International Journal of Neutrosophic Science (IJNS) *Vol. 25, No. 04, PP. 425-432, 2025*

On a convex topological order and neutrosophic continuous sets

Elvis Aponte[1], **Jorge Vielma**[1], **José Sanabria**[2,*], **Ennis Rosas**[3]
[1]Department of Mathematics, Faculty of Natural Sciences and Mathematics, Escuela Superior Politécnica del Litoral, ESPOL, Campus Gustavo Galindo, km. 30.5 vía Perimetral, Guayaquil, Ecuador
[2]Department of Mathematics, Faculty of Education and Sciences, Universidad de Sucre, Sincelejo, Colombia
[3]Department of Natural and Exact Sciences, Universidad de la Costa, Barranquilla, Colombia
Emails: ecaponte@espol.edu.ec; jevielma@espol.edu.ec; jesanabri@gmail.com; ennisrafael@gmail.com

Abstract

In this paper, we employ the classical topological preorder to introduce the concept of topologically bounded sets, in order to relate it to the Collatz conjecture problem. In addition, this preorder allows us to derive some results about topologically convex sets, showing that these form a convex structure. Finally, using this topological preorder, we define the neutrosophic continuous sets and establish the necessary conditions to identify the points that are connected to these sets, which form a topological convex set.

Keywords: Topological preorder; Collatz conjeture; Topological convex set

1 Introduction

In topological theory, it is usual to denote the closure of a set A as $\overline{A}$. This denotation allows us to define the preorder of specialization or topological preorder,[14] from which several results have been derived; for example, the continuous image of a topologically bounded set, is another topologically bounded set because continuous functions preserve the topological preorder.

On the other hand, using the *Collatz map* $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n/2$ if n is even and $f(n) = 3n + 1$ if n is odd, the Collatz conjecture is established as follows: for every $n \in \mathbb{N}$ there exists a natural number r such that the composite r-fold of f with itself gives 1, that is $f^{(r)}(n) = 1$. Although this conjecture has not been demonstrated, it has been characterized in several ways, such as those given in references[3] and,[13] but this has not been studied through the preorder of specialization.

In this manuscript, employing topological preorder and topologically convex sets, we establish some results in four sections. In Section 2, we study the Collatz conjeture problem through the specialization preorder. In Section 3, we explore the notion of topologically convex sets using the preorder of specialization, obtaining among other results, that the collection of these sets is a convex structure, that the kernel and the saturation of a set are topologically convex sets, we compare the cardinality of the family of all topologies defined on a set X with the cardinality of the family of all convex structures on X. In Section 4, we show that topologically convex sets are preserved under homeomorphisms. Finally, in Section 5, we introduce a new class of neutrosophic sets, namely neutrosophic continuous sets, to explore the set of points connected to this new class of sets, showing that this is a topologically convex set. These results could be used to continue with the studies of the Collatz conjecture, and also, to continue studying in depth the topologically convex sets and the neutrosophic continuous sets.

DOI: https://doi.org/10.54216/IJNS.250436

2 Topologies and the preorder of specialization

In this section, we define the preorder of specialization through the concept of topological closure and using this, we introduce the concept of upper τ-bounded set, where τ is a topology. Next, we use the definition of upper τ-bounded set to relate it to the Collatz conjecture problem.

Let (X, τ) be a topological space. For a point $x \in X$, we denote the the set $\{x\}$ by x. It is well known that (X, τ) is T_1 if and only if $\overline{x} = x$ for every $x \in X$.

Definition 2.1. (Preorder of specialization) Let (X, τ) be a topological space. For $x, y \in X$, we say that $x \leq_\tau y$, if $y \in \overline{x}$, that is, y belongs to the closure of x.

Theorem 2.2. *A topological space* (X, τ) *is* T_0 *if and only if* $\leq_\tau$ *is a partial order.*

Proof. ($\Leftarrow$) Suppose that (X, τ) is not T_0. Then, there exist $x_0, y_0 \in X$ such that $x_0 \neq y_0$ and for each $\Theta \in \tau$, we have $\{x_0, y_0\} \subseteq \Theta$. Hence, $x_0 \in \overline{y_0}$ and $y_0 \in \overline{x_0}$, which implies that $x_0 \leq_\tau y_0$ and $y_0 \leq_\tau x_0$. As $\leq_\tau$ is a partial order, it follows that $x_0 = y_0$, which is a contradiction. Therefore, (X, τ) is T_0.

