Fixed Points Results in Algebra Fuzzy Metric Space with an Application to Integral Equations

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Abstract

This paper introduces a new class of mappings termed $(\widehat{\alpha},\widehat{\beta})-\Omega$-contraction mapping (briefly, "$(\widehat{\alpha},\widehat{\beta})-\Omega-$CMap") and establishes certain fixed-point (FP) results in the framework of Algebra fuzzy metric space. Additionally, we expanded our results to include the existence of a nonlinear integral equation solution. Results from this study improve, expand and generalization certain previously published results in the literature.

Keywords: Algebra Fuzzy Metric Space; Fixed-point theorem; Contraction mapping; Cauchy sequence

1. Introduction

In nonlinear functional analysis, the theory of FPs has been crucial over the last three to four decades. Indeed, FP theory is used to deal with several problems in real life. It is commonly recognized that the study of FP theory relies heavily on contractive conditions, and one of the key findings in the analysis is Banach's fixed-point theorem for contraction mappings. Several researchers have generalized and expanded this theorem (see, [1–8]). On the other hand, fuzzy mathematics initially appeared when fuzzy sets were introduced in Zadeh's seminal article [9] to represent the ambiguity in daily life. Numerous mathematical problems have become much fuzzier because of studying fuzzy sets, and they may be used in a variety of fields, including coding theory, image processing, and gaming. Scientific publications on fuzzy sets have been given by several researchers; for instance, see [10-15] Kramosil and Michalek [16] initially presented the concept of a fuzzy metric space (FMs). Grabiec [17] was the first to begin studying FP theory in FMs in 1988. Following that, a large number of researchers investigated FP theories in FMs. In [18], Fang provides several new FP theorems for mappings of contractive type in FMs. Authors in [19] introduce the definition of $\alpha-\varphi$-and $\beta-\psi$- fuzzy contractive mapping and prove some FP theorems for these two types of mappings. Huang et al.[20] introduce fuzzy F-contraction mapping and present some FP results. Additional studies on FPs in various fuzzy spaces can be found in [21-30].

In this paper, we construct a new form of contraction mapping and investigate some FP theorems related to this type of mapping. Furthermore, we provide some particular instances and implementations of our preliminary findings.

2. Preliminaries

This section includes the terms and results used throughout the paper.

Definition 2.1[31]: Consider $\widetilde{\triangle}$: $[0,1]\times[0,1]\rightarrow[0,1]$ be an operation. If this operation meets the following requirements for any $\ell,\mathcal{v},\mathcal{r},\mathrm{e}\in[0,1]$ it is referred to as a t-conorm:

(1) $0\ \widetilde{\triangle}\,\mathcal{v}=\mathcal{v}$,

(2) $\mathcal{v}\,\widetilde{\triangle}\,\mathrm{e}=\mathrm{e}\,\widetilde{\triangle}\,\mathcal{v}$,

DOI: https://doi.org/10.54216/IJNS.250433

(3) $\upsilon \,\tilde{\triangle}\, (\Upsilon \,\tilde{\triangle}\, e) = (\upsilon \,\tilde{\triangle}\, \Upsilon) \,\tilde{\triangle}\, e$,

(4) If $\upsilon \le e$ and $\Upsilon \le \ell$ then $\upsilon \,\tilde{\triangle}\, \Upsilon \le e \,\tilde{\triangle}\, \ell$.

Lemma 2.2 [31]: Consider $\tilde{\triangle}$ be a t-conorm on [0,1] then:

(1) $1 \,\tilde{\triangle}\, 1 = 1$,

(2) $0 \,\tilde{\triangle}\, 1 = 1 \,\tilde{\triangle}\, 0 = 1$,

(3) $0 \,\tilde{\triangle}\, 0 = 0$,

(4) $\Upsilon \,\tilde{\triangle}\, \Upsilon \ge \Upsilon \;\forall \Upsilon \in [0,1]$.

Example 2.3[31]: For all $\ell, \Upsilon \in [0, 1]$, the algebra product $\ell \,\tilde{\triangle}\, \Upsilon = \ell + \Upsilon - \ell\Upsilon$ represents a continuous t-conorm.

Definition 2.4[32]: A triple $(\mathcal{P}, \hat{\mathcal{A}}, \tilde{\triangle})$ is termed as Algebra fuzzy metric space (briefly "AFMS") if $\mathcal{P} \ne \emptyset$, $\tilde{\triangle}$ is a continuous t-conorm and $\hat{\mathcal{A}}: \mathcal{P} \times \mathcal{P} \to [0,1]$meeting the requirements given below:

(a1)$0 \le \hat{\mathcal{A}}(\varpi, \mu) \le 1$;

(a2) $\hat{\mathcal{A}}(\varpi, \mu) = 0$ if and only if $\varpi = \mu$;

(a3) $\hat{\mathcal{A}}(\varpi, \mu) = \hat{\mathcal{A}}(\mu, \varpi)$;

(a4) $\hat{\mathcal{A}}(\varpi, u) \le \hat{\mathcal{A}}(\varpi, \mu) \,\tilde{\triangle}\, \hat{\mathcal{A}}(\mu, u)$.

for all $\varpi, \mu, u \in \mathcal{P}$.

Example 2.5[32]: If $(\mathcal{P}, d)$ is a metric space and $\ell \,\tilde{\triangle}\, \Upsilon = \ell + \Upsilon - \ell\Upsilon \;\forall \ell, \Upsilon \in [0, 1]$. Put $\hat{\mathcal{A}}_d(\varpi, \mu) = \frac{d(\varpi,\mu)}{1+d(\varpi,\mu)}$ for all $\varpi, \mu \in \mathcal{P}$. Then $(\mathcal{P}, \hat{\mathcal{A}}_d, \tilde{\triangle})$ is AFMS.

