New Higher-Order Implicit Method for Approximating Solutions of Boundary-Value Problems

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Abstract

This paper is devoted to introducing a novel numerical approach for approximating solutions to Boundary Value Problems (BVPs). Such an approach will be carried out by using a new version of the shooting method, which would convert the BVP into a linear system of two initial value problems. This system can then be solved by the so-called Obreschkoff approach. The numerical solution of the main BVP will ultimately be a linear combination of the solutions of the two system of equations. Two physical applications will be presented in order to confirm that the suggested numerical technique is valid.

Keywords: Obreschkoff formula; Boundary value problems; Shooting method; Approximations

1 Introduction

Boundary value problems (B.V.P) form the crux of modeling dynamic systems across scientific and engineering disciplines, compelling the quest for precise and efficient solutions through numerical approximation methods.[1–8] For the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha. \tag{1}$$

In,[9] Alomari et al. established a new higher order implicit method for approximating I.V.P. which is described in the difference equation:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\widetilde{B}_1^{(n)}(x_i, w_i) + h\widetilde{B}_2^{(n)}(x_{i+1}, w_{i+1}), \end{aligned} \tag{2}$$

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for each $i = 0, 1, 2, \cdots N - 1$, where

$$\widetilde{B}_1^{(n)}(x_i, w_i) := \sum_{k=1}^{n} \frac{\begin{pmatrix} n \\ k \end{pmatrix}}{\begin{pmatrix} 2n \\ k \end{pmatrix}} \frac{h^{k-1}}{k!} f^{(k-1)}(x_i, w_i),$$

and

$$\widetilde{B}_2^{(n)}(x_{i+1}, w_{i+1}) := \sum_{k=1}^{n} (-1)^{k+1} \frac{\begin{pmatrix} n \\ k \end{pmatrix}}{\begin{pmatrix} 2n \\ k \end{pmatrix}} \frac{h^{k-1}}{k!} f^{(k-1)}(x_{i+1}, w_{i+1}).$$

In particular case, when $n = 4$, we have

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\widetilde{B}_1^{(n)}(x_i, w_i) + h\widetilde{B}_2^{(n)}(x_{i+1}, w_{i+1}), \end{aligned} \tag{3}$$

for each $i = 0, 1, 2, \cdots N - 1$, where

$$\widetilde{B}_1^{(n)}(x_i, w_i) := \frac{1}{2} f(x_i, w_i) + \frac{3h}{28} f'(x_i, w_i) + \frac{h^2}{84} f''(x_i, w_i) + \frac{h^3}{1680} f^{(3)}(x_i, w_i)$$

and

$$\widetilde{B}_2^{(n)}(x_{i+1}, w_{i+1}) := \frac{1}{2} f(x_{i+1}, w_{i+1}) - \frac{3h}{28} f'(x_{i+1}, w_{i+1}) + \frac{h^2}{84} f''(x_{i+1}, w_{i+1}) - \frac{h^3}{1680} f^{(3)}(x_{i+1}, w_{i+1})$$

Let us begin with basic important key well-known results about the uniqueness of the solution of a given Boundary Value Problem (B.V.P). Suppose the function f in B.V.P

$$y'' = f(x, y, y'), \qquad \text{for } a \le x \le b, \text{ with } \qquad y(a) = \alpha, \text{ and } y(b) = \beta$$

is continuous on the set

$$D = \{(x, y, y') \,|\text{for } a \le x \le b, \text{with} - \infty < y < \infty \text{ and} - \infty < y' < \infty\}.$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

1. $f_{y'}(x, y, y') > 0$ for all $(x, y, y') \in D$, and
2. a constant M exists, with

   $$|f_{y'}(x, y, y')| \le M, \qquad \text{for all } (x, y, y') \in D$$

   then the boundary-value problem has a unique solution.

Dealing with the shooting method, a general type of B.V.P. can simplified through the above theorem as such: the differential equation

$$y'' = f(x, y, y')$$

is linear when functions $p(x)$, $q(x)$, and $r(x)$ exist with

$$y'' = p(x)\, y' + q(x)\, y + r(x), \qquad \text{for } a \le x \le b, \text{ with } \qquad y(a) = \alpha, \text{ and } y(b) = \beta \tag{4}$$

In viewing the previous discussion, the above boundary-value problem has a unique solution.

