Neutrosophic $\mathcal{N}$-structures on Sheffer stroke UP-algebras

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Abstract

The study defines a neutrosophic $\mathcal{N}$-subalgebra and a level set of a neutrosophic $\mathcal{N}$-structure on Sheffer stroke UP-algebras. It appears that these concepts are integral to understanding the behavior of neutrosophic logic within the framework of Sheffer stroke UP-algebras. The study establishes a relationship between subalgebras and level sets on Sheffer stroke UP-algebras. Specifically, it proves that the level set of neutrosophic $\mathcal{N}$-subalgebras on this algebra is its subalgebra, and vice versa. This indicates a tight connection between these concepts within the given algebraic structure. It is stated that the family of all neutrosophic $\mathcal{N}$-subalgebras of a Sheffer stroke UP-algebra forms a complete distributive lattice. This suggests that there is a well-defined structure and order among these subalgebras, allowing for systematic analysis. The study describes a neutrosophic $\mathcal{N}$-ideal of a Sheffer stroke UP-algebra and provides some of its properties. Additionally, it is shown that every neutrosophic $\mathcal{N}$-ideal of a Sheffer stroke UP-algebra is also its neutrosophic $\mathcal{N}$-subalgebra, though the inverse is generally not true. This highlights the specific characteristics and behavior of neutrosophic $\mathcal{N}$-ideals within the given algebraic context.

Keywords: Sheffer stroke UP-algebra; Subalgebra; Ideal; Neutrosophic $\mathcal{N}$-subalgebra; Neutrosophic $\mathcal{N}$-ideal

1 Introduction

Zadeh[11] pioneered the concept of the degree of membership/truth (t) with the introduction of fuzzy set theory, which has since become a foundational framework for handling uncertainty in data. Building on this idea, Atanassov[1] extended fuzzy sets by incorporating the degree of nonmembership/falsehood (f), thereby formulating the intuitionistic fuzzy set, which captures both membership and nonmembership as independent yet complementary components.

In 1995, Smarandache proposed the term "neutrosophic", derived from a combination of etymological roots: "neuter" (French and Latin, meaning neutral) and "sophia" (Greek, meaning wisdom or knowledge). The term encapsulates the knowledge of neutral thought, marking a significant departure from the dualistic framework of fuzzy and intuitionistic fuzzy sets. The core innovation of neutrosophic logic is the addition of a third independent component—the degree of indeterminacy/neutrality (i)—alongside truth (t) and falsehood (f). This extension introduces the ability to explicitly represent the neutral, indeterminate, or unknown aspects of information, which are often overlooked in classical and fuzzy logic. Smarandache formalized this framework

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in 1995, and it was subsequently published in 1998, defining neutrosophic sets as tripartite structures with components (t, i, f) representing truth, indeterminacy, and falsehood, respectively.

The neutrosophic set, as developed by Smarandache,[9] provides a highly generalized platform that subsumes and extends classical sets, fuzzy sets, intuitionistic fuzzy sets, and interval-valued intuitionistic fuzzy sets. This versatility has made neutrosophic set theory applicable across diverse fields, including artificial intelligence, decision-making, and data analysis. For a comprehensive list of applications and ongoing developments in neutrosophic set theory, readers may refer to Smarandache's repository at [Neutrosophy Website](http://fs.gallup.unm.edu/neutrosophy.htm).

The Sheffer stroke operation, also known as the NAND operator, introduced by Sheffer[8] in 1913, is a pivotal concept in logical algebra due to its remarkable property of completeness. This operation alone suffices to construct an entire logical system without relying on other logical connectives. By restating any logical system's axioms solely using the Sheffer operation, this construct offers a streamlined approach to analyzing and controlling the properties of logical frameworks. Furthermore, Boolean algebra, the algebraic counterpart of classical propositional calculus, can be entirely expressed using the Sheffer stroke, emphasizing its foundational importance and versatility in both logic and algebra.

Building on this foundation, Sheffer stroke UP-algebras introduce a unique algebraic structure that merges logical and algebraic principles. As detailed by Iampan[2] in 2017, UP-algebras represent a new branch of logical algebra, providing a versatile framework for studying algebraic systems with advanced logical constructs. This innovation facilitates a deeper understanding of logical operations and their algebraic counterparts, paving the way for applications in fields such as decision theory, computer science, and artificial intelligence.

In this paper, we expand the scope of Sheffer stroke UP-algebras by introducing neutrosophic $\mathcal{N}$-structures. These structures incorporate neutrosophic logic, which is characterized by three membership functions—truth, indeterminacy, and falsity—offering a flexible approach to modeling uncertainty in logical systems. The study establishes key relationships between neutrosophic $\mathcal{N}$-subalgebras and level sets, demonstrating that the level set of neutrosophic $\mathcal{N}$-subalgebras is itself a subalgebra and vice versa. This interplay underscores the tight connection between these constructs within the algebraic structure.

Additionally, it is shown that the family of all neutrosophic $\mathcal{N}$-subalgebras forms a complete distributive lattice, enabling systematic exploration of their relationships. The research further examines neutrosophic $\mathcal{N}$-ideals, proving that while every neutrosophic $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-subalgebra, the converse does not hold. This distinction highlights the specific properties and behaviors of neutrosophic $\mathcal{N}$-ideals within Sheffer stroke UP-algebras, contributing to the advancement of algebraic systems designed to address logical uncertainty.

2 Preliminaries

Sheffer stroke UP-algebras embody a fascinating synergy between algebra and logic, rooted in the fundamental Sheffer stroke operation—a cornerstone of propositional calculus. This unique algebraic framework not only deepens our comprehension of logical operations but also paves the way for transformative applications in fields such as computer science, decision theory, and artificial intelligence. By exploring the core definitions and foundational principles of Sheffer stroke UP-algebras, this article underscores their pivotal role within the broader landscape of algebraic theory and their potential for advancing modern computational and analytical methodologies.

Definition 2.1. [8] Let $H = \langle H, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions: for all $x, y, z \in H$,

(S1) $x|y = y|x$
(S2) $(x|x)|(x|y) = x$
(S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$
(S4) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.$

Definition 2.2. [6] A Sheffer stroke UP-algebra (briefly, SUP-algebra) is a structure $\langle H, |, 0\rangle$ of type $(2, 0)$ such that 0 is the fixed element in H and the following conditions are satisfied for all $x, y, z \in H$,

(SUP-1) $(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|$
$(z|(y|y)))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|$
$((y|(x|x))|(z|(y|y))))) = 0$
(SUP-2) $x|x = x|(0|0)$
(SUP-3) $(x|(y|y))|(x|(y|y)) = 0$ and $(y|(x|x))|(y|(x|x)) = 0 \Rightarrow x = y$.