($\Rightarrow$) Let $x_0, y_0 \in X$ such that $x_0 \leq_\tau y_0$ and $y_0 \leq_\tau x_0$, and suppose that $x_0 \neq y_0$. As (X, τ) is T_0, there exists $\Theta \in \tau$ such that $x_0 \in \Theta$ and $y_0 \notin \Theta$. Hence, $x_0 \notin \overline{y_0}$, in this way $y_0 \not\leq_\tau x_0$, which is a contradiction. Therefore, $\leq_\tau$ is a partial order. ☐

Definition 2.3. Let (X, τ) be a topological space. We say that a subset A of X is upper τ-bounded if there exists $b \in X$ such that $a \leq_\tau b$ for each $a \in A$.

Note that if $a \leq_\tau b$ for each $a \in A$, then $b \in \bigcap_{a \in A} \overline{a}$. Therefore, $A \subseteq X$ is upper τ-bounded if and only if $\emptyset \neq \bigcap_{a \in A} \overline{a}$. We denote this as $A \leq_\tau \bigcap_{a \in A} \overline{a} := u_A$.

Example 2.4. Let (X, τ) be any topological space which is T_1. Then, the only sets that are upper τ-bounded are the singleton sets.

Theorem 2.5. *Let* (X, τ) *be a topological space. If* $A \subseteq X$ *is an upper* τ*-bounded and* τ*-closed set, then* A *is a* τ*-connected set.*

Proof. Since A is a τ-bounded set, we have $\emptyset \neq \bigcap_{a \in A} \overline{a}$. Since A is a τ-closed set, it turns out that $A = \bigcap_{a \in A} \overline{a}$. Therefore, A is a τ-connected set. ☐

Corollary 2.6. *Let* (X, τ) *be a topological space. If* X *is an upper* τ*-bounded set, then* (X, τ) *is a connected space.*

Proof. Clearly, X is a τ-closed set. Also, as X is an upper τ-bounded set, then by Theorem 2.5, it follows that (X, τ) is a connected space. ☐

Example 2.7. Let (W, τ_f) be a topological space, where $W = \{2^n : n \in \mathbb{N}\}$ and $\tau_f = \{O \subseteq \mathbb{N} : f^{-1}(O) \subseteq O\}$ is the primal topology[4] on W induced by the function f defined as $f(2^k) = 2^{k-1}$. Observe that for all $n \in \mathbb{N}$, $\overline{\{2^n\}} = \{2^n, 2^{n-1}, 2^{n-2}, ..., 1\}$. Therefore, $1 \in \overline{\{2^n\}}$ for all $n \in \mathbb{N}$. Thus, $2^n \leq_{\tau_f} 1$, for all $n \in \mathbb{N}$ and by Corollary 2.6, it follows that (W, τ_f) is a connected space.

The converse of Corollary 2.6, is not necessarily true, as we can see in the following example.

Example 2.8. Consider $(\mathbb{N}, \tau)$, where $\tau = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, .., n\}, ...\}$. Clearly $\mathbb{N} = \overline{1}$ and so $(\mathbb{N}, \tau)$ is a connected space. Note that for each $n \in \mathbb{N}$, $\overline{n} = \{n, n+1, n+2, ...\}$ and $\bigcap_{n \in \mathbb{N}} \overline{n} = \emptyset$. Therefore, $\mathbb{N}$ is not an upper τ-bounded set.

Definition 2.9. Let (X, τ) be a topological space. We say that a subset A of X is lower τ-bounded if there exists $b \in X$ such that $b \leq_\tau a$ for each $a \in A$.

Note that if $b \leq_\tau a$ for each $a \in A$, then $\bigcup\limits_{a \in A} a \subseteq \overline{b}$. Therefore, if A is a lower τ-bounded set, then $A \subseteq \overline{b}$, so $\overline{A} \subseteq \overline{b}$. We denote this as $l_A := b \leq_\tau A$.

Remark 2.10. Let (X, τ) be a topological space. If $\{A_\alpha : \alpha \in \Delta\}$ is a family of subsets of X such that for every $\alpha \in \Delta$, $A_\alpha \leq_\tau u_A$, then $\bigcup\limits_{\alpha \in \Delta} A_\alpha \leq_\tau u_A$.