Proposition 2.6[32]: Let $(\mathcal{P}, \hat{\mathcal{A}}, \tilde{\triangle})$ be an AFMS. Then $\varpi_n \to \varpi$ if and only if $\hat{\mathcal{A}}(\varpi_n, \varpi) \to 0$.

Definition 2.7[32]: In AFMS $(\mathcal{P}, \hat{\mathcal{A}}, \tilde{\triangle})$ a sequence (ϖ_n) is Cauchy if for each $\delta \in (0,1)$ then there is N such that $\hat{\mathcal{A}}_d(\varpi_n, \varpi_m) < \delta$ for every $m, n \le N$.

Definition 2.8[32]: An AFMS $(\mathcal{P}, \hat{\mathcal{A}}, \tilde{\triangle})$ is called fuzzy complete if (ϖ_n) is a fuzzy Cauchy sequence then $\varpi_n \to \varpi \in \mathcal{P}$.

Definition 2.9[32]: Consider R represents the real number field and let $\tilde{\triangle}$ represent a continuous t-norm. An algebra fuzzy absolute value on R is a fuzzy set ∂_R, which is a function $\partial_R: R \to [0,1]$, provided that it meets the following requirements for every $\mathscr{p}, \mathscr{q} \in R$:

(1) $\partial_R \in [0,1]$,

(2) $\partial_R(\mathscr{p}) = 0$ if and only if $\mathscr{p} = 0$,

(3) $\partial_R(\mathscr{p}\mathscr{q}) \le \partial_R(\mathscr{p}) \cdot \partial_R(\mathscr{q})$,

(4) $\partial_R(\mathscr{p} + \mathscr{q}) \le \partial_R(\mathscr{p}) \,\tilde{\triangle}\, \partial_R(\mathscr{q})$.

Then the triple $(R, \partial_R, \tilde{\triangle})$ is termed as algebra fuzzy absolute value space.

Example 2.10[32]: Let $\mathcal{P}$=C[a,b] and $\ell \,\tilde{\triangle}\, \Upsilon = \ell + \Upsilon - \ell\Upsilon$ for all $\ell, \Upsilon \in [0, 1]$. Define $\hat{\mathcal{A}}(\mathscr{f}, \mathscr{g}) = \max_{t\in[a,b]} \partial_R[\mathscr{f}(t) - \mathscr{g}(t)]$ then $(\mathcal{P}, \hat{\mathcal{A}}, \tilde{\triangle})$ is complete AFMS.

Definition 2.11[33]: Let $\ne \emptyset$, $\Im: \mathcal{P} \to \mathcal{P}$ is mapping and $\hat{\alpha}, \hat{\beta}: \mathcal{P} \times \mathcal{P} \to R^+$ be two functions. A mapping $\Im$ is termed as $(\hat{\alpha}, \hat{\beta})$-cyclic admissible (briefly "$(\hat{\alpha}, \hat{\beta})$-CA ") if for all $\varpi, \mu \in \mathcal{P}$,

1. $\hat{\alpha}(\varpi, \mu) \ge 1 \Rightarrow \hat{\beta}(\Im\varpi, \Im\mu) \ge 1$
2. $\hat{\beta}(\varpi, \mu) \ge 1 \Rightarrow \hat{\alpha}(\Im\varpi, \Im\mu) \ge 1$

Lemma 2.12[33]: Let $\mathcal{P} \ne \emptyset$ and $\Im: \mathcal{P} \to \mathcal{P}$ be $(\hat{\alpha}, \hat{\beta})$-CA mapping. Consider there is $\varpi_\circ \in \mathcal{P}$ with $\hat{\alpha}(\varpi_\circ, \Im\varpi_\circ) \ge 1$ and $\hat{\beta}(\varpi_\circ, \Im\varpi_\circ) \ge 1$. Suppose the sequence (μ_n) define by $\mu_{n+1} = \Im\mu_n$, then $\hat{\alpha}(\varpi_m, \varpi_{m+1}) \ge 1$ implies that $\hat{\beta}(\varpi_n, \Im\varpi_{n+1}) \ge 1$ and $\hat{\beta}(\varpi_m, \varpi_{m+1}) \ge 1$ implies that $\hat{\alpha}(\varpi_n, \Im\varpi_{n+1}) \ge 1$ for each $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.

DOI: https://doi.org/10.54216/IJNS.250433

3. Main Results

In the framework of AFMS, we present $(\hat{\alpha},\hat{\beta})-\Omega-$CMap in this section and establish the existence theorems for FPs of this mapping class.

Let $(\mathcal{P},\hat{\mathcal{A}},\widetilde{\triangle})$ be an AFMS and $\Im:\mathcal{P}\to\mathcal{P}$ be a mapping. A point $\varpi\in\mathcal{P}$ is termed as a FP for $\Im$ if $\Im(\varpi)=\varpi$. A mapping $\Im:\mathcal{P}\to\mathcal{P}$ is referred to as a fuzzy contraction on $\mathcal{P}$ if there is $\delta\in(0,1)$ with $\hat{\mathcal{A}}(\Im(\varpi),\Im(\mu))\leq\delta\,\hat{\mathcal{A}}(\varpi,\mu)$ for all $\varpi,\mu\in\mathcal{P}$.