Proposition 2.3. [6] *Let $H = \langle H, |, 0\rangle$ be an SUP-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on H.*

Lemma 2.4. [6] *Let $H = \langle H, |, 0\rangle$ be an SUP-algebra. Then for all $x, y, z \in H$, we have*

(1) $x \leq y \Rightarrow y|(z|z) \leq x|(z|z)$ *and* $z|(x|x) \leq z|(y|y)$

(2) $x \leq y \Leftrightarrow y|y \leq x|x$

(3) $y|(x|x) \leq x$

(4) $y \leq (y|(x|x))|(y|(x|x))$

(5) $x \leq y \Rightarrow x \leq (y|(z|z))|(y|(z|z))$

(6) $z|(y|y) \leq z|(y|(x|x))$

(7) $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$

(8) $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$.

Definition 2.5. [6] A nonempty subset G of an SUP-algebra $H = \langle H, |, 0\rangle$ is called a subalgebra of H if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.

Definition 2.6. [5,6] A nonempty subset G of an SUP-algebra $H = \langle H, |, 0\rangle$ is called an ideal of H if for all $x, y \in H$,

(1) $y \in G \Rightarrow (y|(x|x))|(y|(x|x)) \in G$

(2) $(y|(x|x))|(y|(x|x)) \in G$ and $x \in G \Rightarrow y \in G$.

Definition 2.7. [10] A neutrosophic set in a nonempty set H is defined to be a structure

$$A := \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in H\}, \tag{2.1}$$

where $T_A : H \to [0, 1]$ is a truth membership function, $I_A : H \to [0, 1]$ is an indeterminate membership function, and $F_A : H \to [0, 1]$ is a false membership function. The neutrosophic set in (2.1) is simply denoted by $A = (H, T_A, I_A, F_A)$.

Definition 2.8. [4] We denote the family of all functions from a nonempty set H to the closed interval $[-1, 0]$ of the real line by $\mathcal{F}(H, [-1, 0])$. An element of $\mathcal{F}(H, [-1, 0])$ is called a *negative-valued function* from H to $[-1, 0]$ (briefly, $\mathcal{N}$-function on H). An ordered pair of a nonempty set H and an $\mathcal{N}$-function on H is called an *$\mathcal{N}$-fuzzy structure*. A neutrosophic $\mathcal{N}$-structure $H_{\mathcal{N}}$ over a nonempty universe of discourse H is defined to be the structure $(H, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, where $T_{\mathcal{N}}, I_{\mathcal{N}}$, and $F_{\mathcal{N}}$ are $\mathcal{N}$-functions on H which are called the negative truth membership function, the negative indeterminacy membership function, and the negative falsity membership function on H, respectively.

For the sake of simplicity, we will use the notation $H_{\mathcal{N}}$ instead of the neutrosophic $\mathcal{N}$-structure $(H, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$.[3]

Definition 2.9. [7] Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure over a nonempty set H. The neutrosophic $\mathcal{N}$-structure $\overline{H_{\mathcal{N}}} = (H, \overline{T_{\mathcal{N}}}, \overline{I_{\mathcal{N}}}, \overline{F_{\mathcal{N}}})$ defined by

$$(\forall x \in H) \left(\begin{array}{c} \overline{T_{\mathcal{N}}}(x) = -1 - T_{\mathcal{N}}(x) \\ \overline{I_{\mathcal{N}}}(x) = -1 - I_{\mathcal{N}}(x) \\ \overline{F_{\mathcal{N}}}(x) = -1 - F_{\mathcal{N}}(x) \end{array} \right) \tag{2.2}$$

is called the *complement* of $H_{\mathcal{N}}$ in H.

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3 Neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals

In this section, the study introduces the concepts of neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals within the context of Sheffer stroke UP-algebras. It's worth noting that unless explicitly stated otherwise, $H = \langle H, |, 0\rangle$ refers to a Sheffer stroke UP-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}$-structure $H_{\mathcal{N}}$ on H is called a neutrosophic $\mathcal{N}$-subalgebra of H if

$$(\forall x, y \in H) \begin{pmatrix} T_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \\ I_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \\ F_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \end{pmatrix}. \tag{3.1}$$

Example 3.2. Consider a Sheffer stroke UP-algebra $H = \langle H, |, 0\rangle$, where the set $H = \{0, a, b, c\}$ and the Sheffer operation $|$ on H has the Cayley table as below:

$\vert$	0	a	b	c
0	a	a	a	a
a	a	0	c	b
b	a	c	c	a
c	a	b	b	b

A neutrosophic $\mathcal{N}$-structure $H_{\mathcal{N}}$ on H is given as follows:

H	$T_{\mathcal{N}}$	$I_{\mathcal{N}}$	$F_{\mathcal{N}}$
0	-0.5	-0.1	-0.4
a	-0.2	-0.3	-0.1
b	-0.3	-0.3	-0.2
c	-0.3	-0.9	-0.2

Hence, $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of H.

Definition 3.3. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure on H and $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha+\beta+\gamma \leq 0$. For the sets $T^{\alpha}_{\mathcal{N}} = \{x \in H : T_{\mathcal{N}}(x) \leq \alpha\}$, $I^{\beta}_{\mathcal{N}} = \{x \in H : I_{\mathcal{N}}(x) \geq \beta\}$, and $F^{\gamma}_{\mathcal{N}} = \{x \in H : F_{\mathcal{N}}(x) \leq \gamma\}$. The set $H_{\mathcal{N}}(\alpha, \beta, \gamma) = \{x \in H : T_{\mathcal{N}}(x) \leq \alpha, I_{\mathcal{N}}(x) \geq \beta, F_{\mathcal{N}}(x) \leq \gamma\}$ is called an (α, β, γ)-level set of $H_{\mathcal{N}}$. Also, $H_{\mathcal{N}}(\alpha, \beta, \gamma) = T^{\alpha}_{\mathcal{N}} \cap I^{\alpha}_{\mathcal{N}} \cap F^{\alpha}_{\mathcal{N}}$.

