Fixed point results in ω_t-distance mappings for Geraghty type contractions

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Abstract

In this study, we establish fixed point theorems for $P\omega_t$-contractions within b-metric spaces by utilizing ω_t-distance mappings. Subsequently, we demonstrate fixed point results pertaining to nonlinear contraction conditions of the Geraghty type, again employing ω_t-distance mappings in the context of a complete b-metric space. Additionally, we bolster our findings with appropriate examples to illustrate the applicability of our results.
Keywords: Fixed point; b-metric space; $P\omega_t$-contraction; Nonlinear contraction

1 Introduction and Preliminary

In mathematics, a fixed-point theorem is a result saying that a self-map function f on a non-empty set $\mho$ will have at least one fixed point in $\mho$ (i.e., there is some $\xi \in \mho$ such that $f\xi = \xi$). This theorem holds true under certain conditions on f, which can be expressed in general terms. In a variety of mathematical contexts, the presence of a solution is often synonymous with the existence of a fixed point for an appropriate mapping. Consequently, the identification of fixed points carries significant ramifications across multiple domains within mathematics and other scientific disciplines. This theory represents a profound integration of analysis (both pure and applied) topology, and geometry. In the past five decades, the fixed point theory has emerged as a highly influential and essential instrument in the exploration of nonlinear analysis.

The Banach fixed-point theorem,[1] known also as the Banach contraction principle, plays a crucial role in the field of mathematics, particularly in the study of metric spaces. This theorem guarantees the presence and uniqueness of fixed points for certain self-maps in metric spaces. Moreover, it presents a methodical approach to determining these fixed points. Essentially, the Banach fixed-point theorem can be seen as a generalized version of Picard's method of successive approximations. It is named after Stefan Banach (1892–1945), who first introduced this theorem in 1922.

Numerous mathematicians have subsequently explored a range of generalizations of Banach's theorem across different contexts see for example.[?,8–12,14–19] A notable example is the concept of b-metric spaces, which was first introduced by Bakhtin[7]and later refined and named by Czerwik.[2] This framework has been employed to examine various fixed point theorems. Additionally, Hussain et al[3] introduced the idea of ωt-distance mappings, utilizing this concept to investigate several fixed point results.

Definition 1.1. [3] Let $\mho$ denote a set. Define a function $d : \mho \times \mho \to [0, \infty)$ that fulfills the following conditions:

1. $d(\xi, \zeta) = 0$ if and only if ξ is equal to ζ;

DOI: https://doi.org/10.54216/IJNS.260101

2. $d(\xi, \zeta) = d(\zeta, \xi)$ for all $\xi, \zeta \in \mho$;
3. There exists a constant $s \geq 1$ such that for any points $\xi, \zeta, \rho \in \mho$, the inequality $d(\xi, \zeta) \leq s(d(\xi, \rho) + d(\rho, \zeta))$ holds.

The triplet $(\mho, d, s)$ is referred to as a b-metric space.

It should be noted that in the case where s equals 1, the triplet $(\mho, d, s)$ forms a metric space. This implies that the properties of a metric space hold true when s is equal to 1.

Definition 1.2. [3]

Let $(\mho, d, s)$ represent a b-metric space.

1. A sequence (ξ_n) is said to converge to an element $\xi \in \mho$ if and only if the limit $\lim_{n\to\infty} d(\xi_n, \xi) = 0$ holds true.
2. The sequence (ξ_n) is classified as Cauchy if and only if the limit $\lim_{n,m\to\infty} d(\xi_n, \xi_m) = 0$ is satisfied.
3. The space $(\mho, d, s)$ is defined as complete if and only if every Cauchy sequence within $\mho$ converges.

Kada et al.[4] presented the notion of ω-distance within a metric space in 1996 and established several fixed point theorems. In the current study, we define the concept of ω_t-distance and articulate a lemma that will be utilized in the principal sections of this research.

Definition 1.3. [3] Let $(\mho, d, s)$ represent a b-metric space where the constant s satisfies $s \geq 1$. A function $p : \mho \times \mho \to [0, \infty)$ is designated as a ω_t on $\mho$ if it fulfills the following conditions:

(a) For any points $\xi, \zeta, \rho \in \mho$, the inequality $p(\xi, \rho) \leq s(p(\xi, \zeta) + p(\zeta, \rho))$ holds;

(b) For every $\xi \in \mho$, the mapping $p(\xi, \cdot) : \mho \to [0, \infty)$ is s-lower semi-continuous;

(c) For any given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $p(\rho, \xi) \leq \delta$ and $p(\rho, \zeta) \leq \delta$, then it follows that $d(\xi, \zeta) \leq \varepsilon$.

A real-valued function f defined on a b-metric space $\mho$ is considered to be s-lower semi-continuous at a point ξ_0 in $\mho$ if one of the following conditions holds: either $\liminf_{\xi_n\to\xi_0} f(\xi_n) = \infty$ or $f(\xi_0) \leq \liminf_{\xi_n\to\xi_0} sf(\xi_n)$. This is applicable for sequences ξ_n that belong to $\mho$ for every $n \in \mathbb{N}$ and converge to ξ_0.

Next, we recall some examples on ω_t mappings.