To approximate the unique solution to this linear problem using the shooting method, we usually transform B.V.P. (4) to two initial value problems, as follows:

$$y'' = p\left(x\right)y' + q\left(x\right)y + r\left(x\right), \qquad \text{for } a \leq x \leq b, \text{ with } \qquad y(a) = \alpha, \text{ and } y'(a) = 0 \tag{5}$$

and

$$y'' = p\left(x\right)y' + q\left(x\right)y, \qquad \text{for } a \leq x \leq b, \text{ with } \qquad y(a) = 0, \text{ and } y'(a) = 1 \tag{6}$$

The main theory of I.V.P. ensures that under the hypotheses mentioned above, both problems have a unique solution.

Let $y_1(x)$ denote the solution to (5), and let $y_2(x)$ denote the solution to (6). Assume that $y_2(b) \neq 0$, and define

$$y\left(x\right) = y_1\left(x\right) + \frac{\beta - y_1\left(b\right)}{y_2\left(b\right)}y_2\left(x\right). \tag{7}$$

It is easy to verify that $y\left(x\right)$ is a solution to the linear boundary problem (5).

The use of the Shooting method to address linear equations involves a sophisticated approach to problem-solving, characterized by a strategic maneuver that replaces the original linear boundary value problem with a pair of initial-value problems. These initial-value problems, aptly referenced as (5) and (6), serve as the foundational building blocks upon which the entire computational framework is constructed. Through this transformative substitution, the problem space is reconfigure4d, allowing for a more nuanced exploration of the solution landscape.

Central to the success of this method is the intricate approximation of the solutions $y_1(x)$ and $y_2(x)$. To achieve this, advanced Oberschoff methods are skillfully deployed, boasting both fourth and eighth orders of accuracy. Renowned for their precision and reliability, these methods navigate the complex terrain of numerical computation with finesse, yielding approximations that converge toward the true solutions with remarkable efficiency.

The acquisition of these approximations marks a critical juncture in the computational process, heralding the transition toward the resolution of the boundary-value problem. Here, Equation (7) emerges as the cornerstone, guiding the meticulous estimation of the solution with adept precision. Through a careful orchestration of mathematical techniques and computational algorithms, the boundary-value problem is dissected and analyzed, culminating in a comprehensive understanding of its underlying dynamics.

Numerical experiments conducted to validate the capabilities of our method further amplify its significance. These experiments not only demonstrate the robustness and accuracy of our analytical method but also underscore its superiority when compared to conventional approaches, including the traditional Taylor method. In summary, our analytical method for approximating B.V.Ps in ODEs represents a pioneering contribution, offering a fresh perspective that extends beyond the limitations of existing techniques and provides a more comprehensive and effective solution for a diverse range of mathematical contexts.

2 The Shooting method through the Obreschkoff approach

Let us begin by transforming (5) into first-order linear I.V.P. by setting $u_1 = y'$ and $u_2 = y''$. Also, for (6) let $v_1 = y'$ and $v_2 = y''$. Thus, we have two systems of equations of the form

$$\begin{cases} u_1' = u_2, & u_1(a) = \alpha \\ u_2' = p\left(x\right)u_2 + q\left(x\right)u_1 + r\left(x\right) & u_2(a) = 0 \end{cases} \tag{8}$$

and

$$\begin{cases} v_1' = v_2, & v_1(a) = 0 \\ v_2' = p\left(x\right)v_2 + q\left(x\right)v_1 & v_2(a) = 1 \end{cases} \tag{9}$$

For the purpose of solving the B.V.P using Oberschoff method of order 8, we consider $\frac{du_1}{dx} = f_1(x, u_1)$ and $\frac{du_2}{dx} = f_2(x, u_1)$. Thus, the corresponding difference equation for the first I.V.P. in (8) is given as follows:

$$\begin{aligned} u_{1,0} &= \alpha \\ u_{1,i+1} &= u_{1,i} + h\widetilde{S}_1^{(2)}(x_i, u_{1,i}) + h\widetilde{S}_2^{(2)}(x_{i+1}, u_{1,i+1}), \end{aligned} \tag{10}$$

for each $i = 0, 1, 2, \cdots N-1$, where

$$S_1^{(2)}(x_i, u_{1,i}) := \frac{1}{2}f_1(x_i, u_{1,i}) + \frac{3h}{28}f_1'(x_i, u_{1,i}) + \frac{h^2}{84}f_1''(x_i, u_{1,i}) + \frac{h^3}{1680}f_1^{(3)}(x_i, u_{1,i}),$$

and

$$\begin{aligned} &S_2^{(2)}(x_{i+1}, u_{1,i+1}) := \\ &\quad \frac{1}{2}f_1(x_{i+1}, u_{1,i+1}) - \frac{3h}{28}f_1'(x_{i+1}, u_{1,i+1}) + \frac{h^2}{84}f_1''(x_{i+1}, u_{1,i+1}) - \frac{h^3}{1680}f_1^{(3)}(x_{i+1}, u_{1,i+1}) \end{aligned}$$