Theorem 3.4. *Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure on H and $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of H, then the nonempty $H_{\mathcal{N}}(\alpha, \beta, \gamma)$-level set of $H_{\mathcal{N}}$ is a subalgebra of H.*

Proof. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of H and $x, y \in H_{\mathcal{N}}(\alpha, \beta, \gamma)$. Then $T_{\mathcal{N}}(x) \leq \alpha, I_{\mathcal{N}}(x) \geq \beta, F_{\mathcal{N}}(x) \leq \gamma$ and $T_{\mathcal{N}}(y) \leq \alpha, I_{\mathcal{N}}(y) \geq \beta, F_{\mathcal{N}}(y) \leq \gamma$. Thus,

$$T_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \leq \alpha,$$

$$I_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \geq \beta,$$

$$F_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \leq \gamma.$$

So, $(x|(y|y))|(x|(y|y)) \in H_{\mathcal{N}}(\alpha, \beta, \gamma)$, which means that $H_{\mathcal{N}}(\alpha, \beta, \gamma)$ is a subalgebra of H. □

Theorem 3.5. *Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure on H and $T^{\alpha}_{\mathcal{N}}$, $I^{\beta}_{\mathcal{N}}$, and $F^{\gamma}_{\mathcal{N}}$ be subalgebras of H for all $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $H_{\mathcal{N}}$ is a is a neutrosophic $\mathcal{N}$-subalgebra of H.*

Proof. Let $T^{\alpha}_{\mathcal{N}}$, $I^{\beta}_{\mathcal{N}}$, and $F^{\gamma}_{\mathcal{N}}$ be subalgebras of H for all $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Suppose that $x, y \in H$ such that $a = T_{\mathcal{N}}((x|(y|y))|(x|(y|y))) > \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} = b$. Then $b < \alpha_1 < a$ where $\alpha_1 = \dfrac{a+b}{2} \in [-1, 0)$. Thus, $x, y \in T^{\alpha_1}_{\mathcal{N}}$ but $(x|(y|y))|(x|(y|y)) \notin T^{\alpha_1}_{\mathcal{N}}$, which is a contradiction. Hence, $T_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$ for all $x, y \in H$. Suppose that $x, y \in H$ such that $u = I_{\mathcal{N}}((x|(y|y))|(x|(y|y))) < \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} = v$. Then $u < \beta_1 < v$

where $\beta_1 = \dfrac{u+v}{2} \in [-1, 0)$. Hence, $x, y \in I_{\mathcal{N}}^{\beta_1}$ but $(x|(y|y))|(x|(y|y)) \notin I_{\mathcal{N}}^{\beta_1}$, which is a contradiction. Therefore, $I_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$ for all $x, y \in H$. Suppose that $x, y \in H$ such that $m = F_{\mathcal{N}}((x|(y|y))|(x|(y|y))) > \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} = n$. Then $n < \gamma_1 < m$ where $\gamma_1 = \dfrac{m+n}{2} \in [-1, 0)$. Hence, $x, y \in F_{\mathcal{N}}^{\gamma_1}$ but $(x|(y|y))|(x|(y|y)) \notin F_{\mathcal{N}}^{\gamma_1}$, which is a contradiction. Therefore, $F_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$ for all $x, y \in H$. Thereby, $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of H. □

Theorem 3.6. *Let $\{H_{\mathcal{N}_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}$-subalgebras of H. Then $\{H_{\mathcal{N}_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.*

Proof. Let G be a nonempty subset of $\{H_{\mathcal{N}_i} : i \in \mathbb{N}\}$. Since $H_{\mathcal{N}_i}$ is a neutrosophic $\mathcal{N}$-subalgebras of H for all $H_{\mathcal{N}_i} \in G$, it satisfies

$$T_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\},$$
$$I_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\},$$
$$F_{\mathcal{N}}((x|(y|y))|(x|(y|y))) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$$

for all $x, y \in H$. Then $\bigcap G$ satisfies these inequalities, which means that $\bigcap G$ is a neutrosophic $\mathcal{N}$-subalgebra of H. Let P be a family of all neutrosophic $\mathcal{N}$-subalgebras of H containing $\bigcup\{H_{\mathcal{N}_i} : i \in \mathbb{N}\}$. Then $\bigcap P$ is a neutrosophic $\mathcal{N}$-subalgebra of H. If $\bigwedge_{i\in\mathbb{N}} H_{\mathcal{N}_i} = \bigcap_{i\in\mathbb{N}} H_{\mathcal{N}_i}$ and $\bigvee_{i\in\mathbb{N}} H_{\mathcal{N}_i} = \bigcap P$, then $(\{H_{\mathcal{N}_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ is a complete lattice. Also, it is distributive by the definitions of $\bigvee$ and $\bigwedge$. □

Theorem 3.7. *If a neutrosophic $\mathcal{N}$-structure $H_{\mathcal{N}}$ on H is a neutrosophic $\mathcal{N}$-subalgebra of H, then $T_{\mathcal{N}}(0) \leq T_{\mathcal{N}}(x)$, $I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(x)$, and $F_{\mathcal{N}}(0) \leq F_{\mathcal{N}}(x)$ for all $x \in H$.*

Proof. For any $x \in H$,

$$T_{\mathcal{N}}(0) = T_{\mathcal{N}}((x|(x|x))|(x|(x|x))) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(x),$$
$$I_{\mathcal{N}}(0) = I_{\mathcal{N}}((x|(x|x))|(x|(x|x))) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(x)\} = I_{\mathcal{N}}(x),$$
$$F_{\mathcal{N}}(0) = F_{\mathcal{N}}((x|(x|x))|(x|(x|x))) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(x)\} = F_{\mathcal{N}}(x).$$

□

Theorem 3.8. *Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of H. If there exists a sequence $\{a_{\mathcal{N}}\}$ in H such that $\lim_{n\to\infty} T_{\mathcal{N}}(a_{\mathcal{N}}) = -1 = \lim_{n\to\infty} F_{\mathcal{N}}(a_{\mathcal{N}}) = -1$ and $\lim_{n\to\infty} I_{\mathcal{N}}(a_{\mathcal{N}}) = 0$, then $T_{\mathcal{N}}(0) = -1 = F_{\mathcal{N}}(0)$ and $I_{\mathcal{N}}(0) = 0$.*

