Some New Results about Neutrosophic KU-Module

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Abstract

In this paper, we present new concept namely neutrosophic algebra. Some types of notions such as KU-module, KU-ideal and KU-submodule. We proved that if AI is minimal submodule, then AI ascending (descending) chain condition. On the other hand, more results about Neutrosophic exact sequence and Neutrosophic homomorphism KU-module have been presented.

Keywords: KU-Module; Maximal ideal; Neutrosophic Set; Neutrosophic ring; Neutrosophic module

1. Introduction

Neutrosophic logic [1], a novel extension of traditional fuzzy logic [2], provides a strong foundation for dealing with uncertainty, indeterminacy, and incomplete data. Neutrosophic modules arise as an important application of this logic in algebra [3, 4], where they broaden standard module theory to include neutrosophic concepts [5,6]. This adaptation enables a more in-depth understanding of mathematical structures, especially in situations characterized by ambiguity and vagueness. Recent research [7-10] has resulted in significant advances in our understanding of neutrosophic modules, showing novel characteristics and linkages that parallel classical algebraic structures. These advancements not only improve theoretical understanding, but also create new opportunities for practical applications in a variety of domains, data analysis, and artificial intelligence. This introduction will summarise some of the important findings from the study of neutrosophic modules, emphasising their relevance for both theoretical research and real-world applications. By investigating these novel findings, we gain insight into neutrosophic modules' capacity to address complicated issues while also contributing to the greater landscape of neutrosophic mathematics. Neutrosophic modules represent an extension of traditional module theory, incorporating the principles of neutrosophic logic. Neutrosophic KU-modules are algebraic structures that extend the idea of classical modules by integrating neutrosophic logic. They provide a framework for analyzing relationships and operations in environments characterized by uncertainty and vagueness. By defining operations on these modules, researchers can explore new avenues for problem solving across various fields, including mathematics, computer science, and artificial intelligence. The study of Neutrosophic KU-modules not only enriches the theoretical landscape of neutrosophic logic but also offers practical applications in modeling real-world systems where information is often incomplete or ambiguous. This introduction aims to outline the foundational concepts of Neutrosophic KU-modules, their properties, and their significance in advancing both theoretical and applied research in the realm of neutrosophic mathematics. Many scientists in this world have been pleased with this idea, so they have presented many works in many magazines. Among these scientists, are Al-Sharqi [7], Abed [16], Al-Quran [13] and others as their research works extended this theory. In this study, we present new concept namely neutrosophic algebra. Some types of notions such as KU-module, KU-ideal and KU-submodule. We proved that if AI is minimal submodule, then AI ascending (descending) chain condition. On the other hand, more results about Neutrosophic exact sequence and Neutrosophic homomorphism KU-module and Neutrosophic concept in [17, 18] have been presented.

2. Basic definitions and background

Definition 2.1: Let R be a commutative ring with identity. An a commutative group (M, +) is called a left R-module (or left module over R) if there is a mapping f: M×M ⟶M ∋ f(r,m) = rm $\forall\, r \in R$ *and* $m \in M$ satisfying the following conditions :-

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1) $f(r, m_1+m_2) = f(r, m_1) + f(r, m_2) \quad \forall r \in R, m_1, m_2 \in M$

2) $f(r_1+r_2, m) = f(r_1,m) + f(r_2,m) \quad \forall r_1,r_2 \in R, m \in M$

3) $f(r_1r_2, m) = r_1f(r_2, m) \quad \forall r_1,r_2 \in R, m \in M$

Definition 2.2: [1] $A_{FS} = \{ x(H_A(c)), c \in y \}$, where $H_A: Y \longrightarrow \wp([0, 1])$ the degree membership reflects the degree to which the element c satisfies the properties of A, and P([0, 1]) of [0, 1], is Fuzzy Set.