The Collatz map induces a primal topology $\tau_f = \{O \subseteq \mathbb{N} : f^{-1}(O) \subseteq O\}$ on $\mathbb{N}$.[4] In[13] it was shown that the Collatz conjecture is true if and only if the primal space $(\mathbb{N}, \tau_f)$ is connected. On the other hand, we see in[3] that the Collatz conjecture is equivalent to assuming that the space $(\mathbb{N}, \tau_f)$ do not satisfies the separation property called weak-R_0. Now, if we consider the topological space $(\mathbb{N}, \tau_f)$, where f is the Collatz function, then by Corollary 2.6, we have:

$$\begin{aligned} \mathbb{N} \text{ is upper } \tau_f\text{-bounded} &\Longrightarrow (\mathbb{N}, \tau_f) \text{ is connected} \\ &\Longleftrightarrow (\mathbb{N}, \tau_f) \text{ is not weak-}R_0 \\ &\Longleftrightarrow \text{Conjecture Collatz is true.} \end{aligned}$$

3 Topologically convex sets

In this section, we define topologically convex sets using the specialization preorder and characterize them in terms of the topological closure and the topological convex hull. Among other results, we show that the saturation and the kernel[5] of a set are topologically convex sets; and also that closed sets are topologically convex. Finally, we find a lower bound for the set of all topologies defined in a set.

Remark 3.1. [11] If τ is a topology on X and $\overline{\tau}$ is the clousure of τ in the topological space $(2^X, \Pi)$, where Π is the product topology on 2^X, then $\overline{\tau}$ is a Alexandroff topology (i.e. the intersection of open sets is an open set) on X. Note that $K \in \overline{\tau}$ if and only if $K = \bigcap\limits_{\alpha \in \Delta} \Theta_\alpha$, where $\Theta_\alpha \in \tau$ for each $\alpha \in \Delta$.

Remark 3.2. [11] If τ is a topology on X and τ^* denotes the family of all τ-closed sets, then $\overline{\tau}^*$ is a topology of Alexandroff on X.

Definition 3.3. [1] (Topologically convex sets) Let (X, τ) be a topological space. A subset A of X is said to be τ-convex, if for each $x, y \in A$ and each $z \in X$ such that $x \leq_\tau z \leq_\tau y$, we have $z \in A$.

Example 3.4. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. Then, $c = X \setminus \{a, b\}$ is a τ-closed set. Note that $F = \{b, c\}$ is not a τ-open set nor is it a τ-closed set. Furthermore, F is not a τ-convex set.

Let (X, τ) be a topological space. The kernel of a subset A of X is defined as $Ker(A) = \bigcap\limits_{A \subseteq \Theta \in \tau} \Theta$ and the saturation of A is defined as $Sat(A) = \bigcup\limits_{x \in A} \overline{x}$. The following characterization of topologically convex sets will allow us, among other things, to see that the closure of a set is a topologically convex set.

Lemma 3.5. *Let (X, τ) be a topological space. A subset A of X is τ-convex if and only if $A = U \cap V$, where $U \in \overline{\tau}^*$ and $V \in \overline{\tau}$.*

Proof. ($\Rightarrow$) Suppose that A is a τ-convex set and let $U = \bigcup\limits_{x \in A} \overline{x}$ and $V = \bigcup\limits_{x \in A} Ker(x)$. If $z \in U \cap V$, then $z \in U$ and $z \in V$, which implies that there exist $x, y \in A$ such that $z \in \overline{x}$ and $z \in Ker(y)$. Thus, $x \leq_\tau z \leq_\tau y$ and as A is a τ-convex set, it follows that $z \in A$, which shows that $U \cap V \subseteq A$. Since $A \subseteq U \cap V$, we get that $A = U \cap V$.