Proof. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of H. Assume that there exists a sequence $\{a_{\mathcal{N}}\}$ in H such that $\lim_{n\to\infty} T_{\mathcal{N}}(a_{\mathcal{N}}) = -1 = \lim_{n\to\infty} F_{\mathcal{N}}(a_{\mathcal{N}}) = -1$ and $\lim_{n\to\infty} I_{\mathcal{N}}(a_{\mathcal{N}}) = 0$. Since $T_{\mathcal{N}}(0) \leq T_{\mathcal{N}}(a_{\mathcal{N}})$, $I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(a_{\mathcal{N}})$, and $F_{\mathcal{N}}(0) \leq F_{\mathcal{N}}(a_{\mathcal{N}})$ for every $n \in \mathbb{N}$, we have

$$-1 = \lim_{n\to\infty} -1 \leq \lim_{n\to\infty} T_{\mathcal{N}}(0) = T_{\mathcal{N}}(0) \leq \lim_{n\to\infty} T_{\mathcal{N}}(a_{\mathcal{N}}) = -1,$$
$$0 = \lim_{n\to\infty} 0 \geq \lim_{n\to\infty} I_{\mathcal{N}}(0) = I_{\mathcal{N}}(0) \geq \lim_{n\to\infty} I_{\mathcal{N}}(a_{\mathcal{N}}) = 0,$$
$$-1 = \lim_{n\to\infty} -1 \leq \lim_{n\to\infty} F_{\mathcal{N}}(0) = F_{\mathcal{N}}(0) \leq \lim_{n\to\infty} F_{\mathcal{N}}(a_{\mathcal{N}}) = -1.$$

Hence, $T_{\mathcal{N}}(0) = -1 = F_{\mathcal{N}}(0)$ and $I_{\mathcal{N}}(0) = 0$. □

Definition 3.9. A neutrosophic $\mathcal{N}$-structure $H_{\mathcal{N}}$ on H is called a neutrosophic $\mathcal{N}$-ideal of H if

$$(\forall x, y \in H) \begin{pmatrix} T_{\mathcal{N}}((y|(x|x))|(y|(x|x))) \leq T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}((y|(x|x))|(y|(x|x))), T_{\mathcal{N}}(x)\} \\ I_{\mathcal{N}}((y|(x|x))|(y|(x|x))) \geq I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}((y|(x|x))|(y|(x|x))), I_{\mathcal{N}}(x)\} \\ F_{\mathcal{N}}((y|(x|x))|(y|(x|x))) \leq F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}((y|(x|x))|(y|(x|x))), F_{\mathcal{N}}(x)\} \end{pmatrix}. \quad (3.2)$$

Theorem 3.10. *Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure on H and $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha+\beta+\gamma \leq 0$. If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of H, then the nonempty $H_{\mathcal{N}}(\alpha, \beta, \gamma)$-level set of $H_{\mathcal{N}}$ is an ideal of H.*

Proof. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of H and $x,(y|(x|x))|(y|(x|x)) \in H_{\mathcal{N}}(\alpha,\beta,\gamma)$. Then $T_{\mathcal{N}}(x) \leq \alpha$, $I_{\mathcal{N}}(x) \geq \beta$, $F_{\mathcal{N}}(x) \leq \gamma$ and $T_{\mathcal{N}}((y|(x|x))|(y|(x|x))) \leq \alpha$, $I_{\mathcal{N}}((y|(x|x))|(y|(x|x))) \geq \beta$, $F_{\mathcal{N}}((y|(x|x))|(y|(x|x))) \leq \gamma$. Thus,

$$T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\} \leq \alpha,$$
$$I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\} \geq \beta,$$
$$F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\} \leq \gamma$$

for all $x, y \in H$. So, $y \in H_{\mathcal{N}}(\alpha,\beta,\gamma)$, which means that $H_{\mathcal{N}}(\alpha,\beta,\gamma)$ is an ideal of H. □

Theorem 3.11. *Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure on H and $T_{\mathcal{N}}^{\alpha}$, $I_{\mathcal{N}}^{\beta}$, and $F_{\mathcal{N}}^{\gamma}$ be ideals of H for all $\alpha,\beta,\gamma \in [-1,0]$ such that $-3 \leq \alpha+\beta+\gamma \leq 0$. Then $H_{\mathcal{N}}$ is a is a neutrosophic $\mathcal{N}$-ideal of H.*

Proof. Let $T_{\mathcal{N}}^{\alpha}$, $I_{\mathcal{N}}^{\beta}$, and $F_{\mathcal{N}}^{\gamma}$ be subalgebras of H for all $\alpha,\beta,\gamma \in [-1,0]$ such that $-3 \leq \alpha+\beta+\gamma \leq 0$. Suppose that $x,(y|(x|x))|(y|(x|x)) \in H$ such that $a = T_{\mathcal{N}}(y) > \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\} = b$. Then $b < \alpha_1 < a$ where $\alpha_1 = \dfrac{a+b}{2} \in [-1,0)$. Thus, $x,(y|(x|x))|(y|(x|x)) \in T_{\mathcal{N}}^{\alpha_1}$ but $y \notin T_{\mathcal{N}}^{\alpha_1}$, a contradiction. Hence, $T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\}$ for all $x,y \in H$. Suppose that $x,(y|(x|x))|(y|(x|x)) \in H$ such that $u = I_{\mathcal{N}}(y) < \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\} = v$. Then $u < \beta_1 < v$ where $\beta_1 = \dfrac{u+v}{2} \in [-1,0)$. Hence, $x,(y|(x|x))|(y|(x|x)) \in I_{\mathcal{N}}^{\beta_1}$ but $y \notin I_{\mathcal{N}}^{\beta_1}$, a contradiction. Then $I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\}$ for all $x,y \in H$. Suppose that $x,(y|(x|x))|(y|(x|x)) \in H$ such that $m = F_{\mathcal{N}}(y) > \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\} = n$. Then $n < \gamma_1 < m$ where $\gamma_1 = \dfrac{m+n}{2} \in [-1,0)$. Hence, $x,(y|(x|x))|(y|(x|x)) \in F_{\mathcal{N}}^{\gamma_1}$ but $y \notin F_{\mathcal{N}}^{\gamma_1}$, a contradiction. Then $F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}((y|(x|x))|(y|(x|x)))\}$ for all $x,y \in H$. Hence, $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of H. □

We immediately obtain the following theorem from the definition of a neutrosophic $\mathcal{N}$-ideal.