Example 1.4. [3] Let $(\mho, d, s)$ represent a b-metric space. In this context, the function $p = d$ serves as an ω_t-distance on the set $\mho$.

Proof. The statements (a) and (b) are self-evident. To demonstrate (c), let us consider any $\epsilon > 0$ and set $\delta = \frac{\epsilon}{2s}$. It follows that if $p(\xi, \rho) \leq \delta$ and $p(\rho, \zeta) \leq \delta$, then it can be concluded that $d(\xi, \zeta) \leq \epsilon$. ☐

Example 1.5. [3] Let $\mho = \mathbb{R}$ and define the distance function as $d(\xi, \zeta) = (\xi - \zeta)^2$. The function $p : \mho \times \mho \to [0, \infty)$, given by $p(\xi, \zeta) = |\xi|^2 + |\zeta|^2$ for all $\xi, \zeta \in \mho$, serves as an ω_t-distance on the set $\mho$.

Proof. The assertions (a) and (b) are self-evident. To demonstrate (c), let us consider any $\epsilon > 0$ and set $\delta = \frac{\epsilon}{4}$. Consequently, we obtain the following inequality:

$d(\xi, \zeta) = (\xi - \zeta)^2 \leq 2|\xi|^2 + 2|\zeta|^2 = 2p(\rho, \xi) + 2p(\rho, \zeta) = 2\delta + 2\delta = \epsilon.$ ☐

Example 1.6. [3] Let $\mho = \mathbb{R}$ and define the distance function $d(\xi, \zeta) = (\xi - \zeta)^2$. The function $p : \mho \times \mho \to [0, \infty)$, given by $p(\xi, \zeta) = |\zeta|^2$ for all $\xi, \zeta \in \mho$, qualifies as a ω_t-distance on $\mho$.

Proof. The assertions (a) and (b) are self-evident. To demonstrate (c), let us consider any $\epsilon > 0$ and set $\delta = \frac{\epsilon}{4}$. Consequently, we obtain the following inequality:

$d(\xi, \zeta) = (\xi - \zeta)^2 \leq 2|\xi|^2 + 2|\zeta|^2 = 2p(\rho, \xi) + 2p(\rho, \zeta) = 2\delta + 2\delta = \epsilon.$ □

Lemma 1.7. [3] *Let $(\mho, d, s)$ represent a b-metric space characterized by a constant $s \geq 1$, and let p denote a ω_t-distance defined on $\mho$. Consider sequences (ξ_n) and (ζ_n) within $\mho$, along with sequences (α_n) and (β_n) in the interval $[0, \infty)$ that converge to zero, and let ξ, ζ, ρ be elements of $\mho$. The following statements are established:*

1. *If for every natural number n, the conditions $p(\xi_n, \zeta) \leq \alpha_n$ and $p(\xi_n, \rho) \leq \beta_n$ hold, then it follows that $\zeta = \rho$. Specifically, if $p(\xi, \zeta) = 0$ and $p(\xi, \rho) = 0$, it can be concluded that $\zeta = \rho$.*
2. *If the inequalities $p(\xi_n, \zeta_n) \leq \alpha_n$ and $p(\xi_n, \rho) \leq \beta_n$ are satisfied for all natural numbers n, then the distance $d(\zeta_n, \rho)$ approaches zero.*
3. *If the condition $p(\xi_n, \xi_m) \leq \alpha_n$ is satisfied for all natural numbers n and m with $m > n$, then the sequence (ξ_n) is identified as a Cauchy sequence.*
4. *If the inequality $p(\zeta, \xi_n) \leq \alpha_n$ holds for every natural number n, then the sequence (ξ_n) is also classified as a Cauchy sequence.*

Proof. (1) Let $\epsilon > 0$ be given. Since $\alpha_n \to 0$, then $\forall\ \delta > 0\ \exists\ N_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta, \forall n \geq N_0$.

Similarly, since $\beta_n \to 0$, then $\forall\ \delta > 0\ \exists\ N_1 \in \mathbb{N}$ such that $\beta_n \leq \delta, \forall n \geq N_1$.

If $N = \max\{N_0, N_1\}$, then:

$$\alpha_n \leq \delta, \text{ and } \beta_n \leq \delta \qquad \text{for } n \geq N$$

so,

$$p(\xi_n, \zeta) \leq \alpha_n < \delta \text{ and } p(\xi_n, \rho) \leq \beta_n < \delta \quad \text{for } n \geq N$$

Thus, we have

$$d(\zeta, \rho) < \epsilon$$

Hence, $d(\zeta, \rho) = 0$ and so, $\zeta = \rho$.

(2) Let $\epsilon > 0$ be given. Since $\alpha_n \to 0$, then $\forall\ \delta > 0\ \exists\ N_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta, \forall n \geq N_0$.

Similarly, since $\beta_n \to 0$, then $\forall\ \delta > 0\ \exists\ N_1 \in \mathbb{N}$ such that $\beta_n \leq \delta, \forall n \geq N_1$.