Definition 2.3: [2] Let T, I, F be standard or non-standard of]−0,1+[,

sup$T$= t_sup, inf T=t_inf,

sup I=i_sup, inf I=i_inf,

sup$F$=f_sup,inf F=f_inf,

and n_sup=t_sup+i_sup+f_sup,

n_inf = t_inf + i_inf +f_inf

T, I, F are called neutrosophic components

Definition 2.4: [3] If (H, ∗) be a group and let (H ∪ I) = {n + mI : n, m ∈ H}. N(H) =

((G ∪ I) , ∗) is neutrosophic group by H and I with the binary operation ∗. I is called the neutrosophic element, I^2 = I.

Definition 2.5: [4] If (R,+, .) be a ring. Then the set (R ∪ I) = {c + dI : c, d ∈ R}

is called a neutrosophic ring generated by R and I under the operations of R.

Definition 2.6: [5] *If we have* $({}_RM,+,\cdot)$ *as a left R− module over a ring R and we have*

PH(${}_RM$*) as a neutrosophic set by* ${}_RM$ *and I , the triple* PH(${}_RM$) =

(${}_RM$,+,·) *is defined as a weak neutrosophic R− module (WNML) over R*

Definition 2.7: [6] If M be an R-module.Then M is called neutrosophic submodule of M if the following conditions are satisfied:

1) K(0) = Z, i.e.,

$g_A(0) = 1, i_A(0) = 1, f_A(0) = 0.$

2) K(a + b) ≥A(a) ∧ A(b), for each a,b ∈ M i.e.,

$g_A(a + b) \geq g_A(a) \wedge g_A(b)$, $i_A(a + b)\ ) \geq i_A(a)\ ) \wedge i_A(b)$ and $f_A(a+b) \geq f_A(a) \wedge f_A(b)$.

3) K(ra)) ≥ K(a), for each a ∈ M; r ∈ R, i.e.,

$g_A(ra) \geq g_A(a)$, $i_A(ra) \geq i_A(a)$ and $f_A(ra) \leq fA(a)$.

And denoted by NSM(M) to the neutrosophic submodule of M.

3. Neutrosophic KU-module

In this section, we present new concept namely Neutrosophic KU-module. However, before that necessary define some tools and introduce known results about the topic.

Definition 3.1: Any triple (A,0,1) where 0 is a binary operation of type (2,0), we can named it neutrosophic KU-algebra (short N-KU-algebra) if:

a) (xI ∘ yI) ∘ [(yI∘zI) ∘ (xI ∘ zI)]=1I

b) xI ∘ 1I=1I

c) 1I ∘ xI=xI

d) xI ∘ yI=yI ∘ xI=1I,then xI=yI, ∀ xI,yI,zI ∈ A

Note that uI ≤ vI reffer to partial order (≤) inside A if and only if vI ∘ uI=1I.By based on definition of module in general and KU-module, we present a definition of Neutrosophic KU-module (short N-KU-module).

Definition 3.2 : For a Neutrosophic KU-algebra (A,*,0), an Neutrosophic abelian group GI with + and a mapping AI × GI ⟶ GI defined by (AI,GI) ⟶ AI × GI ∋ ;

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1)-(AI ∧ BI) GI=AI (BIGI)

2)-aI (G_1I+G_2I)= aIG_1 + aIG_2I

3)-0I . gI = 0I ∀ aI, bI ∈ G, G_1I,G_2I ∈ GI where aI ∧ bI=(aI ∗ bI)*bI, is called neutrosophic KU-module (N-KU-module).

Definition 3.3: Let a_1I and a_2I are two N-KU-modules. Then f:a_1I ⟶ a_2I is called neutrosophic homomorphism KU-module (N-H-KU-module) if :

1-f(G_1I + G_2I)= f(G_1I) + f(G_2I).

2-f(aIG_1I)= aIf(G_1I) ∀ aI ∈ A.