and the corresponding difference equation for the second I.V.P. in (8) is given as follows:

$$\begin{aligned} u_{2,0} &= 0 \\ u_{2,i+1} &= u_{2,i} + h\widetilde{S}_1^{(2)}(x_i, u_{2,i}) + h\widetilde{S}_2^{(2)}(x_{i+1}, u_{2,i+1}), \end{aligned} \tag{11}$$

for each $i = 0, 1, 2, \cdots N-1$, where

$$S_1^{(2)}(x_i, u_{2,i}) := \frac{1}{2}f_2(x_i, u_{2,i}) + \frac{3h}{28}f_2'(x_i, u_{2,i}) + \frac{h^2}{84}f_2''(x_i, u_{2,i}) + \frac{h^3}{1680}f_2^{(3)}(x_i, u_{2,i}),$$

and

$$\begin{aligned} &S_2^{(2)}(x_{i+1}, u_{2,i+1}) := \\ &\quad \frac{1}{2}f_2(x_{i+1}, u_{2,i+1}) - \frac{3h}{28}f_2'(x_{i+1}, u_{2,i+1}) + \frac{h^2}{84}f_2''(x_{i+1}, u_{2,i+1}) - \frac{h^3}{1680}f_2^{(3)}(x_{i+1}, u_{2,i+1}) \end{aligned}$$

In a similar manner, the equation in (9) could be treated as in (10) and (11).

3 Applications

This part intends to provide two physical applications; the first one is about the beam deflection, whereas the second one is about the heat balance for a long thin rod.

Application 3.1. *The Figure 1 shows the beam deflection of the elastic curve for a simply supported, uniformly loaded beam. The general differential equation represents this problem is given as*

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}w\left(Lx - x^2\right) \tag{12}$$

where E:= the modulus of elasticity, I:= The moment of inertia, L:= Length of beam, and w:= the force at one end. The boundary conditions are $y(0) = y(L) = 0$. We solve for the beam deflection problem using the shooting method via Obreschkoff's approach by considering the following values of parameters:

$$E = 200\,GPa, \quad I = 30,000\,cm^4, \quad w = 150\,kN/m, \quad \text{and } L = 3\,m.$$

Transforming (12) *into a system of differential equation yields that*

$$EIu'' = \frac{1}{2}w\left(Lx - x^2\right), \quad u(0) = 0,\ u'(0) = 0. \tag{13}$$

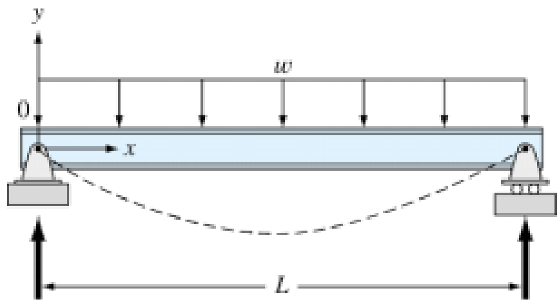

Figure 1: Beam deflection.

$$EIv'' = 0, \quad v(0) = 0, \ v'(0) = 1. \tag{14}$$

To solve the I.V.P. in (13)*, let us assume* $u = \varphi'$*, and therefore* $u' = \varphi'' = \frac{1}{2EI} w\left(Lx - x^2\right)$*. The corresponding system of* (13) *is*

$$\begin{cases} u' = f_1(x,u), & u(0) = 0 \\ \varphi' = f_2(x,\varphi), & \varphi(0) = 0 \end{cases}. \tag{15}$$

Making of use (10)*, then for the system* (15) *we have*

$$\begin{aligned} f_1(x,u) &= u, \\ f_1'(x,u) &= u' = \frac{1}{2EI} w\left(Lx - x^2\right), \\ f_1''(x,u) &= \frac{1}{2EI} w(L - 2x), \\ f_1'''(x,u) &= -\frac{w}{EI}. \end{aligned}$$