If $N = \max\{N_0, N_1\}$, then:

$$\alpha_n \leq \delta, \text{ and } \beta_n \leq \delta \qquad \text{for } n \geq N$$

so,

$$p(\xi_n, \zeta_n) \leq \alpha_n < \delta \text{ and } p(\xi_n, \rho) \leq \beta_n < \delta \quad \text{for } n \geq N$$

$$\Rightarrow d(\zeta_n, \rho) < \epsilon \quad \text{for each } \epsilon > 0$$

$$\Rightarrow d(\zeta_n, \rho) \to 0$$

(3) Let $\epsilon > 0$ be given. Since $\alpha_n \to 0$, then $\forall\ \delta > 0\ \exists\ N_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta, \forall n \geq N_0$.

Hence, $p(\xi_n, \xi_m) \leq \alpha_n < \delta$ and $p(\xi_n, \xi_{n+1})\alpha_n < \delta$ for $m \geq n \geq N_0$.
Thus, by the definition of p, we have

$$d(\xi_{n+1}, \xi_m) < \epsilon \quad \text{for } m \geq n \geq N.$$

This implies that (ξ_n) is a Cauchy sequence.
(4) Let $\epsilon > 0$ be given. Since $\alpha_n \to 0$, then $\forall\ \delta > 0\ \exists\ N_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta, \forall n \geq N_0$

$$\alpha_n \leq \delta \quad \text{for } n \geq N.$$

So,

$$p(\zeta, \xi_n) \leq \alpha_n < \delta \quad \text{for } n \geq N_0$$

So,

$$\alpha_m \leq \delta \quad \text{for } m \geq N_0.$$

Then, $p(\zeta, \xi_n) \leq \alpha_m < \delta \quad \text{for } m \geq N_1$

If $N = \max\{N_0, N_1\}$, then:

$$d(\zeta, \xi_n) < \epsilon \quad \text{for } n, m \geq N.$$

This implies that (ξ_n) is a Cauchy sequence.

□

2 Main Result

Next, we give the definition of $P\omega_t$ contractions in ω_t mappings.

Definition 2.1. Let $d : \mho \times \mho \to [0, \infty)$ be a b-metric on $\mho$, p be a ω_t-distance on $\mho$ there exists $k \in [0, 1)$ such that $f : \mho \to \mho$ satisfying

$$p(f\xi, f\zeta) \leq k\,[p(\xi, \zeta) + |p(\xi, f\xi) - p(\zeta, f\zeta)|] \tag{1}$$

Then f is said to be a $P\omega_t$-contraction.

Theorem 2.2. *Let $(\mho, d, s)$ represent a b-metric space characterized by a constant $s \geq 1$, and let p denotes a ω_t-distance on $\mho$. Additionally, let $f : \mho \to \mho$ be characterized as a $P_{\omega t}$-contraction. We assume that one of the following conditions is satisfied:*

(i) The function f is continuous.

(ii) For every $\zeta \in \mho$ such that $\zeta \neq f\zeta$, it holds that

$$\inf\{p(\xi, \zeta) + p(\xi, f(\xi)) : \xi \in \mho\} > 0.$$

Assuming that $\frac{2ks}{1+k} < 1$, it follows that f possesses a unique fixed point $\rho \in \mho$.

Proof. Let $\xi_0 \in \mho$ represent an arbitrary point. We examine the Picard sequence (ξ_n) characterized by the relation $\xi_{n+1} = f(\xi_n)$ for all $n \geq 0$. We will now analyze two distinct scenarios:

First, assume that there exists an integer $n_0 \in \mathbb{N}$ for which the condition $p(\xi_{n_0}, \xi_{n_0+1}) = 0$ holds. In this case, we claim that $p(\xi_{n_0+1}, \xi_{n_0+2}) = 0$. Indeed, by Condition 1 we have

$$\begin{aligned} p(\xi_{n_0+1}, \xi_{n_0+2}) &= p(f\xi_{n_0}, f\xi_{n_0+1}) \\ &\leq k[p(\xi_{n_0}, \xi_{n_0+1}) + |p(\xi_{n_0}, f\xi_{n_0}) - p(\xi_{n_0+1}, f\xi_{n_0+1})|] \\ &= k[p(\xi_{n_0}, \xi_{n_0+1}) + |p(\xi_{n_0}, \xi_{n_0+1}) - p(\xi_{n_0+1}, \xi_{n_0+2})|] \\ &= kp(\xi_{n_0+1}, \xi_{n_0+2}). \end{aligned}$$

So, $(1-k)p(\xi_{n_0+1}, \xi_{n_0+2}) \leq 0$. Hence $p(\xi_{n_0+1}, \xi_{n_0+2}) = 0$. From the triangular inequality, we have

$$p(\xi_{n_0}, \xi_{n_0+2}) \leq p(\xi_{n_0}, \xi_{n_0+1}) + p(\xi_{n_0+1}, \xi_{n_0+2}) = 0.$$

Now, we have $p(\xi_{n_0}, \xi_{n_0+1}) = 0$ and $p(\xi_{n_0}, \xi_{n_0+2}) = 0$, from Lemma 1.7 (a), we get $\xi_{n_0+1} = \xi_{n_0+2} = f\xi_{n_0+1}$; hence ξ_{n_0+1} is a fixed point of f.