Definition 3.4: Let a_1I and a_2I are two N-KU-module and f:a_1I ⟶ a_2I is called neutrosophic epimorophism. If BI ≤ A_2I and A'= f^{-1}(BI) then $\frac{A_1'I}{A'I} \cong \frac{A_2'I}{BI}$. Moreover; if BI =[0] ,then $\frac{A_1'I}{Ker(f)} \geq A_2'$.

Let AI be an N-KU-module and A_1I,A_2I,A_3I are neutrosophic submodules of AI then $\frac{A_1I+A_2I}{A_3I}$ is isomorphic to $\frac{A_1I}{A_2I+A_3I}$. Next if A_3I ⊂ A_2I ⊂ A_1I, then $\frac{A_2I}{A_3I}$ is a neutrosophic submodule of $\frac{A_3I}{A_1I}$, we have $(\frac{A_1I}{A_3I})/(\frac{A_2I}{A_3I})$is isomorphic to $\frac{A_1I}{A_2I}$.

Theorem 3.5: An ideal in a bounded N-KU-algebra XI is always a submodule.

Proof: x_1I+x_2I = [(x_1I∗x_2I) ∧ (x_2I∗ x_2I)] in a N-KU-algebra I as a bounded.

By ideal of Neutrosophic KU-algebras, we have x_2I∗ x_1I ∈ I and x_1I∗ x_2I ∈ I.In a bounded N-KU-algebra, N-KU-ideals are lattice ideals. Hence x_1I+x_2I ∈ I. Further x_1I+0=0+x_1I=x_1 for all x_1 ∈ I.So, 0 is the identity, x_1I ∈ I, 0∈ I ⟹ x_1I+x_1I=0. That is x_1I is inverse of x_1I.(I,+) is a neutrosophic subgroup of bounded N-KU-algebra. Next x_1I ∈ X, x_2I ∈ I then x_1I ∧ x_2I = x_1Ix_2I ∈ I. Hence I ≤ XI.

Definition 3.6: AI is a N-KU-module. AI satisfy minimal (maximal) conditions for submodules of AI inside AI.

Definition 3.7: AI be N-KU-module of a N-KU-algebra. AI satisfy the decending chain of submodules if A_1I ⊇ A_2I ⊇ ……… ⊇ …….. A_nI ⊇ …….. is terminated in the sense A_kI =A_kI +1 for some k≥0.

Definition 3.8: AI is a module of a N-KU-algebra. AI satisfy maximal (minimal) condition for submodules if a collection of submodules inside AI.

Proposition 3.9: If AI is a N-KU-module then the following are equivalent:

a-AI satisfy the minimal (maximal) condition for submodules.

b-AI ascending (descending) chain are satisfied by AI.

Proof: (a) ⟹ (b). AI satisfies the maximal condition for neutrosophic submodules of a N-KU-module.

Further let A_1I ⊆ A_2I ⊆ A_3I ⊆ …… be given chain of submodules of AI. Next, AI=$\{A_i I | i \in I\}$

∋ A_iI is sequence. AI has maximal element, say A_uI A_mI=A_uI for m ≥u.

(b) ⟹ (a). AI (A.c.c) ∋ ≤for neutrosophic N-KU-module. Let ΩI be a set of neutrosophic submodule of MI. ΩI contain a max number. Let A_1I ∈ ΩI. Since A_1I is not maximal in ΩI, A_2I ∈ ΩI with A_1I ⊂ A_2I. However, A_2I is not maximal in ΩI. Consequently; there exist A_3I ∈ ΩI, for which A_2I ⊂ A_3I .A_1I ⊆ A_2I ⊆ A_3I ⊆ …….. is generated

Theorem 3.10: Let N_1I,N_2I ,K_1I and K_2I are submodules of a N-KU-module so that N_2I ⊂ N_1I and K_2I ⊂ K_1I. Then [N_2I + (N_1I ∩ K_1I)] / [N_2I + N_1I ∩ K_2I]=[K_2I +(K_1I ∩ N_1I)] / [K_2I +(K_1I ∩ N_2I)]