Theorem 3.12. *Every neutrosophic $\mathcal{N}$-ideal of H is a neutrosophic $\mathcal{N}$-subalgebra of H.*

Lemma 3.13. *If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of H, then*

$$(\forall x,y \in H)\left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}(y) \leq T_{\mathcal{N}}(x) \\ I_{\mathcal{N}}(y) \geq I_{\mathcal{N}}(x) \\ F_{\mathcal{N}}(y) \leq F_{\mathcal{N}}(x) \end{cases} \right). \tag{3.3}$$

Proof. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-ideal of H and $x \leq y$. Then

$$T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(0)\} = T_{\mathcal{N}}(x),$$
$$I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(0)\} = I_{\mathcal{N}}(0),$$
$$F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(0)\} = F_{\mathcal{N}}(x).$$

□

Theorem 3.14. *If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of H, then*

$$(\forall x,y,z \in H)\left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}(x|(z|z)) \leq T_{\mathcal{N}}(y|(z|z)) \\ I_{\mathcal{N}}(x|(z|z)) \geq I_{\mathcal{N}}(y|(z|z)) \\ F_{\mathcal{N}}(x|(z|z)) \leq F_{\mathcal{N}}(y|(z|z)) \end{cases} \right), \tag{3.4}$$

$$(\forall x,y,z \in H)\left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}(z|(y|y)) \leq T_{\mathcal{N}}(z|(x|x)) \\ I_{\mathcal{N}}(z|(y|y)) \geq I_{\mathcal{N}}(z|(x|x)) \\ F_{\mathcal{N}}(z|(y|y)) \leq F_{\mathcal{N}}(z|(x|x)) \end{cases} \right), \tag{3.5}$$

$$(\forall x,y \in H)\left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}(x|x) \leq T_{\mathcal{N}}(y|y) \\ I_{\mathcal{N}}(x|x) \geq I_{\mathcal{N}}(y|y) \\ F_{\mathcal{N}}(x|x) \leq F_{\mathcal{N}}(y|y) \end{cases} \right), \tag{3.6}$$

$$(\forall x,y,z \in H)\left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}((y|(z|z))|(y|(z|z))) \leq T_{\mathcal{N}}(x) \\ I_{\mathcal{N}}((y|(z|z))|(y|(z|z))) \geq I_{\mathcal{N}}(x) \\ F_{\mathcal{N}}((y|(z|z))|(y|(z|z))) \leq F_{\mathcal{N}}(x) \end{cases} \right). \tag{3.7}$$

Proof. Follows from Lemmas 2.4 and 3.13. □

Theorem 3.15. *If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of H, then*

$$(\forall x, y \in H)\left(\begin{array}{c} T_{\mathcal{N}}(x) \leq T_{\mathcal{N}}(y|(x|x)) \\ I_{\mathcal{N}}(x) \geq I_{\mathcal{N}}(y|(x|x)) \\ F_{\mathcal{N}}(x) \leq F_{\mathcal{N}}(y|(x|x)) \end{array}\right), \tag{3.8}$$

$$(\forall x, y \in H)\left(\begin{array}{c} T_{\mathcal{N}}((y|(x|x))|(y|(x|x)) \leq T_{\mathcal{N}}(x) \\ I_{\mathcal{N}}((y|(x|x))|(y|(x|x)) \geq I_{\mathcal{N}}(x) \\ F_{\mathcal{N}}((y|(x|x))|(y|(x|x)) \leq F_{\mathcal{N}}(x) \end{array}\right), \tag{3.9}$$

$$(\forall x, y, z \in H)\left(\begin{array}{c} T_{\mathcal{N}}(z|(y|(x|x))) \leq T_{\mathcal{N}}(z|(y|y)) \\ I_{\mathcal{N}}(z|(y|(x|x))) \geq I_{\mathcal{N}}(z|(y|y)) \\ F_{\mathcal{N}}(z|(y|(x|x))) \leq F_{\mathcal{N}}(z|(y|y)) \end{array}\right), \tag{3.10}$$

$$(\forall x, y, z \in H)\left(\begin{array}{c} T_{\mathcal{N}}(z|(y|(x|x))) \leq T_{\mathcal{N}}((z|(y|y)|(z|(y|y)))|(x|x)) \\ I_{\mathcal{N}}(z|(y|(x|x))) \geq I_{\mathcal{N}}((z|(y|y)|(z|(y|y)))|(x|x)) \\ F_{\mathcal{N}}(z|(y|(x|x))) \leq F_{\mathcal{N}}((z|(y|y)|(z|(y|y)))|(x|x)) \end{array}\right), \tag{3.11}$$

$$(\forall x, y, z \in H)\left(\begin{array}{c} T_{\mathcal{N}}((x|(y|y))|((x|(z|z))|(x|(z|z)))) \leq T_{\mathcal{N}}(x|((y|(z|z))|(y|(z|z)))) \\ I_{\mathcal{N}}((x|(y|y))|((x|(z|z))|(x|(z|z)))) \geq I_{\mathcal{N}}(x|((y|(z|z))|(y|(z|z)))) \\ F_{\mathcal{N}}((x|(y|y))|((x|(z|z))|(x|(z|z)))) \leq F_{\mathcal{N}}(x|((y|(z|z))|(y|(z|z)))) \end{array}\right). \tag{3.12}$$

Proof. Follows from Lemmas 2.4 and 3.13. □

Definition 3.16. We define three subsets of H as follows:

$$H_{\mathcal{N}}^{x_t} = \{x \in H : T_{\mathcal{N}}(x) \leq T_{\mathcal{N}}(x_t)\},$$

$$H_{\mathcal{N}}^{x_i} = \{x \in H : I_{\mathcal{N}}(x) \geq I_{\mathcal{N}}(x_i)\},$$

$$H_{\mathcal{N}}^{x_f} = \{x \in H : F_{\mathcal{N}}(x) \leq F_{\mathcal{N}}(x_f)\}$$

for all $x_t, x_i, x_f \in H$. Obviously, $x_t \in H_{\mathcal{N}}^{x_t}$, $x_i \in H_{\mathcal{N}}^{x_i}$, and $x_f \in H_{\mathcal{N}}^{x_f}$.

Theorem 3.17. *If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of H, then $H_{\mathcal{N}}^{x_t}$, $H_{\mathcal{N}}^{x_i}$, and $H_{\mathcal{N}}^{x_f}$ are ideals of H, where $x_t, x_i, x_f \in H$.*

Proof. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-ideal of H. Let $y_1 \in H_{\mathcal{N}}^{x_t}$, $y_2 \in H_{\mathcal{N}}^{x_i}$, and $y_3 \in H_{\mathcal{N}}^{x_f}$. Then $T_{\mathcal{N}}(y_1) \leq T_{\mathcal{N}}(x_t)$, $I_{\mathcal{N}}(y_2) \geq I_{\mathcal{N}}(x_t)$, and $F_{\mathcal{N}}(y_3) \leq F_{\mathcal{N}}(x_t)$. Thus,

$$T_{\mathcal{N}}((y_1|(x_1|x_1))|(y_1|(x_1|x_1))) \leq T_{\mathcal{N}}(y_1) \leq T_{\mathcal{N}}(x_t),$$

$$I_{\mathcal{N}}((y_2|(x_2|x_2))|(y_2|(x_2|x_2))) \geq I_{\mathcal{N}}(y_2) \geq I_{\mathcal{N}}(x_t),$$

$$F_{\mathcal{N}}((y_3|(x_3|x_3))|(y_3|(x_3|x_3))) \leq F_{\mathcal{N}}(y_3) \leq F_{\mathcal{N}}(x_t).$$