Now suppose $p(\xi_n, \xi_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then from 1 we have

$$\begin{aligned} p(\xi_{n+1}, \xi_{n+2}) &= p(f\xi_n, f\xi_{n+1}) \\ &\leq k[p(\xi_n, \xi_{n+1}) + |p(\xi_n, f\xi_n) - p(\xi_{n+1}, f\xi_{n+1})|] \\ &= k[p(\xi_n, \xi_{n+1}) + |p(\xi_n, \xi_{n+1}) - p(\xi_{n+1}, \xi_{n+2})|]. \end{aligned}$$

If $p(\xi_{n+1}, \xi_{n+2}) > p(\xi_n, \xi_{n+1})$, then we have $p(\xi_{n+1}, \xi_{n+2}) \leq kp(\xi_{n+1}, \xi_{n+2})$, which implies that $p(\xi_{n+1}, \xi_{n+2}) = 0$ a contradiction.
If $p(\xi_{n+1}, \xi_{n+2}) = p(\xi_n, \xi_{n+1})$, then we have $p(\xi_{n+1}, \xi_{n+2}) \leq kp(\xi_n, \xi_{n+1}) = kp(\xi_{n+1}, \xi_{n+2})$, which implies that $p(\xi_{n+1}, \xi_{n+2}) = 0$, a contradiction.

So, we just have $p(\xi_{n+1}, \xi_{n+2}) < p(\xi_n, \xi_{n+1})$ for all $n \in \mathbb{N}$. Hence,

$$p(\xi_{n+1}, \xi_{n+2}) \leq \frac{2k}{1+k} p(\xi_n, \xi_{n+1})$$

for all $n \in \mathbb{N}$. Therefore we have

$$p(\xi_{n+1}, \xi_{n+2}) \leq \lambda^n p(\xi_0, x_1),$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{2k}{1+k} < 1$. Furthermore,

$$\lim_{n\to\infty} p(\xi_{n+1}, \xi_{n+2}) = 0. \tag{2}$$

Now for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p(\xi_n, \xi_m) &\leq s\left[p(\xi_n, \xi_{n+1}) + p(\xi_{n+1}, \xi_{n+2})\right] \\ &\leq sp(\xi_n, \xi_{n+1}) + s^2 p(\xi_{n+1}, \xi_{n+2}) + s^3 p(\xi_{n+2}, \xi_{n+3}) + \ldots \\ &\leq s\lambda^n p(\xi_0, \xi_1) + s^2 \lambda^{n+1} p(\xi_0, x_1) + s^3 \lambda^{n+2} p(\xi_0, \xi_1) + \ldots \\ &= s\lambda^n p(\xi_0, x_1)\left[1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + \ldots\right] \\ &= \frac{s\lambda^n p(\xi_0, \xi_1)}{1 - s\lambda} \end{aligned}$$

So, we have

$$p(\xi_n, \xi_m) \leq \frac{s\lambda^n}{1 - s\lambda} p(\xi_0, \xi_1) \tag{3}$$

Now taking limit as $n, m \to \infty$ we get

$$\lim_{n,m\to\infty} p(\xi_n, \xi_m) = 0 \tag{4}$$

and so from Lemma 1.7 (c), (ξ_n) is a Cauchy sequence. Due to the completeness of $\mho$, there exists $\rho \in \mho$ such that $\xi_n \to \rho$ as $n \to \infty$. Since p is s-lower semicontinuous in the second variable and $\xi_m \to \rho$ as $m \to \infty$, from 4 we get

$$p(\xi_n, \rho) \leq \liminf_{m\to\infty} s \cdot p(\xi_n, \xi_m) \leq s^2 \frac{\lambda^n}{1 - s\lambda} p(\xi_0, \xi_1) \to 0 \text{ as } n \to \infty. \tag{5}$$

Now, if f is continuous, then $\xi_{n+1} = f\xi_n \to f\rho$ and so by the uniqueness of the limit we get $\rho = f\rho$.

Finally, assume (ii) holds and $\rho \neq f\rho$. Then from 2 and 5 we have

$$\begin{aligned} 0 &< \inf\{p(\xi, \rho) + p(\xi, f\xi) : \xi \in \mho\} \\ &\leq \inf\{p(\xi_n, \rho) + p(\xi_n, f\xi_n) : n \in \mathbb{N}\} \\ &= \inf\{p(\xi_n, \rho) + p(\xi_n, \xi_{n+1}) : n \in \mathbb{N}\} \to 0 \end{aligned}$$

as $n \to \infty$, which is a contradiction. Hence $\rho = f\rho$.

To show the uniqueness of the fixed point, first, we have $p(\rho, \rho) = 0$. Indeed, from Condition 1, we get

$$p(\rho, \rho) = p(f\rho, f\rho) \leq kp(\rho, \rho).$$

So, $(1 - k)p(\rho, \rho) \leq 0$. Hence, $p(\rho, \rho) = 0$. Now, suppose u is also a fixed point of f. Then from Condition 1 we have

$$p(\rho, u) = p(f\rho, fu) \leq kp(\rho, u),$$

which implies $p(\rho, u) = 0$. Hence from Lemma 1.7 (a) we have $u = \rho$.