Lemma 3.1. Suppose $(\mathcal{P},\hat{\mathcal{A}},\widetilde{\triangle})$ is AFMS and consider (ϖ_n) be a sequence in $\mathcal{P}$ such that $\hat{\mathcal{A}}(\varpi_{\mathfrak{n}},\varpi_{\mathfrak{n}+1})\to 0$. If (ϖ_n) is not a Cauchy then there exists $\sigma>0$ and sequences of positive integers $(\mathfrak{m}_\kappa)$ and $(\mathfrak{n}_\kappa)$ where $\mathfrak{m}_\kappa>\mathfrak{n}_\kappa>\kappa$ with $\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})\geq\sigma$, $\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa})<\sigma$ and,

(a) $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})=\sigma$

(b) $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})=\sigma$

(c) $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa})=\sigma$

Proof: If (ϖ_n) is not a Cauchy then there is $\sigma>0$ and sequences $(\mathfrak{m}_\kappa)$ and $(\mathfrak{n}_\kappa)$ with $\mathfrak{m}_\kappa>\mathfrak{n}_\kappa>\kappa$ satisfying:

$$\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})\geq\sigma \tag{1}$$

We choose $\mathfrak{m}_\kappa$, the least positive integer satisfying (1). Then we have

$$\mathfrak{m}_\kappa>\mathfrak{n}_\kappa>\kappa \text{ with } \hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})\geq\sigma \text{ and } \hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa})<\sigma \tag{2}$$

We now prove (a). By using the triangle inequality, we have

$$\sigma\leq\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})\leq\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{m}_\kappa-1})\,\widetilde{\triangle}\,\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})\,\widetilde{\triangle}\,\hat{\mathcal{A}}(\varpi_{\mathfrak{n}_\kappa+1},\varpi_{\mathfrak{n}_\kappa})$$

By taking the limit as $\kappa\to\infty$, we get:

$$\sigma\leq\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{m}_\kappa-1})\,\widetilde{\triangle}\,\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})\,\widetilde{\triangle}\,\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{\mathfrak{n}_\kappa+1},\varpi_{\mathfrak{n}_\kappa})$$

Now, on using $\hat{\mathcal{A}}(\varpi_{\mathfrak{n}},\varpi_{\mathfrak{n}+1})\to 0$ as $\mathfrak{n}\to\infty$ and using condition $(0\,\widetilde{\triangle}\,v=v \text{ for all } v\in[0,1])$ one get:

$$\sigma\leq\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1}) \tag{3}$$

Now

$$\begin{aligned}\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})&\leq\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa})\,\widetilde{\triangle}\,\hat{\mathcal{A}}(\varpi_{\mathfrak{n}_\kappa},\varpi_{\mathfrak{n}_\kappa+1})\\&<\sigma\,\widetilde{\triangle}\,\hat{\mathcal{A}}(\varpi_{\mathfrak{n}_\kappa},\varpi_{\mathfrak{n}_\kappa+1})\end{aligned}$$

Now taking limit as $\kappa\to\infty$:

$$\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})\leq\sigma\,\widetilde{\triangle}\,\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})=\sigma.$$

Therefore, $\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})\leq\sigma.$ (4)

From (3) and (4), we get

$$\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})=\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})=\sigma$$

So that $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})$ exists and $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa+1})=\sigma$

Thus (a) holds.

(b) We have, $\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})\geq\sigma$, and hence

$$\sigma\leq\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa}) \tag{5}$$

Now $\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{n}_\kappa})\leq\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{m}_\kappa-1})\,\widetilde{\triangle}\,\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa-1},\varpi_{\mathfrak{n}_\kappa})$

$$\leq\hat{\mathcal{A}}(\varpi_{\mathfrak{m}_\kappa},\varpi_{\mathfrak{m}_\kappa-1})\,\widetilde{\triangle}\,\sigma$$

This implies,

$$\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{n_\kappa})\le\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{m_\kappa-1})\,\tilde{\triangle}\,\sigma$$

So that

$$\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{n_\kappa})\le\sigma \qquad (6)$$

From (5) and (6), we get

$$\sigma\le\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{n_\kappa})\le\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{n_\kappa})\le\sigma$$

Hence $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{n_\kappa})=\sigma$. Thus (b) holds.

Now to prove (c) . We have, $\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})<\sigma$, and hence

$$\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})\le\sigma \qquad (7)$$

Now,

$$\sigma\le\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{n_\kappa})\le\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{m_\kappa-1})\,\tilde{\triangle}\,\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})$$

Thus $\sigma\le\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{m_\kappa},\varpi_{m_\kappa-1})\,\tilde{\triangle}\,\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})$

using $\hat{\mathcal{A}}(\varpi_n,\varpi_{n+1})\to 0$ as $n\to\infty$ one get

$$\sigma\le\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa}) \qquad (8)$$

From (7) and (8), one gets

$$\sigma\le\lim_{\kappa\to\infty}\sup\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})\le\lim_{\kappa\to\infty}\inf\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})\le\sigma.$$

Hence $\lim_{\kappa\to\infty}\hat{\mathcal{A}}(\varpi_{m_\kappa-1},\varpi_{n_\kappa})=\sigma$. Thus (c) holds.

Definition 3.2. Let $(\mathcal{P},\hat{\mathcal{A}},\tilde{\triangle})$ is AFMS , $\hat{\alpha},\hat{\beta}:\mathcal{P}\times\mathcal{P}\to[0,1)$ be two functions and $\Im:\mathcal{P}\to\mathcal{P}$ be a mapping. $\Im$ is termed as $(\hat{\alpha},\hat{\beta})-\Omega-\text{CMap}$, if there exists $\mathcal{G}>0$ such that for all $\varpi,\mu\in\mathcal{P}$ with $\Im\varpi\ne\Im\mu$; we have

$$\hat{\alpha}(\varpi,\Im\varpi)\hat{\beta}(\mu,\Im\mu)\ge 1\ \Rightarrow\hat{\mathcal{A}}(\Im\varpi,\Im\mu)\ \le\Omega\left(\hat{\mathcal{A}}(\varpi,\mu)\right)+\mathcal{G}\ \ \min\{\hat{\mathcal{A}}(\varpi,\Im\varpi),\hat{\mathcal{A}}(\mu,\Im\mu),\hat{\mathcal{A}}(\varpi,\Im\mu)\hat{\mathcal{A}}(\mu,\Im\varpi)\} \qquad (9)$$

where $\Omega:\ [0,1)\to\ [0,1)$ is a continuous and $\Omega(w)<w$ for any $w>0$ and $\Omega(0)=0$.