Proof: Let A_1I = N_2I +(N_1I ∩ K_2I), A_2I =N_1I ∩ K_1I.Then A_1I+A_2I = N_2I + N_1I ∩ K_1I. Know, we have A_1I ∩ A_2I =[N_2I +(N_1I ∩ K_2I)] ∩ (N_1I ∩ K_1I)= N_2I ∩ K_1I + N_1I ∩ K_2I. By Isomorphism theorem, we have A_1I + A_2I /A_1I≅ A_2I/A_1I ∩ A_2I[N_2I + N_1I ∩ K_1I] / N_2I + (N_1I ∩ K_2I) ≅ [N_1I ∩ K_1I] / [N_1I ∩ K_2I + (N_2I ∩ K_1I). Also, [K_2I + (K_1I ∩ N_1I)] / [K_2 I+ (K_1I ∩ N_2I)] ≅ [K_1I ∩ N_1I] / [N_1I ∩ K_2I +(N_2I ∩ K_1I)].

Theorem 3.11: For two chains of a N-KU-module we have,

NI=A_0I ⊂ A_1I ⊂ … ⊂ A_tI = MI. Moreover, NI = $A_0' I \subset A_1' I \subset \ldots . \subset A_s' I$ = MI, have equal length and are divisible by a given factor. A_{1+I} I / A_iI of isomorphism to M_{i+1}'I / $M_j' I$.

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Definition 3.12: suppose that $AI \neq 0$ is a neurtosophic KU-module, a finite descending chain of neutorosophic submodule of AI starting with AI with (0I) is a N-KU-composition series. i.e. $MI = A_0I \supset A_1I \supset .. \supset A_mI = (0I)$ in such $A_iI / A_{i+1}I$ are simple for all $i, 0 < I < m-1$.

4-Exact Sequences in N-KU-module:

In a homology of N-KU-modules exact sequences is mentioned in this part.

Definition 4.1: Let $f_1: A_1I \longrightarrow A_2I$ and $f_2: A_2I \longrightarrow A_3I$ be a homomorphism of N-KU-modules. Then $A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I$ is exact sequence if $Im(f_1)= \ker f_2$. It is semi exact sequence, $Im(f_1) \supset Ker(f_2)$.

Remark 4.2: i-If f_1I is one-one at A_1I, then $0I \longrightarrow A_1I \xrightarrow{f_1} A_2I$ is exact sequence.

ii-If f_1I is onto at A_2I then $A_1I \xrightarrow{f_1} A_2I \longrightarrow 0I$ is exact sequence.

Theorem 4.3: In the homology of a N-KU-module with an exae.seq. $A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \xrightarrow{f_3} A_4I$, the below are equivalent:

i-f_1I is an epi.

ii-f_2I is the trivial homo.

iii-f_3I is monomorohism.

Theorem 4.4: Let $f_1 : A_1I \longrightarrow A_2I$ and $g: A_2I \longrightarrow A_3I$ be two homo. of N-KU-module. Then it is trivial iff $Im(f_1) \subseteq Ker(f_2)$.

Proof : Let f_3 is trivial homo. and let A_2I be any element of $Im(f_1)$. We have that there exists $A_1I \in A_1I$ such that $f_1(A_1I)= A_2I$ and hence $f_2(A_2I) = f_2(f1(A_1I))=f_3(A_1I) = 0I$. Thus $A_2I \in Ker(h_2)$. This proves that $Im(f_1) \supset Ker(f_2)$.

Consider $Im(f_1) \subset Ker(f_2)$ and let A_1I be any element of A_1I. Then we have $f_3(A_1I)= f_2(f_1(A_1I))$.

Since $f_1(A_1I) \in Im(f_1) \supset Ker(f_2)$. Then we have $f_2(f_1(A_1I)) =0 \Longrightarrow$ that $f_3(A_1I) = 0$ for all $A_1I \in A_1I$.

