vartheta,\mu}(\varepsilon)q$ on the subfamily $C^*(\beta_1,\beta_2)$.

Theorem 3.2. *For $q \in \mathcal{ML}^{\gamma_3}(\gamma_1,\gamma_2)$, if the inequality*

$$\left[\frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\right]\leq\frac{\gamma_1\beta_2(1-\beta_1)}{|\gamma_3|(1-\gamma_2)}$$

is satisfied, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon)q \in \mathrm{C}^*(\beta_1,\beta_2)$.

Proof. Let $q \in ML^{\gamma_3}(\gamma_1,\gamma_2)$ and given by (2), by virtue of Lemma 1.3 it suffices to show that

$$\sum_{\upsilon=2}^{\infty}\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}|C_\upsilon|\leq 2\beta_2(1-\beta_1).$$

Since $q \in \mathcal{ML}^{\gamma_3}(\gamma_1,\gamma_2)$, then by Lemma 3.1

$$\begin{aligned}&\sum_{\upsilon=2}^{\infty}\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}|C_\upsilon|\\ \leq\;& 2|\gamma_3|(1-\gamma_2)\sum_{\upsilon=2}^{\infty}\frac{\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]}{\gamma_1(\upsilon-1)+1}\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}.\end{aligned}$$

Since $\gamma_1(\upsilon-1)+1\geq\upsilon\gamma_1$

$$\begin{aligned}&\sum_{\upsilon=2}^{\infty}\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}|C_\upsilon|\\ &\leq\frac{2|\gamma_3|(1-\gamma_2)}{\gamma_1}\sum_{\upsilon=2}^{\infty}\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}.\end{aligned}$$

Proceeding as in Theorem 2.3, we get

$$\begin{aligned}&\sum_{\upsilon=2}^{\infty}\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}|C_\upsilon|\\ &\leq\frac{2|\gamma_3|(1-\gamma_2)}{\gamma_1}\left[\frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\right].\end{aligned}$$

But the last inequality is bounded above by $2\beta_2(1-\beta_1)$, this concludes the proof of Theorem 3.2. □

Corollary 3.3. *For $q \in \mathcal{ML}^{\gamma_3}(\gamma_1,\gamma_2)$, if the inequality*

$$\left[\frac{\mu}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right)\right]\leq\frac{\gamma_1(1-\beta_1)}{2|\gamma_3|(1-\gamma_2)}$$

is satisfied, then $\mathcal{R}_{\vartheta,\mu}(\varepsilon)q \in \mathrm{C}^*(\beta_1,1)$.

Theorem 3.4. *The functions $\mathcal{Q}_{\vartheta,\mu}(\varepsilon)=\int_0^{\varepsilon}\frac{\mathcal{R}_{\vartheta,\mu}(s)}{s}ds$ in the subfamily $\mathrm{C}^*(\beta_1,\beta_2)$ if and only if satisfied the inequality (11).*

Proof. Since

$$\mathcal{Q}_{\vartheta,\mu}(\varepsilon)=\varepsilon+\sum_{\upsilon=2}^{\infty}\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}\frac{\varepsilon^\upsilon}{\upsilon},$$

then by Theorem 2.4 it suffices to show that

$$\sum_{\upsilon=2}^{\infty}\upsilon\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)\upsilon}\leq 2\beta_2(1-\beta_1).$$

That is,

$$\sum_{\upsilon=2}^{\infty}\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)} \leq 2\beta_2(1-\beta_1).$$

As a proceeding in Theorem 2.3, we get

$$\sum_{\upsilon=2}^{\infty}\left[(\upsilon(\beta_2+1)+\beta_2(1-2\beta_1)-1\right]\frac{\mu^{\upsilon-1}\Gamma(1+\vartheta)}{\Gamma((1+\vartheta)\upsilon)}$$
$$\leq \frac{\mu(\beta_2+1)}{\vartheta+1}e^{\frac{\mu}{\vartheta+1}}+2\beta_2(1-\beta_1)\left(e^{\frac{\mu}{\vartheta+1}}-1\right).$$

But the last inequality is bounded above by $2\beta_2(1-\beta_1)$, this concludes the proof of Theorem 3.4. □

Corollary 3.5. *The functions* $\mathcal{Q}_{\vartheta,\mu}(\varepsilon)=\int_0^{\varepsilon}\frac{\mathcal{R}_{\vartheta,\mu}(s)}{s}ds$ *in the subfamily* $\mathrm{C}^*(\beta_1,1)$ *if and only if satisfied the inequality (14).*

4 Conclusions

Using of the Rabotnov function $\mathcal{R}_{\vartheta,\mu}(\varepsilon)$, we examined some inclusion properties, necessary and sufficient condition for this function to be in the subfamilies $L^*(\beta_1,\beta_2)$, $S^*(\beta_1,\beta_2)$ and $C^*(\beta_1,\beta_2)$. Furthermore, some corollaries will be implied by our findings. After this work, the function $\mathcal{R}_{\vartheta,\mu}(\varepsilon)$ is applicable to the derivation of novel inclusion relations and conditions for analytic functions in different subclasses in the open unit disk.

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References

[1] Y. N. Rabotnov, Equilibrium of an elastic medium with after-effect (in Russian), Prikladnaya Matematika i Mekhanika (*J. Appl. Math. Mech.*), **12**(1) (1948), 53-62.

[2] V. P. Gupta, P. K. Jain, Certain classes of univalent functions with negative coefficents II, *Bull. Aust. Math. Soc.*, **15** (1976), 467-473.

[3] V. P. Gupta, P. K. Jain, Certain classes of univalent functions with negative coefficents I, *Bull. Aust. Math. Soc.*, **14** (1976), 409-416.

[4] H. Silverman, Univalent functions with negative coefficents, *Proc. Amer. Math. Soc.*, **51** (1975), 109-116.

[5] A. Hussen, A. Zeyani, Coefficients and Fekete–Szegö functional estimations of bi-univalent subclasses based on Gegenbauer polynomials, *Mathematics*, **11**(13) (2023), 2852.

[6] M. Illafe, A. Amourah, M. Haji Mohd, Coefficient estimates and Fekete–Szegö functional inequalities for a certain subclass of analytic and bi-univalent functions, *Axioms*, **11**(4) (2022), 147.

[7] S. S. Eker, S. Ece, Geometric properties of normalized Rabotnov function, *Hacettepe Journal of Mathematics and Statistics*, **51**(5) (2022), 1248-1259.

[8] T. Al-Hawary, I. Aldawish, B. A. Frasin, O. Alkam, F. Yousef, Necessary and Sufficient Conditions for Normalized Wright Functions to Be in Certain Classes of Analytic Functions, *Mathematics*, **10** (24)(2022), 4693.

[9] A. Swaminathan, Certain sufficient conditions on Gaussian hypergeometric functions, *Journal of Inequalities in Pure and Applied Mathematics*, **5**(4) (2004), Art. 83, 1-10.