□

Example 2.3. Let $\mho = [0, 1]$ with the b-metric space d by $d(\xi, \zeta) = (\xi - \zeta)^2$. Define $f : \mho \to \mho$ by $f(\xi) = \frac{1-\xi^3}{4-2\xi^3}$ and consider the ω_t-distance in $\mho$ as $p(\xi, \zeta) = \frac{1}{4}(\xi - \zeta)^2$. Then we have

$$\begin{aligned} p(f\xi, f\zeta) &= \frac{1}{4}\left(\frac{1-\xi^3}{4-2\xi^3} - \frac{1-\zeta^3}{4-2\zeta^3}\right)^2 \\ &= \frac{1}{4}\frac{(\xi^3-\zeta^3)^2}{(4-2\xi^3)^2(4-2\zeta^3)^2} \\ &\leq \frac{1}{4}\frac{((\xi-\zeta)(\xi^2+\xi\zeta+\zeta^2))^2}{(4-2\xi^3)^2(4-2\zeta^3)^2} \\ &\leq \frac{9}{64}(\xi-\zeta)^2 \\ &= \frac{9}{64}p(\xi, \zeta). \end{aligned}$$

Now,

$$
\begin{aligned}
p(\xi,\zeta) &= \frac{1}{4}(\xi-\zeta)^2 \\
p(\xi,f\xi) &= \frac{1}{4}\left(\xi-\frac{1-\xi^3}{4-2\xi^3}\right)^2 = \frac{1}{4}\left(\frac{4\xi-2\xi^4-1+\xi^3}{4-2\xi^3}\right)^2 \geq 0 \\
p(\zeta,f\zeta) &= \frac{1}{4}\left(\zeta-\frac{1-\zeta^3}{4-2\zeta^3}\right)^2 = \frac{1}{4}\left(\frac{4\zeta-2\zeta^4-1+\zeta^3}{4-2\zeta^3}\right)^2 \geq 0
\end{aligned}
$$

for all $\xi,\zeta \in \mho$. Hence we get

$$
\begin{aligned}
p(f\xi,f\zeta) &\leq \tfrac{9}{64}(\xi-\zeta)^2 \\
&= \tfrac{9}{16}\tfrac{1}{4}(\xi-\zeta)^2 \\
&\leq \tfrac{9}{16}\left[\, p(\xi,\zeta)+|\, p(\xi,f\xi)-\ p(\zeta,f\zeta)|\right]
\end{aligned}
$$

Clearly, f is continuous. So, for all $\xi,\zeta \in \mho$, that is, f is a P_{ω_t}-contraction, therefore f has a unique fixed point.

Example 2.4. Let $\mho = \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \cup \{0\}$ with the b-metric space d by $d(\xi,\zeta) = (\xi-\zeta)$. Define $f : \mho \to \mho$ by $f\left(\frac{1}{3^n}\right) = \frac{1}{3^{n+1}}$ for $n \in \mathbb{N}$ and $f(0) = 0$. This defines a mapping f on $\mho$ such that each element is mapped to one-third of its current value, and 0 is mapped to itself , and consider the ω_t distance in $\mho$ as $p(\xi,\zeta) = \zeta$. Then f is $P\omega_t$-contraction.

Proof. Case1: If $\xi = 0,\ \zeta = 0$ or $\xi = \frac{1}{3^n},\ \zeta = 0$, then

$$0 = p(f\xi,f\zeta) \leq \frac{1}{3}\left[p(\xi,\zeta)+|p(\xi,fx)-p(\zeta,f\zeta)|\right]$$

Case2: If $\xi = \frac{1}{3^n},\ \zeta = \frac{1}{3^m}$,

$$
\begin{aligned}
p(f\xi,f\zeta) &= p\left(f\left(\tfrac{1}{3^n}\right), f\left(\tfrac{1}{3^m}\right)\right) \\
&= p\left(\tfrac{1}{3^{n+1}}, \tfrac{1}{3^{m+1}}\right) \\
&= \tfrac{1}{3^{m+1}}
\end{aligned}
$$

$$
\begin{aligned}
p(\xi,\zeta)+|\, p(\xi,f\xi)-\ p(y,f\zeta)| &= p\left(\tfrac{1}{3^n},\tfrac{1}{3^m}\right)+\left|\, p\left(\tfrac{1}{3^n}, f\left(\tfrac{1}{3^n}\right)\right)-\ p\left(\tfrac{1}{3^m}, f\left(\tfrac{1}{3^m}\right)\right)\right| \\
&= \tfrac{1}{3^m}+\left|\, p\left(\tfrac{1}{3^n},\tfrac{1}{3^{n+1}}\right)-\ p\left(\tfrac{1}{3^m},\tfrac{1}{3^{m+1}}\right)\right| \\
&= \tfrac{1}{3^m}+\left|\tfrac{1}{3^{n+1}}-\tfrac{1}{3^{m+1}}\right| \\
&= \tfrac{1}{3^m}+\tfrac{1}{3}\left|\tfrac{1}{3^n}-\tfrac{1}{3^m}\right|.
\end{aligned}
$$

So,

$$p(f\xi,f\zeta) \leq \frac{1}{3}\left[\, p(\xi,\zeta)+|\, p(\xi,f\xi)-\ p(\zeta,f\zeta)|\right] \leq \frac{1}{2}\left[\, p(\xi,\zeta)+|\, p(\xi,f\xi)-\ p(\zeta,f\zeta)|\right].$$