($\Leftarrow$) Suppose that $A = U \cap V$, where $U \in \overline{\tau}^*$ and $V \in \overline{\tau}$. Let $x, y \in A$ and $z \in X$ such that $x \leq_\tau z \leq_\tau y$. Then, $z \in \overline{x}$ and $z \in Ker(y)$, whence $z \in U$ and $z \in V$. Therefore, $z \in A$ and so A is a τ-convex set. $\square$

Corollary 3.6. *Let (X, τ) be a topological space and $A \subseteq X$. Then:*

1. *A is a τ-convex set, if A is a τ-closed set.*
2. *$Sat(A)$ is a τ-convex set.*
3. *$Ker(A)$ is a τ-convex set.*

Proof. (1) Consider $x, y \in A$ and $z \in X$ such that $x \leq_\tau z \leq_\tau y$. By definition $z \in \overline{y} \subseteq \overline{A} = A$. Hence, A is a τ-convex set.

(2) Since $X \in \overline{\tau}$ and $Sat(A) \in \overline{\tau}^*$, we have $Sat(A) = X \cap Sat(A)$ is a τ-convex set, by Lemma 3.5. Note that in particular, each closed set is τ-convex.

(3) Since $Ker(A) \in \overline{\tau}$ and $X \in \overline{\tau}^*$, we have $Ker(A) = Ker(A) \cap X$ is a τ-convex set, by Lemma 3.5. Note that in particular, each τ-open set is τ-convex. ☐

Corollary 3.7. *Let (X, τ) be a topological space. For any $A \subseteq X$, $\overline{A}$ is a τ-convex set.*

Remark 3.8. For any topological space (X, τ) that is T_1, we have $\overline{\tau}$ and $\overline{\tau}^*$ are equal to 2^X.[11] Therefore, by Lemma 3.5, every subset of X is τ-convex. So we will study spaces that are not T_1.

The following result leads us to define the convex hull of a set, which will later serve to give a characterization of sets that are topologically convex.

Theorem 3.9. *Let (X, τ) be a topological space. If $\{A_\alpha : \alpha \in \Delta\}$ is a family of τ-convex subsets of X, then $\bigcap_{\alpha \in \Delta} A_\alpha$ is a τ-convex set.*

Proof. Consider $x, y \in \bigcap_{\alpha \in \Delta} A_\alpha$ and $z \in X$ such that $x \leq_\tau z \leq_\tau y$. Note that for all $\alpha \in \Delta$, we have $z \in A_\alpha$, because each A_α is a convex set. Hence, $z \in \bigcap_{\alpha \in \Delta} A_\alpha$ and this implies that $\bigcap_{\alpha \in \Delta} A_\alpha$ is a convex set. ☐

Definition 3.10. Let (X, τ) be a topological space and A be a non-empty subset of X. The convex hull of A, denoted by $\Gamma(A)$, is the smallest τ-convex set containing A.

Clearly, if (X, τ) is a topological space and $x \in X$, then for $F = \{x\}$ is holds that $x \in \Gamma(F)$.

Lemma 3.11. *Let (X, τ) be a topological space. $A \subseteq X$ is a τ-convex set if and only if for all finite set $F \subseteq A$ it turns out that $\Gamma(F) \subseteq A$.*

Proof. ($\Leftarrow$) Let $x, y \in A$ and $z \in X$ such that $x \leq_\tau z \leq_\tau y$. We consider the set $F = \{x, y\}$ contained in A. By hypothesis, $\Gamma(F) \subseteq A$, and as $z \in \Gamma(F)$, we have $z \in A$. Therefore, A is a τ-convex set.

($\Rightarrow$) It is clear, because A is a τ-convex set. ☐

Remark 3.12. Let (X, τ) be a topological space. A finite subset F of X is said to be a convex locality of $x \in X$, if $x \in \Gamma(F)$. Thus, $K \subseteq X$ is a τ-convex set if and only if K contains every convex locality of its points.

Definition 3.13. [12] A family $\mathcal{A} \subseteq 2^X$ is a convexity on X, if $\mathcal{A}$ satisfies the following conditions:

1. $\emptyset, X \in \mathcal{A}$.
2. If $\{A_i : i \in I\} \subseteq \mathcal{A}$ is non-empty, then $\bigcap_{i \in I} A_i \in \mathcal{A}$.

3. If $\{A_i : i \in I\} \subseteq \mathcal{A}$ is non-empty and totally ordered by inclusion, then $\bigcup_{i\in I} A_i \in \mathcal{A}$.