Example 3.3. Assume $\mathcal{P}=[0,\infty)$ and let $(\mathcal{P},d)$ be a metric space where $d(\varpi,\mu)=|\varpi-\mu|$. Put $\hat{\mathcal{A}}_d(\varpi,\mu)=\frac{d(\varpi,\mu)}{1+d(\varpi,\mu)}$ for all $\varpi,\mu\in\mathcal{P}$. Then $(\mathcal{P},\hat{\mathcal{A}}_d,\tilde{\triangle})$ is AFMS where $\tilde{\triangle}$ is defined by $\ell\ \tilde{\triangle}\ ɤ=\ell+ɤ-\ellɤ$ for all $\ell,ɤ\in[0,1]$(see [32]). Suppose a mapping $\Im:\mathcal{P}\to\mathcal{P}$ specified as follows:

$$\Im(\varpi)=\begin{cases}\dfrac{\varpi}{5} & \text{if } \varpi\in[0,1]\\ 3\varpi & \text{if } \varpi\in(1,\infty)\end{cases}$$

$\hat{\alpha},\hat{\beta}:\mathcal{P}\times\mathcal{P}\to[0,1)$ given as

$$\hat{\alpha}(\varpi,\mu)=\begin{cases}1 & \text{if } \varpi,\mu\in[0,1]\\ 0 & \text{if } \varpi,\mu\in(1,\infty)\end{cases}$$

$$\hat{\beta}(\varpi,\mu)=\begin{cases}2 & \text{if } \varpi,\mu\in[0,1]\\ 0 & \text{if } \varpi,\mu\in(1,\infty)\end{cases}$$

and $\Omega:\ [0,1)\to\ [0,1)$ by $\Omega(w)=\frac{w}{2}$. Then $\Im$ is $(\hat{\alpha},\hat{\beta})-\Omega-\text{CMap}$.

Proof: Obviously for any $\varpi,\mu\in[0,1]$ we have that $\hat{\alpha}(\varpi,\Im\varpi)=1$ and $\hat{\beta}(\mu,\Im\mu)=2$ as such we have that $\hat{\alpha}(\varpi,\Im\varpi)\hat{\beta}(\mu,\Im\mu)>1$. We must demonstrate that,

$$\hat{\mathcal{A}}(\Im\varpi,\Im\mu)\ \le\Omega\left(\hat{\mathcal{A}}(\varpi,\mu)\right)+\mathcal{G}\ \min\{\hat{\mathcal{A}}(\varpi,\Im\varpi),\hat{\mathcal{A}}(\mu,\Im\mu),\hat{\mathcal{A}}(\varpi,\Im\mu)\hat{\mathcal{A}}(\mu,\Im\varpi)\}$$

DOI: https://doi.org/10.54216/IJNS.250433

for any $\varpi, \mu \in [0,1]$ .Observe that for $\mathcal{G} > 0$ we have:

$$\hat{\mathcal{A}}(\mathfrak{I}\varpi, \mathfrak{I}\mu) = \frac{\left|\frac{\varpi}{5} - \frac{\mu}{5}\right|}{1 + \left|\frac{\varpi}{5} - \frac{\mu}{5}\right|}$$

$$< \frac{1}{5}\left[\frac{|\varpi - \mu|}{5 + |\varpi - \mu|}\right]$$

$$\leq \frac{1}{2}\left[\frac{|\varpi-\mu|}{1+|\varpi-\mu|}\right] + \mathcal{G}\min\{\frac{\left|\varpi-\frac{\varpi}{5}\right|}{1+\left|\varpi-\frac{\varpi}{5}\right|}, \frac{\left|\mu-\frac{\mu}{5}\right|}{1+\left|\mu-\frac{\mu}{5}\right|}, (\frac{\left|\varpi-\frac{\mu}{5}\right|}{1+\left|\varpi-\frac{\mu}{5}\right|})(\frac{\left|\mu-\frac{\varpi}{5}\right|}{1+\left|\mu-\frac{\varpi}{5}\right|})\}$$

$$\leq \Omega\left(\hat{\mathcal{A}}(\varpi, \mu)\right) + \mathcal{G}\min\{\hat{\mathcal{A}}(\varpi, \mathfrak{I}\varpi), \hat{\mathcal{A}}(\mu, \mathfrak{I}\mu), \hat{\mathcal{A}}(\varpi, \mathfrak{I}\mu)\hat{\mathcal{A}}(\mu, \mathfrak{I}\varpi)\}$$

Thus $\mathfrak{I}$ is $(\hat{\alpha}, \hat{\beta}) - \Omega -$ CMap .

Theorem 3.4. Assume $(\mathcal{P}, \hat{\mathcal{A}}, \widetilde{\triangle})$ is a complete AFMS and $\mathfrak{I}: \mathcal{P} \to \mathcal{P}$ is $(\hat{\alpha}, \hat{\beta}) - \Omega -$ CMap . Assume the following assumptions are true:

a) $\mathfrak{I}$ is $(\hat{\alpha}, \hat{\beta})$-CA mapping,

b) there is $\varpi_\circ \in \mathcal{P}$ with $\hat{\alpha}(\varpi_\circ, \mathfrak{I}\,\varpi_\circ) \geq 1$ and $\hat{\beta}(\varpi_\circ, \mathfrak{I}\,\varpi_\circ) \geq 1$.

c) $\mathfrak{I}$ is continuous.

Then $\mathfrak{I}$ possess FP.