Case3: If $\xi = 0,\ \zeta = \frac{1}{3^m}$,then

$$\begin{aligned} p(f\xi, f\zeta) &= p\left(f(0), f\left(\tfrac{1}{3^m}\right)\right) \\ &= p\left(0, \tfrac{1}{3^{m+1}}\right) \\ &= \tfrac{1}{3^{m+1}}. \end{aligned}$$

$$\begin{aligned} p(\xi,\zeta) + |\, p(\xi, f\xi) - p(\zeta, f\zeta)| &= p\left(0, \tfrac{1}{3^m}\right) + \left|\, p\left(0, f\,(\,0)\right) - p\left(\tfrac{1}{3^m}, f\left(\tfrac{1}{3^m}\right)\right)\right| \\ &= \tfrac{1}{3^{m+1}} + \left|\, p\left(0,0\right) - p\left(\tfrac{1}{3^m}, \tfrac{1}{3^{m+1}}\right)\right| \\ &= \tfrac{1}{3^{m+1}} + \left|0 - \tfrac{1}{3^{m+1}}\right| \\ &= \tfrac{2}{3^{m+1}}. \end{aligned}$$

So,

$$p(f\xi, f\zeta) \leq \frac{1}{2}\left[p(\xi,\zeta) + |p(\xi, f\xi) - p(\zeta, f\zeta)|\right].$$

If $\zeta \neq f\zeta$ then $y \neq 0$, so

$$\inf\{p(\xi,\zeta) + p(\xi, f\xi) : \xi \in \mho\} > 0$$

Case1: If $\xi = \frac{1}{3^n}$, so,

$$\inf\{p(\xi,\zeta) + p(\xi, f\xi) : \xi \in \mho\} = \inf\{\zeta + \frac{1}{3^{n+1}} : \xi \in \mho\} = \zeta > 0.$$

Case2: If $\xi = 0$, so,

$$\inf\{p(\xi,\zeta) + p(\xi, f\xi) : \xi \in \mho\} = \inf\{\zeta + 0 : \xi \in \mho\} = \zeta > 0.$$

Then, f fulfilled the conditions of Theorem 2.2, and so, f has a unique fixed point. □

Definition 2.5. [5] The function $\varphi : [0,\infty) \to [0,\infty)$ is classified as an altering distance function if it fulfills the subsequent criteria:

1. The function φ is continuous and nondecreasing;
2. The condition $\varphi(t) = 0$ holds true if and only if $t = 0$.

From this point forward, we will refer to the set of all altering distance functions as Ψ.

Definition 2.6. [6] Let S be the class of all functions $\alpha : \mathbb{R}^+ \to [0,1)$ that satisfy the following implication:

$$\alpha(t_n) \to 1 \implies t_n \to 0.$$

Now, we are ready to introduce our definition.

Definition 2.7. Let $(\mho, d)$ denote a b-metric space that is endowed with the ω_t-distance denoted by p. A self-mapping $f : \mho \to \mho$ is classified as a (φ, α)-Geraghty contraction if there exist functions $\varphi \in \Psi$ and $\alpha \in S$ such that the following condition holds:

$$\varphi sp(f\xi, f\zeta) \leq \alpha(p(\xi,\zeta))\varphi p(\xi,\zeta), \quad \forall \xi,\zeta \in \mho. \tag{6}$$

The concept of Geraghty contraction extends the famous Banach contraction principle.[1]

Remark 2.8. If f is (φ,α)-Geraghty contraction, then for all $\xi,\zeta \in \mho$, we have

$$p(f\xi, f\zeta) < \frac{1}{s}p(\xi,\zeta).$$

Remark 2.9. According to Lemma 1.7, if a sequence (ξ_n) is not a Cauchy sequence, then $\exists\, \varepsilon > 0$ such that for each $K \in \mathbb{N}$, there exist $m > n > K$ such that $p(\xi_n, \xi_m) \geq \varepsilon$.

Lemma 2.10. *Assume that the sequence (x_n) does not qualify as a Cauchy sequence. Consequently, there exists a positive constant $\varepsilon > 0$ such that we can identify subsequences (x_{n_k}) and (x_{m_k}) from the sequence (x_n), where the indices satisfy $n_k > m_k > k$, and it holds that $p(x_{n_k}, x_{m_k}) \geq \varepsilon$ for every natural number K. Additionally, for each m_k, we can select n_k to be the smallest integer greater than m_k that meets the aforementioned condition. It follows that $p(x_{n_k-1}, x_{m_k}) < \varepsilon$.*

Proof. By remark 2.9, there is $\varepsilon > 0$ such that for each $K \in \mathbb{N}$, $\exists m > n > K$ such that $p(x_n, x_m) \geq \varepsilon$ for $m > n > K$.

For $K = 1$, there are $n_1 > m_1 > 1$ such that $p(\xi_{n_1}, \xi_{m_1}) \geq \varepsilon$. If we choose n_1 corresponding to m_1 such that it is the smallest integer with $n_1 > m_1$, then we get $p(\xi_{n_1-1}, \xi_{m_1}) < \varepsilon$.

Again for $K = 2$, there are $n_2 > m_2 > 2$ such that $p(\xi_{n_2}, \xi_{m_2}) \geq \varepsilon$. If we choose n_2 corresponding to m_2 such that it is the smallest integer with $n_2 > m_2$, then we get $p(\xi_{n_2-1}, \xi_{m_2}) < \varepsilon$.