Theorem 3.14. *Let (X, τ) be a topological space. The family $\mathcal{C}_0 = \{A \subseteq X : A \text{ is a } \tau\text{-convex set}\}$ is a convexity on X.*

Proof. We will verify that the family $\mathcal{C}_0$ is a convexity on X:
(1) Follows from the definition of τ-convex set.
(2) Follows from Theorem 2.10.
(3) Suppose that $\{A_i : i \in I\} \subseteq \mathcal{C}_0$ is non-empty and totally ordered by inclusion. Let $x, y \in \bigcup_{i\in I} A_i$ and $z \in X$ such that $x \leq_\tau z \leq_\tau y$. Then, there exist $i_1, i_2 \in I$ such that $x \in A_{i_1}$ and $y \in A_{i_2}$. Since $\{A_i : i \in I\}$ is non-empty and totally ordered by inclusion, there exists $i_3 \in I$ such that $A_{i_1} \subseteq A_{i_3}$ and $A_{i_2} \subseteq A_{i_3}$. Thus, $x, y \in A_{i_3}$ and as A_{i_3} is a τ-convex set, it follows that $z \in A_{i_3} \subseteq \bigcup_{i\in I} A_i$. Therefore, $\bigcup_{i\in I} A_i$ is a τ-convex set and so $\bigcup_{i\in I} A_i \in \mathcal{C}_0$. ☐

Theorem 3.15. *Let (X, τ) be a topological space and ρ a topology on X such that $\tau \subseteq \rho \subseteq \overline{\tau}$. Then, a subset B of X is τ-convex if and only if it is ρ-convex.*

Proof. The hypothesis $\tau \subseteq \rho \subseteq \overline{\tau}$ implies that $\overline{\rho} = \overline{\tau}$. Therefore, the proof easily follows by Lemma 3.5. ☐

The following result shows that the cardinality of the set formed by all topologies on X is greater than the cardinality of the set formed by all convexities on X.

Corollary 3.16. *Let $\mathcal{C}(X)$ be the family of all convexities on X and $Top(X)$ be the family of all topologies on X. Then, $|Top(X)| \geq |\mathcal{C}(X)|$.*

4 Preservation of preorder under functions

In this section, we show that direct images of continuous functions between topological spaces preserve the preorder of specialization and send upper (resp. lower) bounded sets into upper (resp. lower) bounded sets. Furthermore, we show that the preorder is preserved via inverse image of an injective open function. Finally, we prove that topologically convex sets are preserved under homeomorphisms, which is natural because it is a topological definition. To see this, we first present some useful results.

Theorem 4.1. *Let (X, τ) and (Y, ρ) be two topological spaces and $f : (X, \tau) \to (Y, \rho)$ be a continuous function. If $x \leq_\tau y$, then $f(x) \leq_\rho f(y)$.*

Proof. Suppose that $x \leq_\tau y$. Then, $y \in \overline{x}$ and so, $f(y) \in f(\overline{x})$. Since $f(\overline{x}) \subseteq \overline{f(x)}$, we have $f(y) \in \overline{f(x)}$. Therefore, $f(x) \leq_\rho f(y)$. ☐

Corollary 4.2. *Let (X, τ) and (Y, ρ) be two topological spaces and $f : (X, \tau) \to (Y, \rho)$ be an injective open function. If $f(x) \leq_\rho f(y)$, then $x \leq_\tau y$.*

Proof. Let $x, y \in X$ such that $f(x) \leq_\rho f(y)$ and V be a τ-open set containing y. Since f is an open function, $f(V)$ is a ρ-open set containing $f(y)$ and as $f(x) \leq_\rho f(y)$, we have $f(y) \in \overline{f(x)}$. Since f is an injective function, we have $x \in f^{-1}(f(V)) = V$ and so, $y \in \overline{x}$. Hence, $x \leq_\tau y$. ☐

Corollary 4.3. *Let (X, τ) and (Y, ρ) be two topological spaces and $f : (X, \tau) \to (Y, \rho)$ be an injective, continuous and open function. Then, $f(x) \leq_\rho f(y)$ if and only if $x \leq_\tau y$.*

Theorem 4.4. *Let (X, τ) and (Y, ρ) be two topological spaces. If $f : (X, \tau) \to (Y, \rho)$ is a continuous function, then $f(A)$ is an upper (resp. lower) ρ-bounded set whenever A is an upper (rep. lower) τ-bounded set.*