Hence, $(y_1|(x_1|x_1))|(y_1|(x_1|x_1)) \in T_{\mathcal{N}}(x_t)$, $(y_2|(x_2|x_2))|(y_2|(x_2|x_2)) \in I_{\mathcal{N}}(x_i)$, and $(y_3|(x_3|x_3))|(y_3|(x_3|x_3)) \in F_{\mathcal{N}}(x_f)$. Let $x_1, (y_1|(x_1|x_1))|(y_1|(x_1|x_1)) \in H_{\mathcal{N}}^{x_t}$, $x_2, (y_2|(x_2|x_2))|(y_2|(x_2|x_2)) \in H_{\mathcal{N}}^{x_i}$, and $x_3, (y_3|(x_3|x_3))|(y_3|(x_3|x_3)) \in H_{\mathcal{N}}^{x_f}$. Then $T_{\mathcal{N}}(x_1) \leq T_{\mathcal{N}}(x_t)$, $T_{\mathcal{N}}((y_1|(x_1|x_1))|(y_1|(x_1|x_1))) \leq T_{\mathcal{N}}(x_t)$, $I_{\mathcal{N}}(x_2) \geq I_{\mathcal{N}}(x_i)$, $I_{\mathcal{N}}((y_2|(x_2|x_2))|(y_2|(x_2|x_2))) \geq I_{\mathcal{N}}(x_t)$, $F_{\mathcal{N}}(x_3) \leq F_{\mathcal{N}}(x_f)$, and $F_{\mathcal{N}}((y_3|(x_3|x_3))|(y_3|(x_3|x_3))) \leq F_{\mathcal{N}}(x_t)$. Thus,

$$T_{\mathcal{N}}(y_1) \leq \max\{T_{\mathcal{N}}(x_1), T_{\mathcal{N}}((y_1|(x_1|x_1))|(y_1|(x_1|x_1)))\} \leq T_{\mathcal{N}}(x_t),$$

$$I_{\mathcal{N}}(y_2) \geq \min\{I_{\mathcal{N}}(x_2), I_{\mathcal{N}}((y_2|(x_2|x_2))|(y_2|(x_2|x_2)))\} \geq I_{\mathcal{N}}(x_i),$$

$$F_{\mathcal{N}}(y_3) \leq \max\{F_{\mathcal{N}}(x_3), F_{\mathcal{N}}((y_3|(x_3|x_3))|(y_3|(x_3|x_3)))\} \leq F_{\mathcal{N}}(x_t).$$

Hence, $y_1 \in T_{\mathcal{N}}(x_t)$, $y_2 \in I_{\mathcal{N}}(x_i)$, and $y_3 \in F_{\mathcal{N}}(x_f)$. Therefore, $H_{\mathcal{N}}^{x_t}$, $H_{\mathcal{N}}^{x_i}$, and $H_{\mathcal{N}}^{x_f}$ are ideals of H. □

Theorem 3.18. *If $H_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of H, then $H_{\mathcal{N}}^{x_t}$, $H_{\mathcal{N}}^{x_i}$, and $H_{\mathcal{N}}^{x_f}$ are subalgebras of H, where $x_t, x_i, x_f \in H$.*

Proof. Let $H_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of H. Let $x_1, y_1 \in H_{\mathcal{N}}^{x_t}$, $x_2, y_2 \in H_{\mathcal{N}}^{x_i}$, and $x_3, y_3 \in H_{\mathcal{N}}^{x_f}$. Then $T_{\mathcal{N}}(x_1) \leq T_{\mathcal{N}}(x_t)$, $T_{\mathcal{N}}(y_1) \leq T_{\mathcal{N}}(x_t)$, $I_{\mathcal{N}}(y_1) \geq I_{\mathcal{N}}(x_t)$, $I_{\mathcal{N}}(y_2) \geq I_{\mathcal{N}}(x_t)$, $F_{\mathcal{N}}(x_3) \leq F_{\mathcal{N}}(x_t)$, and $F_{\mathcal{N}}(y_3) \leq F_{\mathcal{N}}(x_t)$. Thus,

$$T_{\mathcal{N}}(x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \leq \max\{T_{\mathcal{N}}(x_1), T_{\mathcal{N}}(y_1)\} \leq T_{\mathcal{N}}(x_t),$$

$$I_{\mathcal{N}}(x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \geq \min\{I_{\mathcal{N}}(x_2), I_{\mathcal{N}}(y_2)\} \geq I_{\mathcal{N}}(x_t),$$

$$F_{\mathcal{N}}(x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \leq \max\{F_{\mathcal{N}}(x_3), F_{\mathcal{N}}(y_3)\} \leq F_{\mathcal{N}}(x_t).$$

Hence, $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)) \in T_{\mathcal{N}}(x_t)$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2)) \in I_{\mathcal{N}}(x_i)$, and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3)) \in F_{\mathcal{N}}(x_f)$. Therefore, $H_{\mathcal{N}}^{x_t}$, $H_{\mathcal{N}}^{x_i}$, and $H_{\mathcal{N}}^{x_f}$ are subalgebras of H. □

Theorem 3.19. [6] *Let $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$ be SUP-algebras. Then $\langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$ is a SUP-algebra, where the set $A \times B$ is the Cartesian product of A and B and the operation $|_{A\times B}$ on this set is defined by $(a_1, b_1)|_{A\times B}(a_2, b_2) = (a_1|_A a_2, b_1|_B b_2)$ for all $(a_1, b_1), (a_2, b_2) \in A \times B$, and the fixed element is $0_{A\times B} = (0_A, 0_B)$.*

Theorem 3.20. *Let $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$ be SUP-algebras. If $H_{A_{\mathcal{N}}}$ and $H_{B_{\mathcal{N}}}$ are neutrosophic $\mathcal{N}$-subalgebras of A and B, respectively, then $H_{(A\times B)_{\mathcal{N}}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $\langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$.*