Proof: Consider (μ_n) defined by $\mu_{n+1} = \mathfrak{I}\mu_n$, $\forall n \in \mathbb{N} \cup \{0\}$. If $\mu_{n+1} = \mu_n$ then the intended outcome is achieved. Assume that $\mu_{n+1} \neq \mu_n$. Since $\mathfrak{I}$ is $(\hat{\alpha}, \hat{\beta})$-CA mapping and $\hat{\alpha}(\varpi_\circ, \varpi_1) \geq 1$, we have $\hat{\beta}(\mathfrak{I}\varpi_\circ, \mathfrak{I}\,\varpi_1) = \hat{\beta}(\varpi_1, \varpi_2) \geq 1$ and this implies that $\hat{\alpha}(\varpi_2, \varpi_3) = \hat{\alpha}(\mathfrak{I}\varpi_1, \mathfrak{I}\,\varpi_2) \geq 1$, proceeding with the procedure, we have

$$\hat{\alpha}(\varpi_{2\kappa}, \varpi_{2\kappa+1}) \geq 1 \text{ and } \hat{\beta}(\varpi_{2\kappa+1}, \varpi_{2\kappa+2}) \geq 1 \text{ for each } \kappa \in \mathbb{N} \cup \{0\} \qquad (10)$$

By applying a similar reasoning, we have that

$$\hat{\beta}(\varpi_{2\kappa}, \varpi_{2\kappa+1}) \geq 1 \text{ and } \hat{\alpha}(\varpi_{2\kappa+1}, \varpi_{2\kappa+2}) \geq 1 \qquad (11)$$

From (10) and (11), it is evident that $\hat{\alpha}(\varpi_n, \varpi_{n+1}) \geq 1$ and $\hat{\beta}(\varpi_n, \varpi_{n+1}) \geq 1$. Since $\hat{\alpha}(\varpi_n, \varpi_{n+1})$ $\hat{\beta}(\varpi_{n+1}, \varpi_{n+2}) \geq 1$; we obtain from (9)

$$\hat{\mathcal{A}}(\varpi_{n+1}, \varpi_{n+2}) = \hat{\mathcal{A}}(\mathfrak{I}\varpi_n, \mathfrak{I}\varpi_{n+1})$$

$$\leq \Omega\left(\hat{\mathcal{A}}(\varpi_n, \varpi_{n+1})\right) + \mathcal{G}\min\{\hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}), \hat{\mathcal{A}}(\varpi_{n+1}, \varpi_{n+2}), \hat{\mathcal{A}}(\varpi_n, \varpi_{n+2})\hat{\mathcal{A}}(\varpi_{n+1}, \varpi_{n+1})\} \qquad (12)$$

$$= \Omega\left(\hat{\mathcal{A}}(\varpi_n, \varpi_{n+1})\right) + \mathcal{G}.0$$

$$< \hat{\mathcal{A}}(\varpi_n, \varpi_{n+1})$$

Thus $\hat{\mathcal{A}}(\varpi_{n+1}, \varpi_{n+2}) < \hat{\mathcal{A}}(\varpi_n, \varpi_{n+1})$

By using the same approach, it is simple to observe that $\hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}) < \hat{\mathcal{A}}(\varpi_{n-1}, \varpi_n)$. Hence the sequence $(\hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}))$ is strictly decreasing and bounded below. Thus there is $\gamma \geq 0$ such that $\lim_{n\to\infty} \hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}) = \gamma$. Utilizing the continuity of Ω and the reality that $\Omega(w) < w$ for any $w > 0$, and $\Omega(0) = 0$, we may conclude from (12) that

$$\gamma = \lim_{n\to\infty} \hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}) \leq \lim_{n\to\infty} \Omega(\hat{\mathcal{A}}(\varpi_{n-1}, \varpi_n)) < \gamma$$

This is a contradiction. As a result, $\gamma = 0$, and therefore

$$\lim_{n\to\infty} \hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}) = 0 \qquad (13)$$

Now to show that (ϖ_n) is a Cauchy. Suppose that (ϖ_n) is not a Cauchy, then by Lemma 3.1, there exists an $\sigma > 0$ and sequences (m_κ) and (n_κ) with $m_\kappa > n_\kappa > \kappa$ such that $\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa}) \geq \sigma$. We can select n_κ as the least

DOI: https://doi.org/10.54216/IJNS.250433

positive integer such that $\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa}) \geq \sigma$, $\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa - 1}) < \sigma$ and (a)-(c) of Lemma 3.1 hold. Since $\hat{\alpha}(\varpi_\circ, \Im\,\varpi_\circ) \geq 1$ and $\hat{\beta}(\varpi_\circ, \Im\,\varpi_\circ) \geq 1$, Applying Lemma 2.12, we see that,

$\hat{\alpha}(\varpi_{m_\kappa}, \varpi_{m_\kappa + 1})\ \hat{\beta}(\varpi_{n_\kappa}, \varpi_{n_\kappa + 1}) \geq 1$. Thus for all $\kappa \geq n_\circ$,

$$\hat{\mathcal{A}}(\varpi_{m_\kappa + 1}, \varpi_{n_\kappa + 1}) \leq \hat{\mathcal{A}}(\Im\varpi_{m_\kappa}, \Im\varpi_{n_\kappa})$$

$$\leq \Omega\left(\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa})\right) + \mathcal{G}\min\{\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{m_\kappa+1}), \hat{\mathcal{A}}(\varpi_{n_\kappa}, \varpi_{n_\kappa+1}), \hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa+1})\hat{\mathcal{A}}(\varpi_{n_\kappa}, \varpi_{m_\kappa+1})\} \quad (14)$$