This prove that f_3 is a trivial homo.

Lemma 4.5: Considering A_1I, A_2I, A_3I be N-KU-modules and let $f_3: A_1I \longrightarrow A_2I$ be an epi., and let $f_2: A_1I \longrightarrow A_3I$ be a homo. If $Ker(f_3) \subseteq Ker(f_2)$, then there is a unique homo. $f_1: A_2I \longrightarrow A_3I$ satisfying $f_1 * f_2 = f_3$.

Proof: Let $A_2I \in A_2I$. Since f_3 is an epimorphism hence $A_1I \in A_1I$ and $A_2I \in f_3(A_1I)$. We have the following mapping equations:

$$A_1I \xrightarrow{f_3} A_2I \xrightarrow{f_1} A_3I \text{ and } A_1I \xrightarrow{f_2} A_3I \text{ ....1}$$

$$A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_3} A_3I \text{ and } A_2I \xrightarrow{f_2} A_3I \text{ ....2}$$

$$A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \longrightarrow 0I \text{ and } A_2I \xrightarrow{f_3} A_4I \text{ and } A_3I \xrightarrow{f_4} A_4I \text{ ....3}$$

$$A_4 \xrightarrow{f_4} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \text{ and } 0I \longrightarrow A_1I \text{ and } A_4I \xrightarrow{f_2} A_1I \text{.......4}$$

$$A_1I \xrightarrow{\alpha} A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \text{ and } A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \xrightarrow{y} A_3I \text{ and } A_2I \xrightarrow{\beta} A_2I \text{.....5}$$

$$A_1I \xrightarrow{\alpha} A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \text{ and } A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \xrightarrow{Y} A_3I \text{ and } A_2I \xrightarrow{\beta} A_2' \text{ ....6}$$

Define $f_1 : A_2I \longrightarrow A_3I$ by $f_1(A_2I)= f_2(A_1I)$, where $A_2I = f_3(A_1I)$.

Next, $f_1 \circ f_3 = f_2$. Let $f_1: A_2I \longrightarrow A_3I$ so that $f_1f_3 = f_1' f_3$. Then $f_1 = f_1'$. Hence f_1 is unique.

Proposition 4.6: Let A_1I, A_2I, A_3I be N-KU-modules and let $f_2: A_1I \longrightarrow A_3I$ is a homo. And

$f_3 : A_2I \longrightarrow A_3I$ is a homo. with $Im(f_2) \subset Im(f_3)$. Then $\exists$ a homo.

$f_1 : A_1I \longrightarrow A_2I$ that is unique and satisfy $f_2 = f_3 \circ f_1$ i.e. equation (2) commutes.

Proof: For every $A_1I \in A_1I$, $f_2(A_1I) \in A_3I$ and hence $f_2(A_1I) \in Im(f_2)$. Since $Im(f_2) \subseteq Im(f_3)$ and f_3 is a monomorphism, $\exists$ a unique $A_2I \in A_2I$ so that $f_3(A_2I) = f_2(A_1I)$. Therefore, there is a function $f_1 : A_1I \longrightarrow A_2I$, $A_1I \longrightarrow A_2I$ so that $f_2 = f_1 \circ f_3$. Hence f_3 is a unique homomorphism so that $f_2 = f_1 \circ f_3$.

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Theorem 4.7: In a N-KU-module with f_1 and f_2 as exact homo. With equation (5), and $f_1 \circ f_3 = 0$, we have a unique homomorphism $f_4 : A_3I \longrightarrow A_4I$ so that $f_2 \circ f_4 = f_3$.

Proof : Since $f_1 \circ f_3 = 0$. We get that Ker(f_2) = Im(f_1) ⊂ Ker(f_3). By using proposition **(5.6)**, there is a unique homomorphism $f_4 : A_4 I \longrightarrow A_1I$ satisfying $k \circ f = f_3$.