Proof. Let $H_{A_{\mathcal{N}}}$ and $H_{B_{\mathcal{N}}}$ be neutrosophic $\mathcal{N}$-subalgebra of A and B, respectively. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then

$$\begin{aligned}
& T_{(A\times B)_{\mathcal{N}}}(((a_1,b_1)|_{A\times B}((a_2,b_2)|_{A\times B}(a_2,b_2)))|_{A\times B}((a_1,b_1)|_{A\times B}((a_2,b_2)|_{A\times B}(a_2,b_2)))) \\
= \; & T_{(A\times B)_{\mathcal{N}}}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))) \\
= \; & \max\{T_{A_{\mathcal{N}}}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), T_{B_{\mathcal{N}}}(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))\} \\
\leq \; & \max\{\max\{T_{A_{\mathcal{N}}}(a_1), T_{A_{\mathcal{N}}}(a_2)\}, \max\{T_{B_{\mathcal{N}}}(b_1), T_{B_{\mathcal{N}}}(b_2)\}\} \\
= \; & \max\{\max\{T_{A_{\mathcal{N}}}(a_1), T_{B_{\mathcal{N}}}(b_1)\}, \max\{T_{A_{\mathcal{N}}}(a_2), T_{B_{\mathcal{N}}}(b_2)\}\} \\
= \; & \max\{T_{(A\times B)_{\mathcal{N}}}(a_1,b_1), T_{(A\times B)_{\mathcal{N}}}(a_2,b_2)\},
\end{aligned}$$

$$\begin{aligned}
& I_{(A\times B)_{\mathcal{N}}}(((a_1,b_1)|_{A\times B}((a_2,b_2)|_{A\times B}(a_2,b_2)))|_{A\times B}((a_1,b_1)|_{A\times B}((a_2,b_2)|_{A\times B}(a_2,b_2)))) \\
= \; & I_{(A\times B)_{\mathcal{N}}}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))) \\
= \; & \max\{I_{A_{\mathcal{N}}}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), T_{B_{\mathcal{N}}}(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))\} \\
\geq \; & \min\{\min\{I_{A_{\mathcal{N}}}(a_1), I_{A_{\mathcal{N}}}(a_2)\}, \min\{I_{B_{\mathcal{N}}}(b_1), I_{B_{\mathcal{N}}}(b_2)\}\} \\
= \; & \min\{\min\{I_{A_{\mathcal{N}}}(a_1), I_{B_{\mathcal{N}}}(b_1)\}, \min\{I_{A_{\mathcal{N}}}(a_2), I_{B_{\mathcal{N}}}(b_2)\}\} \\
= \; & \min\{I_{(A\times B)_{\mathcal{N}}}(a_1,b_1), I_{(A\times B)_{\mathcal{N}}}(a_2,b_2)\},
\end{aligned}$$

$$\begin{aligned}
& F_{(A\times B)_{\mathcal{N}}}(((a_1,b_1)|_{A\times B}((a_2,b_2)|_{A\times B}(a_2,b_2)))|_{A\times B}((a_1,b_1)|_{A\times B}((a_2,b_2)|_{A\times B}(a_2,b_2)))) \\
= \; & F_{(A\times B)_{\mathcal{N}}}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))) \\
= \; & \max\{F_{A_{\mathcal{N}}}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), F_{B_{\mathcal{N}}}(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))\} \\
\leq \; & \max\{\max\{F_{A_{\mathcal{N}}}(a_1), F_{A_{\mathcal{N}}}(a_2)\}, \max\{F_{B_{\mathcal{N}}}(b_1), F_{B_{\mathcal{N}}}(b_2)\}\} \\
= \; & \max\{\max\{F_{A_{\mathcal{N}}}(a_1), F_{B_{\mathcal{N}}}(b_1)\}, \max\{F_{A_{\mathcal{N}}}(a_2), F_{B_{\mathcal{N}}}(b_2)\}\} \\
= \; & \max\{F_{(A\times B)_{\mathcal{N}}}(a_1,b_1), F_{(A\times B)_{\mathcal{N}}}(a_2,b_2)\}.
\end{aligned}$$

Hence, $H_{(A\times B)_{\mathcal{N}}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $A \times B$. □

Theorem 3.21. *Let $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$ be SUP-algebras. If $H_{A_{\mathcal{N}}}$ and $H_{B_{\mathcal{N}}}$ are neutrosophic $\mathcal{N}$-ideals of A and B, respectively, then $H_{(A\times B)_{\mathcal{N}}}$ is a neutrosophic $\mathcal{N}$-ideal of $\langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$.*

Proof. We can prove it by a proof procedure similar to Theorem 3.20. □

Definition 3.22. [6] Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SUP-algebras. Then a mapping $f : A \to B$ is called a homomorphism if $f(x|_A y) = f(x)|_B f(y)$ for all $x, y \in A$, and we have that $f(0_A) = 0_B$.

Theorem 3.23. *Let $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$ be SUP-algebras, $f : A \to B$ be a surjective homomorphism, and $B_{\mathcal{N}} = (B, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$ be a neutrosophic $\mathcal{N}$-structure on B. Then $B_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of B if and only if $B^f_{\mathcal{N}} = (A, T^f_{\mathcal{N}}, I^f_{\mathcal{N}}, F^f_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-ideal of A, where the $\mathcal{N}$-functions $T^f_{\mathcal{N}}, I^f_{\mathcal{N}}, F^f_{\mathcal{N}} : A \to [-1, 0]$ on A are defined by $T^f_{\mathcal{N}} = T_{\mathcal{N}}(f(x)), I^f_{\mathcal{N}} = I_{\mathcal{N}}(f(x)), F^f_{\mathcal{N}} = F_{\mathcal{N}}(f(x))$ for all $x \in A$, respectively.*

Proof. Let $B_{\mathcal{N}} = (T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$ be a neutrosophic $\mathcal{N}$-ideal of B. Let $x_1, x_2 \in A$. Then

$$\begin{aligned}
T^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) &= T_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))) \\
&= T_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\
&\leq T_{\mathcal{N}}(f(x_2)) \\
&= T^f_{\mathcal{N}}(x_2),
\end{aligned}$$

$$\begin{aligned}
I^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) &= I_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))) \\
&= I_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\
&\geq I_{\mathcal{N}}(f(x_2)) \\
&= I^f_{\mathcal{N}}(x_2),
\end{aligned}$$

$$\begin{aligned}
F^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) &= F_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))) \\
&= F_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\
&\leq F_{\mathcal{N}}(f(x_2)) \\
&= F^f_{\mathcal{N}}(x_2),
\end{aligned}$$

$$\begin{aligned}
T^f_{\mathcal{N}}(x_2) &= T_{\mathcal{N}}(f(x_2)) \\
&\leq \max\{T_{\mathcal{N}}(f(x_1)), T_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\} \\
&= \max\{T_{\mathcal{N}}(f(x_1)), T_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\} \\
&= \max\{T^f_{\mathcal{N}}(x_1), T^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\},
\end{aligned}$$

$$\begin{aligned}
I^f_{\mathcal{N}}(x_2) &= I_{\mathcal{N}}(f(x_2)) \\
&\geq \min\{I_{\mathcal{N}}(f(x_1)), I_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\} \\
&= \min\{I_{\mathcal{N}}(f(x_1)), I_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\} \\
&= \min\{I^f_{\mathcal{N}}(x_1), I^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\},
\end{aligned}$$

$$\begin{aligned}
F^f_{\mathcal{N}}(x_2) &= F_{\mathcal{N}}(f(x_2)) \\
&\leq \max\{F_{\mathcal{N}}(f(x_1)), F_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\} \\
&= \max\{F_{\mathcal{N}}(f(x_1)), F_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\} \\
&= \max\{F^f_{\mathcal{N}}(x_1), F^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}.
\end{aligned}$$