By using the continuity of Ω with $\Omega(w) < w$ for any $w > 0$ and Lemma 3.2, Lemma2.12, we obtain that:

$$\sigma = \lim_{\kappa \to \infty} \hat{\mathcal{A}}(\Im\varpi_{m_\kappa}, \Im\varpi_{n_\kappa})$$

$$\leq \lim_{\kappa \to \infty} [\Omega\left(\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa})\right) + \mathcal{G}\ \min\{\hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{m_\kappa+1}), \hat{\mathcal{A}}(\varpi_{n_\kappa}, \varpi_{n_\kappa+1}), \hat{\mathcal{A}}(\varpi_{m_\kappa}, \varpi_{n_\kappa+1})\hat{\mathcal{A}}(\varpi_{n_\kappa}, \varpi_{m_\kappa+1})\}\ ]$$

$$< \sigma.$$

This is a contradiction. Hence (ϖ_n) is Cauchy. Since $(\mathcal{P}, \hat{\mathcal{A}}_d, \widetilde{\triangle})$ is complete, then there is $\varpi \in \mathcal{P}$ such that $\lim_{n \to \infty} \varpi_n = \varpi$. Since $\Im$ is continuous, we have that:

$\varpi = \lim_{n \to \infty} \varpi_n = \lim_{n \to \infty} \varpi_{n+1} = \lim_{n \to \infty} \Im\varpi_n = \Im \lim_{n \to \infty} \varpi_n = \Im\varpi$. Thus $\Im$ possess FP.

Theorem 3.5. Suppose $(\mathcal{P}, \hat{\mathcal{A}}, \widetilde{\triangle})$ is complete AFMS and $\Im: \mathcal{P} \to \mathcal{P}$ is $(\hat{\alpha}, \hat{\beta}) - \Omega -$ CMap . Assume the following assumptions are true:

a) $\Im$ is $(\hat{\alpha}, \hat{\beta})$-CA mapping,

b) there exists $\varpi_\circ \in \mathcal{P}$ such that $\hat{\alpha}(\varpi_\circ, \Im\,\varpi_\circ) \geq 1$ and $\hat{\beta}(\varpi_\circ, \Im\,\varpi_\circ) \geq 1$.

c) If for any (ϖ_n) in $\mathcal{P}$ with $\varpi_n \to \varpi$ as $n \to \infty$ then $\hat{\beta}(\varpi, \Im\,\varpi) \geq 1$ and $\hat{\alpha}((\varpi, \Im\,\varpi) \geq 1$.

Then $\Im$ possess FP.

Proof: Assume the sequence (μ_n) defined by $\mu_{n+1} = \Im\mu_n$ for each $n \in \mathbb{N} \cup \{0\}$. It is established in Theorem 3.4 that (ϖ_n) is Cauchy and since $(\mathcal{P}, \hat{\mathcal{A}}_d, \widetilde{\triangle})$ is complete, then there is $\varpi \in \mathcal{P}$ with $\lim_{n \to \infty} \varpi_n = \varpi$. Assuming hypothesis (c) is true, we demonstrate that $\Im$ possess FP.Since $\hat{\alpha}(\varpi_n, \varpi_{n+1}) \geq 1$ and $\hat{\beta}(\varpi, \Im\,\varpi) \geq 1$ we obtain from (9)that

$$\hat{\mathcal{A}}(\varpi_{n+1}, \Im\varpi) = \hat{\mathcal{A}}(\Im\varpi_n, \Im\varpi)$$

$$\leq \Omega\left(\hat{\mathcal{A}}(\varpi_n, \varpi)\right) + \mathcal{G}\min\{\hat{\mathcal{A}}(\varpi_n, \varpi_{n+1}), \hat{\mathcal{A}}(\varpi, \Im\varpi), \hat{\mathcal{A}}(\varpi_n, \Im\varpi)\hat{\mathcal{A}}(\varpi, \Im\varpi_n)\} \quad (15)$$

Using (13) and the assumptions on Ω, and taking the limit as $n \to \infty$, it is simple to demonstrate that

$\hat{\mathcal{A}}(\varpi, \Im\varpi) = \lim_{n \to \infty} \hat{\mathcal{A}}(\varpi_{n+1}, \Im\varpi) = \lim_{n \to \infty} \hat{\mathcal{A}}(\Im\varpi_n, \Im\varpi) = 0$ and this indicate that$\Im\varpi = \varpi$. Thus $\Im$ possess FP.

Now to support the outcomes mentioned above, we provide an example.

Example 3.6. Let $\mathcal{P} = [0, \infty)$ and let $(\mathcal{P}, d)$ be a metric space where $d(\varpi, \mu) = |\varpi - \mu|$. Put $\hat{\mathcal{A}}_d(\varpi, \mu) = \frac{d(\varpi,\mu)}{1+d(\varpi,\mu)}$ for all $\varpi, \mu \in \mathcal{P}$. Then $(\mathcal{P}, \hat{\mathcal{A}}_d, \widetilde{\triangle})$ is AFMS . Suppose a mapping $\Im: \mathcal{P} \to \mathcal{P}$ specified as follows:

$$\Im(\varpi) = \begin{cases} \dfrac{\varpi}{7} & \text{if } \varpi \in [0,2) \\ 2\varpi - \dfrac{25}{3} & \text{if } \varpi \in [2, \infty) \end{cases}$$

$\hat{\alpha}, \hat{\beta}: \mathcal{P} \times \mathcal{P} \to [0,1)$ given as

$$\hat{\alpha}(\varpi, \mu) = \begin{cases} \dfrac{5}{2} & \text{if } \varpi, \mu \in [0,1] \\ 0 & \text{if } \varpi, \mu \in (1, \infty) \end{cases}$$

$$\hat{\beta}(\varpi, \mu) = \begin{cases} 1 & \text{if } \varpi, \mu \in [0,1] \\ 0 & \text{if } \varpi, \mu \in (1, \infty) \end{cases}$$

and $\Omega : [0,1) \to [0,1)$ by $\Omega(w) = \frac{w}{3}$. Then $\Im$ is $(\hat{\alpha}, \hat{\beta}) - \Omega -$ CMap and all conditions of theorem3.5 are hold.