Theorem 4.8: Equation 4 of homo. of N-KU-module, f_1 and f_2 are exact homomorphism and $f_3 \circ f_2 = 0$, then there is a unique homo. $f_4 : A_4I \longrightarrow A_1I$ that satisfy $f_4 \circ f_1 = f_3$.

Proof: $f_3 \circ f_2 = 0$, so we have Im(f_3) ⊂ Ker (f_2) = Imf. Eq(5), there is a unique homomorphism $f_4 : A_4I \longrightarrow A_1I$ satisfying $k \circ f_1 = f_3$.

Theorem 4.8: consider A_1I, A_2I, A_3I and A_1', A_2', A_3' be N-KU-module over XI in equation (6) given homomorphism commutes and exact. If α,y and f' are homomorphism, then β is also monomorphism.

Proof: We see the homomorphism commutes, $f'\alpha = \beta f$ and $g'\alpha = yg$ and it's exact.

Thus Im(f)= Ker(g) and Im(f') = Ker (g').

Let A_2I ∈ Ker(β). Since they are commutative, we have $yg(A_2I) = g'\beta(A_2I) = 0$, $g(A_2I) = 0$, A_2I ⊆ Ker g = Im f.

$A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I \xrightarrow{\alpha_3} A_3'I \xrightarrow{\beta_3} A_3I$ and $A_1I \xrightarrow{\alpha_1} A_1'I \xrightarrow{\beta_1} A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I$ …………..7

$A_2I \xrightarrow{\alpha_2} A_2'I \xrightarrow{\beta_2} A_2I$ and $A_1'I \xrightarrow{f_1'} A_2'I \xrightarrow{f_2'} A_3'I$ …………8

By equation(7) and the equation(8) commutes, f_1 is a monomorphism, hence $A_2I = 0$.

Theorem 4.9: Let A_1I, A_2I, A_3I and $A_1'I$, $A_2'I$, $A_3'I$ are N-KU-modules over XI, as in equation (6) and α,β,y are isomorphism. If equation (6) is exact so the third using, is exact.

Proof : $f_1'\alpha = \beta f_1$ and $f_2'\beta = y f_2$. As Im f_1' = Ker f_2'. By α,β,y are isomorphisms, Im f1 = Ker f_2.

Theorem 4.10: Let the Equation (7)and (8) be equations related to N-KU-modules and homomorphism with $\beta_i\alpha_i$ = I_i, where I_i' denote associated, from I = 1,2,3. If the first part of the equation (8) is exact, then $A_1I \xrightarrow{f_1} A_2I \xrightarrow{f_2} A_3I$ is also exact.

Proof: Since the equation (7)and (8) commutes, hence

$$f_1'\alpha_1 = \alpha_1 f_1 and\ f_2'\alpha_2 = \alpha_3 f_2 \ldots\ldots\ldots\ldots\ldots\ldots.9$$

$$f_1\beta_1 = \beta_2 f_1' and\ f_2\beta_2 = \beta_3 f_2'$$

As the middle row $A_1I \xrightarrow{f_1'} A_2I \xrightarrow{f_2'} A_3I$ is exact, therefore Imf_1' = ker f_2'. To satisfy equation (7) and equation (8) are also exact, Im(f_1) = ker(f_2). Let A_i ∈ A_iI, and consider $\alpha_3 f_2 f_1(A_1I) = f_2'(\alpha_2) f_1(A_1I) = f_2' f_1' \alpha_1(A_1)$ because $\alpha_2 f_1 = f_1'\alpha_1$ (by equation 9). Since Ker f_2' = Im f_1, $f_2' f_1'(\alpha_1(A_1)) = 0$, and hence $\alpha_3 f_2 f_1(A_1I) = 0$. Thus $\alpha_3 f_2 f_1 = 0$ and hence $\beta_3\alpha_3 f_2 f_1 = 0$. Which shows that $f_2 f_1 = 0(\beta_3\alpha_3 = I_3)$. Thus shows that Im$f_1$ ⊆ Kerf_2. Consider xI∈ Ker f_2, we get then f_2(xI) = 0 ; $\alpha_3 f_2$(xI) = 0. As $f_2'\alpha_2 = \alpha_3 f_2$, therefore, α_2(xI) ∈ Ker f_2' and hence α_2(XI) ∈ ker f_2' = Imf_1', therefore α_2(xI) ∈ Imf_1' and hence $\alpha_2(xI) = f_1'(m_1'I)$ for some $m_1' \in A_1'$.