By continuing this process, we get that for any K there are $n_k > m_k > k$ such that $p(\xi_{n_k}, \xi_{m_k}) \geq \varepsilon$. Further, corresponding to m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ such that $p(\xi_{n_k-1}, \xi_{m_k}) < \varepsilon$.

□

Theorem 2.11. *Let $(\mho, d)$ represent a complete b-metric space, with p denoting an ω_t-distance on $\mho$, and let $f : \mho \to \mho$ be characterized as a (φ,α)-Geraghty mapping. We consider one of the following conditions:*

1. *If $u \neq fu$, then the infimum $\inf\{p(\xi, u) + p(f\xi, u) : \xi \in \mho\}$ is greater than 0.*
2. *The function f is continuous.*

Under these assumptions, it can be concluded that f possesses a unique fixed point.

Proof. Let $\xi_0 \in \mho$. We define a sequence $\xi_n = f(\xi_{n-1})$ for $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, based on the contractive condition, it follows that

$$\varphi sp(\xi_n, \xi_{n+1}) = \varphi sp(f(\xi_{n-1}), f(\xi_n)) \leq \alpha(p(\xi_{n-1}, \xi_n))\varphi p(\xi_{n-1}, \xi_n).$$

Given that $\alpha(t) < 1$ for all $t > 0$, we can conclude that $\varphi sp(\xi_n, \xi_{n+1}) < \varphi p(\xi_{n-1}, \xi_n)$. Since φ is a nondecreasing function, this leads us to the desired result.

$$p(\xi_n, \xi_{n+1}) < \frac{1}{s}p(\xi_{n-1}, \xi_n).$$

By induction, we have

$$p(\xi_n, \xi_{n+1}) < \left(\frac{1}{s}\right)^n p(\xi_0, \xi_1).$$

So,

$$\lim_{n\to\infty} p(\xi_n, \xi_{n+1}) = 0. \tag{7}$$

Similarly, we can show that

$$\lim_{n\to\infty} p(\xi_{n+1}, \xi_n) = 0. \tag{8}$$

We aim to demonstrate that:

$$\lim_{n,m\to+\infty} p(\xi_n, \xi_m) = 0,$$

which indicates that the sequence $(\mho_n)$ is a Cauchy sequence.

To explore this, we will assume the opposite, specifically that:

$$\lim_{n,m\to+\infty} p(\xi_n, \xi_m) \neq 0.$$

Consequently, there exists an $\epsilon > 0$ along with two subsequences (ξ_{n_k}) and (ξ_{m_k}) derived from (ξ_n), where (m_k) is selected as the smallest index satisfying the condition.

$$p(\xi_{n_k}, \xi_{m_k}) \geq \epsilon, \quad m_k > n_k > k. \tag{9}$$

This implies that

$$p(\xi_{n_k}, \xi_{m_k-1}) < \epsilon. \tag{10}$$

Set $\delta_k = p(\xi_{n_k-1}, \xi_{m_k})$. By Remark 2.8, Equations 9 and 10, and (a) of the definition 1.3, we get

$$\begin{aligned} \epsilon &\leq p(\xi_{n_k}, \xi_{m_k}) \\ &< \tfrac{1}{s} p(\xi_{n_k-1}, \xi_{m_k-1}) \\ &\leq [p(\xi_{n_k-1}, \xi_{m_k}) + p(\xi_{m_k}, \xi_{m_k-1})]. \end{aligned}$$

By considering the limit inferior as k approaches $+\infty$ and referencing Equation 8, we obtain

$$\epsilon \leq \liminf_{k\to+\infty} \delta_k. \tag{11}$$

In addition,

$$\begin{aligned} p(\xi_{n_k-1}, \xi_{m_k}) &< \tfrac{1}{s} p(\xi_{n_k-2}, \xi_{m_k-1}) \\ &\leq [p(\xi_{n_k-2}, \xi_{n_k}) + p(\xi_{n_k}, \xi_{m_k-1})] \\ &< s[p(\xi_{n_k-2}, \xi_{n_k-1}) + p(\xi_{n_k-1}, \xi_{n_k})] + \epsilon. \end{aligned}$$

By considering the limit inferior as k approaches $+\infty$ and referencing Equation 8, we obtain

$$\limsup_{k\to+\infty} \delta_k \leq \epsilon. \tag{12}$$

By Equations 11 and 12, we get

$$\lim_{k\to+\infty} \delta_k = \epsilon. \tag{13}$$

Now, set $\gamma_k = p(\xi_{n_k}, \xi_{m_k+1})$. By Remark 2.8, we get

$$p(\xi_{n_k}, \xi_{m_k+1}) \leq \frac{1}{s} p(\xi_{n_k-1}, \xi_{m_k}).$$

By applying the limit superior to both sides, we obtain

$$\limsup_{k\to+\infty} \gamma_k \leq \frac{\epsilon}{s}. \tag{14}$$

On the other hand, we have:

$$\epsilon \leq p(\xi_{n_k}, \xi_{m_k}) \leq s[p(\xi_{n_k}, \xi_{m_k+1}) + p(\xi_{m_k+1}, \xi_{m_k})].$$