DOI: https://doi.org/10.54216/IJNS.250433

Proof:For any $\varpi,\mu \in [0,1]$ we have that $\widehat{\alpha}(\varpi,\mu)=\frac{5}{2}>1$ and $\widehat{\beta}(\varpi,\mu)=1$. Also $\Im(\varpi)=\frac{\varpi}{7}$ and $\Im(\mu)=\frac{\mu}{7}$ are in [0,1]. In addition $\widehat{\alpha}(\varpi,\mu)=\frac{5}{2}>1 \Rightarrow \widehat{\beta}(\Im\varpi,\Im\mu)=1$ and $\widehat{\beta}(\varpi,\mu)=1 \Rightarrow \widehat{\alpha}(\Im\varpi,\Im\mu)=\frac{5}{2}>1$ therfore $\Im$ is $(\widehat{\alpha},\widehat{\beta})$-CA mapping.For every $\varpi_\circ \in [0,1]$, it is evident that $\widehat{\alpha}(\varpi_\circ,\Im\varpi_\circ)=\frac{5}{2}>1$ and $\widehat{\beta}(\varpi_\circ,\Im\varpi_\circ)=1$: Additionally, every hypothesis outlined in Theorems 3.5 and 3.4 is met. It remains now to prove that $\Im$ is $(\widehat{\alpha},\widehat{\beta})-\Omega-$CMap . For any $\varpi,\mu \in [0,1]$ we have that $\widehat{\alpha}(\varpi,\Im\varpi)\widehat{\beta}(\mu,\Im\mu)=\frac{5}{2}>1$ and for $\mathcal{G}>0$ its clear that:

$$\begin{aligned}\hat{\mathcal{A}}(\Im\varpi,\Im\mu) &= \frac{\left|\frac{\varpi}{7}-\frac{\mu}{7}\right|}{1+\left|\frac{\varpi}{7}-\frac{\mu}{7}\right|}\\ &< \frac{1}{7}\left[\frac{|\varpi-\mu|}{7+|\varpi-\mu|}\right]\\ &\le \frac{1}{3}\left[\frac{|\varpi-\mu|}{1+|\varpi-\mu|}\right]+\mathcal{G}\min\{\frac{\left|\varpi-\frac{\varpi}{7}\right|}{1+\left|\varpi-\frac{\varpi}{7}\right|},\frac{\left|\mu-\frac{\mu}{7}\right|}{1+\left|\mu-\frac{\mu}{7}\right|},(\frac{\left|\varpi-\frac{\mu}{7}\right|}{1+\left|\varpi-\frac{\mu}{7}\right|})(\frac{\left|\mu-\frac{\varpi}{7}\right|}{1+\left|\mu-\frac{\varpi}{7}\right|})\}\\ &\le \Omega\left(\hat{\mathcal{A}}(\varpi,\mu)\right)+\mathcal{G}\min\{\hat{\mathcal{A}}(\varpi,\Im\varpi),\hat{\mathcal{A}}(\mu,\Im\mu),\hat{\mathcal{A}}(\varpi,\Im\mu)\hat{\mathcal{A}}(\mu,\Im\varpi)\}\end{aligned}$$

Hence $\Im$ is $(\widehat{\alpha},\widehat{\beta})-\Omega-$CMap and $\varpi=0$, $\varpi=\frac{25}{3}$ are two FP of $\Im$.

Corollary 3.7. Suppose $(\mathcal{P},\hat{\mathcal{A}},\widetilde{\triangle})$ is complete AFMS and $\Im:\mathcal{P}\to\mathcal{P}$ satisfying the following inequality

$$\widehat{\alpha}(\varpi,\Im\varpi)\widehat{\beta}(\mu,\Im\mu)\ge 1 \Rightarrow \hat{\mathcal{A}}(\Im\varpi,\Im\mu) \le \Omega\left(\hat{\mathcal{A}}(\varpi,\mu)\right)$$

for all $\varpi,\mu \in \mathcal{P}$, where $\Omega:[0,\infty)\to[0,\infty)$, and for all $w>0$, $\Omega(w)<w$ and $\Omega(0)=0$. Assume the following assumptions are true:

a) $\Im$ is $(\widehat{\alpha},\widehat{\beta})$-CA mapping,

b) there is $\varpi_\circ \in \mathcal{P}$ such that $\widehat{\alpha}(\varpi_\circ,\Im\varpi_\circ)\ge 1$ and $\widehat{\beta}(\varpi_\circ,\Im\varpi_\circ)\ge 1$.

(c) $\Im$ is continuous .

(d) If for any sequence (ϖ_n) in $\mathcal{P}$ such that $\varpi_n\to\varpi$ then $\widehat{\beta}(\varpi,\Im\varpi)\ge 1$ and $\widehat{\alpha}((\varpi,\Im\varpi)\ge 1$. Then $\Im$ possess FP.