$0 \longrightarrow A_1' \xrightarrow{\alpha'} A_1 \xrightarrow{\beta'} A_1'' \longrightarrow 0$ and $0 \longrightarrow A_2' \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_2'' \longrightarrow 0$ and $0 \longrightarrow A_2' \xrightarrow{\alpha'} A_3 \xrightarrow{\beta''} A_3'' \longrightarrow 0$ …10

And;

$0 \longrightarrow A_1' \xrightarrow{f'} A_2' \xrightarrow{g'} A_3' \longrightarrow 0$ and $0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$ and $0 \longrightarrow A_1'' \xrightarrow{f''} A_2'' \xrightarrow{g''} A_3'' \longrightarrow 0$ …..11 $\beta_2\ \alpha_2$ (xI) = $\beta_2 f'(m'_1)$.As $\beta_2\alpha_2 = I_2$. Therefore x = $\beta_2 f'(m_1')$ = $f(\beta_1(m_1'))$ by Eq. (9)

This shows that xI∈ Imf. Therefore; Ker g ⊆ Im f that ⟹ Ker g = Im f.

Theorem 4.11: Let A_1I, $A_1'I$, $A_1''I$, A_2I, $A_2'I$, $A_2''I$, A_3I, $A_3'I$, $A_3''I$ be an N-KU-modules over XI as in the equation (10) and (11). If eq.(11) is exact, and eq.(10) are exact, then ∃ homo. α'': $A_3'I \longrightarrow A_3I$ and β'': $A_3I \longrightarrow A_3''I$ such that the sequence $0 \longrightarrow A_3'I \xrightarrow{\alpha''} A_3I \xrightarrow{\beta''} A_3''I \longrightarrow 0$ is semi-exact.

Proof: Since eq.(11)is exact, we have Im(f_1') = ker (f_2'), Im(f_1) = Ker(f_2), Im(f_1'') = ker(f_2'')(2). Also eq. (5) is exact, we have Imα'= kerβ' and Imα = ker β (3). $\alpha f_1' = f_1\alpha'$, $f_1\beta = f_1''\beta'$ (4). Im(α'') ⊂ ker(β''). α'', α and f_1', f_1, f_1'' are one to one. Similarly f_2', f_2, f_2'' and β', β are onto. Now we have $f_2\alpha f_1' = f_2 f_1\alpha' = 0$ $\alpha' = 0$(Imf = ker f_2). α'':A_3' I ⟶ A_3I by $\alpha''(m_3')$ = A_3I where $f_2\alpha(m_2')$ = A_3I and $f_2'(m_2')$ = (m_1')(f_2' is an epimorphism). Then α'' is well-defined and indeed homo., $f_2\alpha(m_2') = \alpha'' f_2'(m_2')$and hence $f_2\alpha = \alpha'' f_2'$. Now we prove the uniqueness of α''.

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Since $\alpha'' f_2' = f_2\alpha$, ∃ homo. $\alpha_0 : A_3'I \longrightarrow A_3I \ni f_2\alpha = \alpha_0 f_2'$. We will show that $\alpha_0 = \alpha''$.