Proof. Suppose that A is an upper τ-bounded set. Then, $\emptyset \neq \bigcap_{a\in A} \overline{a}$ and hence, $\emptyset \neq f\left(\bigcap_{a\in A} \overline{a}\right) \subseteq \bigcap_{a\in A} f(\overline{a}) \subseteq \bigcap_{a\in A} \overline{f(a)}$, which shows that $f(A)$ is an upper ρ-bounded set. On the other hand, assume that A is a lower τ-bounded set. Then, $A \subseteq \overline{b}$ for some $b \in X$ and so, $f(A) \subseteq \overline{f(b)}$. Therefore, $f(A)$ is a lower ρ-bounded set. ☐

Theorem 4.5. *Let (X,τ) and (Y,ρ) be two topological spaces. If $f:(X,\tau)\to(Y,\rho)$ is a continuous function, then $f^{-1}(B)$ is a τ-convex set whenever B is a ρ-convex set.*

Proof. Let B be a ρ-convex set. By Lemma 3.5, we have $B = K\cap S$, where $K\in\overline{\rho}$ and $S\in\overline{\rho}^*$. Now, by Remark 3.1, K and S can be written as $K=\bigcap_{\alpha\in\Delta} U_\alpha$ and $S=\bigcup_{\omega\in\Omega} V_\omega$, where $U_\alpha\in\rho$ for each $\alpha\in\Delta$ and $V_\omega\in\rho^*$ for each $\omega\in\Omega$. Hence, $f^{-1}(B)=f^{-1}(K)\cap f^{-1}(S)=\left(\bigcap_{\alpha\in\Delta} f^{-1}(U_\alpha)\right)\bigcap\left(\bigcup_{\omega\in\Omega} f^{-1}(V_\omega)\right)$. By the continuity of f and Lemma 3.5, we get that $f^{-1}(B)$ is a τ-convex set. ☐

Theorem 4.6. *Let (X,τ) and (Y,ρ) be two topological spaces. If $f:(X,\overline{\tau})\to(Y,\overline{\rho})$ is a continuous function, then $f^{-1}(B)$ is a τ-convex set whenever B is a ρ-convex set.*

Proof. Suppose that B is a ρ-convex set. By Lemma 3.5, we have $B=K\cap S$, where $K\in\overline{\rho}$ and $S\in\overline{\rho}^*$. Then, $f^{-1}(B)=f^{-1}(K)\cap f^{-1}(S)$ and as f is continuous, $f^{-1}(K)\in\overline{\tau}$ and $f^{-1}(S)\in\overline{\tau}^*$. Again, by Lemma 3.5, we obtain that $f^{-1}(B)$ is τ-convex. ☐

Theorem 4.7. *Let (X,τ) and (Y,ρ) be two topological spaces. If $f:(X,\tau)\to(Y,\rho)$ is a bijective open function, then $f(A)$ is a ρ-convex set whenever A is a τ-convex set.*

Proof. Suppose that A is a τ-convex set and let $u,v\in f(A)$ and $v\in Y$ such that $u\leq_\rho v\leq_\rho w$. By the definition of direct image, there exist $x,y\in A$ such that $u=f(x)$ and $w=f(y)$. Moreover, as f is surjective, there exists $z\in X$ such that $v=f(z)$, which implies that $f(x)\leq_\rho f(z)\leq_\rho f(y)$. By Corollary 4.2, we get that $x\leq_\tau z\leq_\tau y$ and as A is a τ-convex set, it follows that $z\in A$. Therefore, $v=f(z)\in f(A)$, which shows that $f(A)$ is a ρ-convex set. ☐

Corollary 4.8. *Let (X,τ) and (Y,ρ) be two topological spaces and $f:(X,\tau)\to(Y,\rho)$ be a homeomorphism. Then, $f(A)$ is a ρ-convex set if and only if A is a τ-convex set.*

Example 4.9. Let X be a non-empty set and $f:X\to X$ be a function. Consider on X the topology $\tau_f := \{A : f^{-1}(A)\subseteq A\}$. Then, $f:(X,\tau_f)\to(X,\tau_f)$ is continuous and hence, $f^{-1}(B)$ is a τ_f-convex set whenever B is a τ_f-convex set.