4. Applications

We apply our major finding to the existence of the integral equation's solution in this section. We provide a solution to the following nonlinear integral problem employing Corollary 3.7.

$$\varpi(\mathfrak{t}) = \mathcal{k}(\mathfrak{t}) + \int_{a}^{b}\mathcal{Q}(\mathfrak{t},s)\,K(\mathfrak{t},\varpi(s))ds \qquad (16)$$

where $:[a,b]\times[a,b]\to R^{+}$, $K:[a,b]\times R\to R$ and $\mathcal{k}:[a,b]\to R$ are continuous functions. Let $\mathcal{P}=C([a,b],R)$ be the space of all continuous real-valued functions defined on $[a,b]$. We defined $\hat{\mathcal{A}}(\varpi,\mu)$as follows

$$\hat{\mathcal{A}}(\varpi,\mu)=\max_{\mathfrak{t}\in[a,b]}\partial_{R}(\varpi(\mathfrak{t})-\mu(\mathfrak{t})) \text{ for all } \varpi,\mu\in\mathcal{P}.$$

Then $(\mathcal{P},\hat{\mathcal{A}},\widetilde{\triangle})$ is complete AFMS where $\ell\,\widetilde{\triangle}\,ɤ=\ell+ɤ-\ellɤ$ for all $\ell,ɤ \in [0,1]$ (see [32]).

Theorem 4.1. Suppose $\mathcal{P}=C([a,b],R)$ and assume $\Im:\mathcal{P}\to\mathcal{P}$ is operator define by

$$\Im\varpi(\mathfrak{t}) = \mathcal{k}(\mathfrak{t}) + \int_{a}^{b}\mathcal{Q}(\mathfrak{t},s)\,K(\mathfrak{t},\varpi(s))ds \text{ for any } \mathfrak{t},s \in [a,b]$$

where $:[a,b]\times[a,b]\to R^{+}$, $K:[a,b]\times R\to R$ and $\mathcal{k}:[a,b]\to R$ are continuous functions. Consider $=C([a,b],R)$. Assume the following assumptions are true:

1) there is a continuous mapping $g:\mathcal{P}\times\mathcal{P}\to[0,\infty)$ such that

$$K(s,\varpi(s))-K(s,\mu(s))\le g(\varpi,\mu)[\varpi(s)-\mu(s)]$$

for each $s \in [a,b]$ and $\varpi,\mu\in\mathcal{P}$.

DOI: https://doi.org/10.54216/IJNS.250433

2)there is $\vartheta \in (0,1)$ and $\hat{\alpha}, \hat{\beta}: \mathcal{P} \times \mathcal{P} \to [0,\infty)$ such that $\hat{\alpha}(\varpi,\mu) \geq 1 \Rightarrow \hat{\beta}(\Im\varpi,\Im\mu) \geq 1$ and $\hat{\beta}(\varpi,\mu) \geq 1 \Rightarrow \hat{\alpha}(\Im\varpi,\Im\mu) \geq 1$ for all $\varpi,\mu \in \mathcal{P}$ and $\int_{a}^{b} \mathcal{Q}(\mathfrak{t},s)\, g(\varpi,\mu) \leq \vartheta$.

3) there is $\varpi_{\circ} \in \mathcal{P}$ such that $\hat{\alpha}(\varpi_{\circ},\Im\varpi_{\circ}) \geq 1$ and $\hat{\beta}(\varpi_{\circ},\Im\varpi_{\circ}) \geq 1$.

4.) If for any $(\varpi_{\mathfrak{n}})$ in $\mathcal{P}$ such that $\varpi_{\mathfrak{n}} \to \varpi$ as $\mathfrak{n} \to \infty$ then $\hat{\beta}(\varpi,\Im\varpi) \geq 1$ and $\hat{\alpha}((\varpi,\Im\varpi) \geq 1$. Then, there is a solution to the integral equation (16).

Proof : First we define $\hat{\alpha}, \hat{\beta}: \mathcal{P} \times \mathcal{P} \to [0,\infty)$ as follows

$$\hat{\alpha}(\varpi,\mu) = \begin{cases} 1 & \text{if } \varpi \preccurlyeq \mu \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\beta}(\varpi,\mu) = \begin{cases} 2 & \text{if } \varpi \preccurlyeq \mu \\ 0 & \text{otherwise} \end{cases}$$

and consider a mapping Ω given by $\Omega(\mathfrak{t}) = \vartheta\mathfrak{t}$ for all $\mathfrak{t} \in [a,\mathrm{b}]$. We define $\varpi,\mu \in \mathcal{P}, \varpi \preccurlyeq \mu$ if and only if $\varpi(\mathfrak{t}) \leq \mu(\mathfrak{t})$. It is clear that if $\varpi \preccurlyeq \mu$ then $\hat{\alpha}(\varpi,\Im\varpi)\hat{\beta}(\mu,\Im\mu) > 1$. Thus, we have

$$\begin{aligned} [\Im\mu(s) - \Im\varpi(s)] &\leq \int_{a}^{b} [\mathcal{Q}(\mathfrak{t},s)[\mathrm{K}(\mathfrak{t},\mu(s)) - \mathrm{K}(\mathfrak{t},\varpi(s))]]\, ds \\ &\leq \int_{a}^{b} \mathcal{Q}(\mathfrak{t},s) g(\varpi,\mu)[\mu(s) - \varpi(s)]\, ds \\ &\leq \max_{s\in[a,b]} \partial_{\mathrm{R}}(\mu(s) - \varpi(s)) \int_{a}^{b} \mathcal{Q}(\mathfrak{t},s) g(\varpi,\mu)\, ds \\ &\leq \vartheta\hat{\mathcal{A}}(\varpi,\mu) \\ &\leq \Omega(\hat{\mathcal{A}}(\varpi,\mu)) \end{aligned}$$

Hence $\hat{\mathcal{A}}(\Im\varpi,\Im\mu) \leq \Omega\left(\hat{\mathcal{A}}(\varpi,\mu)\right)$. All requirements in Corollary 3.7 are met, indicating that $\Im$ has an FP. Thus, the equation (16) has a solution.

5. Conclusion

In the context of Algebra Fuzzy Metric Space, this paper introduces a novel class of contraction mappings and proves the existence of theorems for their FPs. Additionally; we offer some examples that support our results. Eventually, the fixed-point theorem will be directly applied to the solution of integral equations.

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