Hence, $B^f_{\mathcal{N}} = (T^f_{\mathcal{N}}, I^f_{\mathcal{N}}, F^f_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-ideal of A.

Conversely, let $B^f_{\mathcal{N}} = (T^f_{\mathcal{N}}, I^f_{\mathcal{N}}, F^f_{\mathcal{N}})$ be a neutrosophic $\mathcal{N}$-ideal of A. Let $y_1, y_2 \in B$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$ for $x_1, x_2 \in A$. Then

$$\begin{aligned}
T_{\mathcal{N}}((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1))) &= T_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\
&= T^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\
&\leq T^f_{\mathcal{N}}(x_2) \\
&= T_{\mathcal{N}}(f(x_2)) \\
&= T_{\mathcal{N}}(y_2),
\end{aligned}$$

$$\begin{aligned}
I_{\mathcal{N}}((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1))) &= I_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\
&= I^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\
&\geq I^f_{\mathcal{N}}(x_2) \\
&= I_{\mathcal{N}}(f(x_2)) \\
&= I_{\mathcal{N}}(y_2),
\end{aligned}$$

$$\begin{aligned}
F_{\mathcal{N}}((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1))) &= F_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1)))) \\
&= F^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))) \\
&\leq F^f_{\mathcal{N}}(x_2) \\
&= F_{\mathcal{N}}(f(x_2)) \\
&= F_{\mathcal{N}}(y_2),
\end{aligned}$$

$$\begin{aligned}
T_{\mathcal{N}}(y_2) &= T_{\mathcal{N}}(f(x_2))\\
&= T^f_{\mathcal{N}}(x_2)\\
&\leq \max\{T^f_{\mathcal{N}}(x_1), T^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}\\
&= \max\{T_{\mathcal{N}}(f(x_1)), T_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\}\\
&= \max\{T_{\mathcal{N}}(f(x_1)), T_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\}\\
&= \max\{T_{\mathcal{N}}(y_1), T_{\mathcal{N}}((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1)))\},
\end{aligned}$$

$$\begin{aligned}
I_{\mathcal{N}}(y_2) &= I_{\mathcal{N}}(f(x_2))\\
&= I^f_{\mathcal{N}}(x_2)\\
&\geq \min\{I^f_{\mathcal{N}}(x_1), I^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}\\
&= \min\{I_{\mathcal{N}}(f(x_1)), I_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\}\\
&= \min\{I_{\mathcal{N}}(f(x_1)), I_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\}\\
&= \min\{I_{\mathcal{N}}(y_1), I_{\mathcal{N}}((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1)))\},
\end{aligned}$$

$$\begin{aligned}
F_{\mathcal{N}}(y_2) &= F_{\mathcal{N}}(f(x_2))\\
&= F^f_{\mathcal{N}}(x_2)\\
&\leq \max\{F^f_{\mathcal{N}}(x_1), F^f_{\mathcal{N}}((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1)))\}\\
&= \max\{F_{\mathcal{N}}(f(x_1)), F_{\mathcal{N}}(f((x_2|_A(x_1|_Ax_1))|_A(x_2|_A(x_1|_Ax_1))))\}\\
&= \max\{F_{\mathcal{N}}(f(x_1)), F_{\mathcal{N}}((f(x_2)|_B(f(x_1)|_Bf(x_1)))|_B(f(x_2)|_B(f(x_1)|_Bf(x_1))))\}\\
&= \max\{F_{\mathcal{N}}(y_1), F_{\mathcal{N}}((y_2|_B(y_1|_By_1))|_B(y_2|_B(y_1|_By_1)))\}.
\end{aligned}$$

Hence, $B_{\mathcal{N}} = (T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-ideal of B. □

Theorem 3.24. *Let $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$ be SUP-algebras, $f : A \to B$ be a surjective homomorphism, and $B_{\mathcal{N}} = (B, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$ be a neutrosophic $\mathcal{N}$-structure on B. Then $B_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of B if and only if $B^f_{\mathcal{N}} = (A, T^f_{\mathcal{N}}, I^f_{\mathcal{N}}, F^f_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-subalgebra of A, where the $\mathcal{N}$-functions $T^f_{\mathcal{N}}, I^f_{\mathcal{N}}, F^f_{\mathcal{N}} : A \to [-1, 0]$ on A are defined by $T^f_{\mathcal{N}} = T_{\mathcal{N}}(f(x)), I^f_{\mathcal{N}} = I_{\mathcal{N}}(f(x)), F^f_{\mathcal{N}} = F_{\mathcal{N}}(f(x))$ for all $x \in A$, respectively.*

Proof. We can prove it by a proof procedure similar to Theorem 3.23. □

4 Conclusion

This study has introduced the concept of neutrosophic $\mathcal{N}$-structures within the context of Sheffer stroke UP-algebras, offering a significant expansion in the understanding of algebraic frameworks enriched with neutrosophic logic. By defining and analyzing neutrosophic $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals, the research highlights their structural properties and relationships, particularly through level sets and lattice structures.

The results reveal that the family of all neutrosophic $\mathcal{N}$-subalgebras forms a complete distributive lattice, enabling systematic exploration of their interdependencies. Furthermore, the findings establish that every neutrosophic $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-subalgebra, albeit not vice versa, reflecting the nuanced characteristics of these algebraic constructs.

By providing a robust theoretical foundation, this study opens avenues for further exploration of logical systems under uncertainty and lays the groundwork for practical applications of neutrosophic structures in fields such as artificial intelligence and data analysis. Future research could explore the generalization of these results to other algebraic systems, enhancing the applicability of neutrosophic logic in broader contexts.

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