Let $m_3'I \in A_3I$. Then as f_2' is onto, we have $m_3'I = f_2'(m_2'I)$ for some $m_2'I \in A_2I$. Thus, $a_0(m_3'I) = a_0 f_2'(m_2'I)$. Similarly $a''(m_3'I) = a'' f_2''(m_2''I)$. Since $f_2a = a_0 f_2', f_2 a = a'' f_2'$ and $f_2 a(m_2'I) = a'' f_2'(m_2'I) f_2 a(m_2'I) = a_0 f_2'(m_2'I)$ which that $a_0(m_2'I) = a''(m_3'I)$ for all $m_3'I \in A_3I$. Thus $a_0 = a''$. Thus prove the uniqueness of a''.

Define $\beta'' : A_3I \longrightarrow A_3''I$ by $\beta''(A_3I) = m_3''I$, where $f_2''\beta(A_2I) = m_3''$ and $f_2(A_2I) = A_3I$. Then β'' is well-defined and indeed homomorphism. Moreover, $f_2''\beta(A_2I) = \beta'' f_2(A_2I)(by (3))$ therefore, $f_2''\beta = \beta'' f_2$.

The uniqueness of β'' can be established on the lines similar to α'' .

Let $m_3' \in Ker(\alpha'')$, since f_2' is surjective, we see that there exists $m_2'I \in M_2'I$ such that $f_2'(m_2'I) = m_3'I$. Thus, by definition of α', $0 = \alpha''(m_3'I) = \alpha'' f_2'(m_2'I) = f_2\alpha(m_2'I)$. Thus shows that $\alpha(m_2'I) \in \ker(f_2) = Im(f_1)$(by the sequence).Therefore $\alpha(m_2'I) = f_1(A_1I)$ for some $A_1I \in A_1I$. By the exactness of middle row $\beta\alpha(m_2'I) = 0$. Since $0 = \beta\alpha(m_2'I) = \beta f(A_1I) = f_1''\beta'(A_1I)(by (3))$. Therefore $\beta'(A_1I) \in \ker(f_1'') = 0$ and hence $A_1 \in \ker(\beta') = Im(\alpha')$(by the exactness of the sequence)which that $A_1I = \alpha'(m_1'I)$ for some $m_1'I \in A_1'I$. Therefore, we have $\alpha(m_2'I) = f_1(A_1I) = f_1\alpha'(m_1'I) = \alpha f_1'(m_1'I)$.

Since α is a monomorphism, $m_2'I = f_1'(m_1'I)$, it follows that $m_3'I = f_2'(m_2'I) = f_2'f_1'(m_1'I) = 0$ ($Imf' =$ Ker$(\alpha'') = \{0I\}$.i.e. α'' is monomorphism.

Next, since $\beta'' f_2 = f_2''\beta$ and since f_2'' , are each surjective, so $\beta'' f_2$ is surjective, which indeed that β'' is surjective . Finally, we have $\beta''\alpha'' f_2' = f_2''\beta\alpha = f_2''0 = 0(ker\beta = Im\alpha)$. Since f_2' is surjective, then indeed $f_2'(A_2') = A_3'$ and $\beta''\alpha'' f_2'(A_2'I) = \beta''\alpha''(A_3'I) = 0$.So $\beta''\alpha''$is zero mapping, $\beta''\alpha'' = 0$. Hence Im$(\alpha'') \subseteq \ker(\beta'')$.

5. Conclusion

In this work, a new algebraic formula is presented by linking several previous algebraic theorems and concepts. This formula is called Neutrosophic KU-Module. Many results were presented in this work, where many algebraic theorems, theorems and algebraic definitions were presented that helped in understanding these developed algebraic concepts. We proved that if AI is minimal submodule, then AI ascending (descending) chain condition. On the other hand, more results about Neutrosophic exact sequence and Neutrosophic homomorphism KU-module have been presented. Finally, to provide many future works, these algebraic definitions can be combined with other works that can be viewed through [20-22].

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