Theorem 4.10. *Let (X,τ) and (Y,ρ) be two topological spaces. If $f:(X,\tau)\to(Y,\rho)$ is an injective, closed and open function, then $f(A)$ is a ρ-convex set whenever A is a τ-convex set.*

Proof. Let A be a τ-convex set. By Lemma 3.5, we have $A=K\cap S$, where $K\in\overline{\tau}$ and $S\in\overline{\tau}^*$. Observe that K and S can be written as $K=\bigcap_{\alpha\in\Delta} U_\alpha$ and $S=\bigcup_{\omega\in\Omega} V_\omega$, where $U_\alpha\in\tau$ for each $\alpha\in\Delta$ and $V_\omega\in\tau^*$ for each $\omega\in\Omega$. By the injectivity of f, we have

$$\begin{aligned} f(A) &= f(K\cap S) = f(K)\cap f(S) \\ &= f\left(\bigcap_{\alpha\in\Delta} U_\alpha)\right)\cap f\left(\bigcup_{\omega\in\Omega} V_\omega)\right) \\ &= \left(\bigcap_{\alpha\in\Delta} f(U_\alpha)\right)\cap\left(\bigcup_{\omega\in\Omega} f(V_\omega)\right). \end{aligned}$$

Now, as f is a closed and open function, by Lemma 3.5, it follows that $f(A)$ is a ρ-convex set. ☐

5 Neutrosophic continuous sets

According,[9] we say that an object B is a neutrosophic set on a non-empty set X, if B has the form $B = \{\langle x, \mu_{1B}(x), \mu_{2B}(x), \mu_{3B}(x)\rangle; x \in X\}$, where $\mu_{iB} : X \to [0,1]$ is a function for each $i = 1,2,3$ and $0 \leq \sum_{i=1}^{3} \mu_{iB}(x) \leq 3$. Several researches related to topologies have been generated from the concept of neutrosophic set, among which we can highlight some recent works such as.[2,6–8,10] Note that when the interval $[0,1]$ is considered with the usual topology inherited from $\mathbb{R}$ and X is endowed with a suitable topology (in particular, the initial topology on X induced by the functions μ_{iB}), it may be the case that the functions μ_{iB} are continuous. This fact motivates the definitions and results of this section, where $[0,1]$ is endowed with usual topology inherited from $\mathbb{R}$, which we denote by $\sigma_{[0,1]}$.

Definition 5.1. Let (X,τ) be a topological space. A neutrosophic τ-continuous set B on X, is a neutrosophic set $B = \{\langle x, \mu_{1B}(x), \mu_{2B}(x), \mu_{3B}(x)\rangle : x \in X\}$, where $\mu_{iB} : X \to [0,1]$ is a continuous function for each $i = 1,2,3$.

Examples of neutrosophic τ-continuous sets are the neutrosophic universal set and the neutrosophic empty set, because the associated functions are constant and so continuous. Similarly, the neutrosophic complement of a neutrosophic τ-continuous set is a neutrosophic τ-continuous set. Since the supremum and infimum of continuous functions is a continuous function, we have the union and intersection of neutrosophic τ-continuous sets is another neutrosophic τ-continuous set.

Definition 5.2. Let B be a neutrosophic set on X. A point $y \in X$ is said to be connected to B, if there exists $x \in X$, $x \neq y$, such that $\mu_{iB}(x) = \mu_{iB}(y)$ for each $i = 1,2,3$.

Example 5.3. [8, Example 3.7] Let $X = \{x,y,z\}$ and consider the following neutrosophic sets:

$$\begin{aligned}
A_1 &= \{\langle x, 0.1, 0, 0.9\rangle, \langle y, 0, 0, 1\rangle, \langle z, 0, 0, 1\rangle\},\\
A_2 &= \{\langle x, 0, 0, 1\rangle, \langle y, 0.3, 0, 0.7\rangle, \langle z, 0, 0, 1\rangle\},\\
A_3 &= \{\langle x, 0, 0, 1\rangle, \langle y, 0, 0, 1\rangle, \langle z, 0.4, 0, 0.6\rangle\},\\
A_4 &= \{\langle x, 0.6, 0, 0.4\rangle, \langle y, 0, 0, 1\rangle, \langle z, 0.8, 0, 0.2\rangle\}.
\end{aligned}$$

Then:

- The points y and z are connected to A_1, but the point x is not connected A_1.
- The points x and z are connected to A_2, but the point y is not connected A_2.
- The points x and y are connected to A_3, but the point z is not connected A_3.
- The points x, y and z are not connected to A_4.