By taking the limit inferior on both sides, we get:

$$\frac{\epsilon}{s} \leq \liminf_{k\to+\infty} \gamma_k. \tag{15}$$

By Equations 14 and 15, we get

$$\lim_{k\to+\infty} \gamma_k = \frac{\epsilon}{s}. \tag{16}$$

By substituting $\xi = \xi_{n_k-1},\ \zeta = \xi_{m_k}$ in Condition 6, we have

$$\varphi sp(f\xi_{n_k-1}, f\xi_{m_k}) \leq \alpha(p(\xi_{n_k-1}, \xi_{m_k}))\varphi p(\xi_{n_k-1}, \xi_{m_k}).$$

So,

$$\frac{\varphi sp(\xi_{n_k}, \xi_{m_k+1})}{\varphi p(\xi_{n_k-1}, \xi_{m_k})} \leq \alpha(p(\xi_{n_k-1}, \xi_{m_k})).$$

By taking the limit as $k \to \infty$, we get

$$\frac{\varphi(s\frac{\epsilon}{s})}{\varphi(\epsilon)} \leq \lim_{k\to\infty} \alpha(p(\xi_{n_k-1}, \xi_{m_k})).$$

So, we have $\lim_{k\to\infty} \alpha(p(\xi_{n_k-1}, \xi_{m_k})) = 1$, hence, $\lim_{k\to\infty} p(\xi_{n_k-1}, \xi_{m_k}) = 0$ a contradiction since $\epsilon > 0$. Hence $(\mho_n)$ is a Cauchy sequence. Since $(\mho, d)$ is a complete b-metric space. Thus, there is $u \in \mho$ such that $(\xi_n) \to u$.

Given that

$$\lim_{n,m\to\infty} p(\xi_n, \xi_m) = 0,$$

then for a specified $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $p(\xi_n, \xi_m) \leq \epsilon$ for all $n, m \geq k$. By virtue of the s-lower semi-continuity of p, it follows that

$$p(\xi_n, u) \leq \lim_{m\to\infty} \inf sp(\xi_n, x_m) \leq s\epsilon, \quad \forall n \geq k.$$

Next, let us assume that condition (1) is satisfied. If $u \neq fu$, then we have

$$\inf\{p(\xi, u) + p(f\xi, u) : \xi \in \mho\} \leq \inf\{p(\xi_n, u) + p(f\xi_n, u) : n \in \mathbb{N}\}$$

$$= \inf\{p(\xi_n, u) + p(\xi_{n+1}, u) : n \in \mathbb{N}\} \leq s\epsilon,$$

for all $\epsilon > 0$, which leads to a contradiction.
Hence, $fu = u$. If (2) holds, the mapping f is continuous, and it is obvious that $fu = u$.

To establish the uniqueness, let us assume the existence of an element v such that $v = fv$. According to Condition 6 and the characteristics of φ, we can derive the following

$$\varphi p(u, v) \leq \varphi sp(u, v) = \varphi sp(fu, fv) \leq \alpha(p(u, v))\varphi(p(u, v)).$$

So, $(1 - \alpha(p(u, v)))p(u, v) \leq 0$, and hence, $p(u, v) = 0$. In a similar manner, we can show that $p(u, u) = 0$, Thus, $d(u, v) = 0$, which implies that $u = v$.

□

Example 2.12. Let $\mho = [0, 1]$ with the b-metric space defined by $d(\xi, \zeta) = (\xi - \zeta)^2$, and consider the ω_t distance $p(\xi, \zeta) = |\zeta|^2$. Define a function $f : \mho \to \mho$ by $f(\xi) = \frac{1}{2}\xi$. Assume the altering distance function $\varphi(t) = t$ and $\alpha(t) = \frac{2}{3}$. Then we have

$$p(f\xi, f\zeta) = |f\zeta|^2 = |\frac{1}{2}\zeta|^2 = \frac{1}{4}\zeta^2,$$
$$sp(f\xi, f\zeta) = \frac{1}{2}|\zeta|^2,$$

$$\varphi(sp(f\xi, f\zeta)) = \frac{1}{2}\zeta^2,$$

$$p(\xi, \zeta)) = |\zeta|^2,$$

$$\varphi p(\xi, \zeta)) = |\zeta|^2,$$

$$\alpha(p(\xi, \zeta)) = \frac{2}{3},$$

$$\alpha(p(\xi, \zeta))\varphi p(\xi, \zeta)) = \frac{2}{3}\zeta^2.$$

So,

$$\varphi sp(f\xi, f\zeta) \leq \alpha(p(\xi, \zeta))\varphi p(\xi, \zeta)$$

Thus, all conditions of Theorem 2.11 hold true, and so, f has a unique fixed point.

Conclusion

The concept of fixed points is fundamental in both pure and applied mathematics, with numerous applications in various contexts. Following Banach's results in metric spaces, many researchers have expanded upon the Banach contraction principle in diverse ways. In our study, we established several fixed point results within the framework of the ωt distance and presented various illustrative examples. Future research could focus on generalizing our contraction results or exploring outcomes in broader distance spaces. Also, we aim to incorporate our research with other disciplines, particularly fuzzy set theory, as demonstrated in the studies conducted by.[20–25]

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