Theorem 5.4. *Let (X,τ) be a topological space and $B = \{(x, \mu_{1B}(x), \mu_{2B}(x), \mu_{3B}(x)) : x \in X\}$ be a neutrosophic τ-continuous set on X. If $y \in X$ is such that $y \leq_\tau x$ for some $x \in X$, then y is connected to B.*

Proof. Let $y \in X$ be such that $y \leq_\tau x$ for some $x \in X$. Since $B = \{(x, \mu_{1B}(x), \mu_{2B}(x), \mu_{3B}(x)) : x \in X\}$ is a neutrosophic τ-continuous set on X, we have $\mu_{iB} : X \to [0,1]$ is a continuous function for each $i = 1,2,3$. By Theorem 4.1, it follows that $\mu_{iB}(y) \leq_{\sigma_{[0,1]}} \mu_{iB}(x)$ for each $i = 1,2,3$. Thus, $\mu_{iB}(x) \in \overline{\mu_{iB}(y)}$ for each $i = 1,2,3$ and as $([0,1], \sigma_{[0,1]})$ is T_1, we get $\overline{\mu_{iB}(y)} = \mu_{iB}(y)$ for each $i = 1,2,3$. Therefore $\mu_{iB}(x) = \mu_{iB}(y)$ for each $i = 1,2,3$, which shows that y is connected to B. □

Finally, using the definitions and the transitive property of equality and the procedure of the proof of Theorem 5.4, we obtain the following result.

Corollary 5.5. *Let (X,τ) be a topological space. If B is a neutrosophic τ-continuous set on X, then the set of connected points to B is a τ-convex set.*

References

[1] M.L. Colasante, C. Uzcátegui, J. Vielma, Boolean algebras and low separation axioms, *Topology Proceedings* **34** (2009), 1-15.

[2] K. Damodharan, M. Vigneshwaran, N_δ-closure and N_δ-interior in neutrosophic topological spaces, *Neutrosophic Sets and Systems* **66** (2024), 108-118.

[3] G. Di Maio, A separation axiom weaker than R_0, *Indian Journal of Pure and Applied Mathematics* **4** (1985), 373-375.

[4] O. Echi, On the product of primal spaces, *Quaestiones Mathematicae* **45**(1) (2022), 1-10.

[5] H. Maki, Generalized Λ-sets and the associated closure operator, *The Special Issue in Commemoration of Prof. Kazuada IKEDA's Retirement* (1986), 139-146.

[6] A. A. Salama, H. E. Khalid, A. K. Essa, and A. G. Mabrouk, "Exploring the Potential of Neutrosophic Topological Spaces in Computer Science," Neutrosophic Systems with Applications, vol. 21, pp. 63–97, Sep. 2024.

[7] J. Sanabria, C. Granados, L. Sánchez, Properties of co-local function and related Φ-operator in ideal neutrosophic topological space, *Neutrosophic sets and Systems* **63** (2024), 49-61.

[8] M. Kirişci and N. Şimşek, "Neutrosophic Metric Spaces," arXiv preprint arXiv:1907.00798, Jun. 2019.

[9] F. Smarandache, *A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic*, American Research Press: Rehoboth, USA, 1999.

[10] F. Smarandache, Foundation of revolutionary topologies: An overview, examples, trend analysis, research issues, challenges, and future directions, *Neutrosophic Systems with Applications* **13** (2024), 45-66.

[11] C. Uzcátegui, and J. Vielma, Alexandroff topologies viewed as closed sets in the Cantor cube, *Divulgaciones Matemáticas* **13** (2005), 45-53.

[12] M. Van de Vel, *Theory of convex structures*, North-Holland Mathematical Library: Amsterdan, The Netherlands, 1993.

[13] J. Vielma and Á. Guale, A topological approach to the Ulam-Kakutani-Collatz conjecture, *Topology and its Applications* **256** (2019), 1-6.

[14] T. Yokoyama, Preorder characterizations of lower separation axioms and their applications to foliations and flows, *arXiv 1708.06626* (2017), preprint.