The corresponding difference equation of the first I.V.P. in (15) *could be obtained by substituting in* (10) *with* $h = 0.1$

$$\begin{aligned} u_{1,0} &= 0 \\ u_{1,i+1} &= u_{1,i} + h\widetilde{S}_1^{(2)}(x_i, u_{1,i}) + h\widetilde{S}_2^{(2)}(x_{i+1}, u_{1,i+1}), \end{aligned}$$

for each $i = 0, 1, 2, \cdots 30$*, where*

$$S_1^{(2)}(x_i, u_{1,i}) := \frac{1}{2} u_{1,i} + \frac{3h}{28} \cdot \frac{w}{2EI}\left(Lx_i - x_i^2\right) + \frac{h^2}{84} \cdot \frac{w}{2EI}(L - 2x_i) - \frac{h^3}{1680} \cdot \frac{w}{EI},$$

and

$$S_2^{(2)}(x_{i+1}, u_{1,i+1}) := \frac{1}{2} u_{1,i+1} - \frac{3h}{28} \cdot \frac{w}{2EI}\left(Lx_{i+1} - x_{i+1}^2\right) + \frac{h^2}{84} \cdot \frac{w}{2EI}(L - 2x_{i+1}) + \frac{h^3}{1680} \cdot \frac{w}{EI}.$$

Similarly, for the second I.V.P., we have

$$\begin{aligned} f_2(x,\varphi) &= \frac{1}{2EI} w\left(Lx - x^2\right), \\ f_2'(x,\varphi) &= \frac{1}{2EI} w(L - 2x), \\ f_2''(x,\varphi) &= -\frac{w}{EI}, \\ f_2'''(x,\varphi) &= 0. \end{aligned}$$

Thus, the corresponding difference equation of the first I.V.P. in (15) *could be obtained by substituting in* (10) *with* $h = 0.1$

$$\begin{aligned} u_{2,0} &= 0 \\ u_{2,i+1} &= u_{2,i} + h\widetilde{S}_1^{(2)}(x_i, u_{2,i}) + h\widetilde{S}_2^{(2)}(x_{i+1}, u_{2,i+1}), \end{aligned}$$

for each $i = 0, 1, 2, \cdots 30$*, where*

$$S_1^{(2)}(x_i, u_{2,i}) := \frac{1}{2}u_{2,i} + \frac{3h}{28} \cdot \frac{w}{2EI}\left(Lx_i - x_i^2\right) + \frac{h^2}{84} \cdot \frac{w}{2EI}(L - 2x_i) - \frac{h^3}{1680} \cdot \frac{w}{EI},$$

and

$$S_2^{(2)}(x_{i+1}, u_{2,i+1}) := \frac{1}{2}u_{2,i+1} - \frac{3h}{28} \cdot \frac{w}{2EI}\left(Lx_{i+1} - x_{i+1}^2\right) + \frac{h^2}{84} \cdot \frac{w}{2EI}(L - 2x_{i+1}) + \frac{h^3}{1680} \cdot \frac{w}{EI}.$$

By substituting in (10)*, we get an approximation solution of the first solution* $y_1(x)$ *of the first D.E. in* (13)*. Now, to solve the I.V.P. in* (14)*, let us assume* $z = v' = f_3(x, v)$ *and* $z' = v'' = 0$*. The corresponding system of* (14) *is*

$$\begin{cases} v' = f_3(x, v) \\ z' = f_4(x, z) = 0 \end{cases}. \tag{16}$$

Making of use (10)*, the for the system* (16) *we have*

$$\begin{aligned} f_3(x, v) &= z, \\ f_3'(x, v) &= z' = 0, \\ f_3''(x, v) &= 0, \\ f_3'''(x, v) &= 0, \end{aligned}$$

and

$$\begin{aligned} f_4(x, z) &= 0, \\ f_4'(x, z) &= 0, \\ f_4''(x, z) &= 0, \\ f_4'''(x, z) &= 0. \end{aligned}$$

By substituting in (11)*, we get an approximation solution of the first solution* $y_2(x)$ *of the first D.E. in* (14)*. Now, by applying the shooting method, we have the general solution*

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)}y_2(x).$$

To see the numerical solution approximated by using the Obreschkoff–Shooting method for Application 3.1, we compare it with its analytical solution

$$y(x) = \frac{wLx^3}{12EI} - \frac{wx^4}{24EI} - \frac{wL^3x}{24EI}$$

as shown in Table 1 and Figure 2. In addition, we plot in Figure 3 the absolute error involving the numerical and analytical solutions for Application 3.1.

Application 3.2. *A heat balance for a long, thin rod can be created using the conservation of heat (see Figure 4). When the system is in a steady state and the rod is not insulated along its length, the following equation emerges:*

$$\frac{d^2T}{dx^2} + h^*(T_a - T) = 0, \tag{17}$$

where T_a *is the ambient air temperature (*$^\circ C$*), and* h^* *is a heat transfer coefficient (*m^{-2}*) that parameterizes the rate of heat dissipation to the surrounding air. Maintaining constant temperatures at the end of the bar is the most straightforward scenario for suitable boundary conditions. Such conditions might be given mathematically as*

$$T(0) = T_1, \quad T(1) = T_2. \tag{18}$$

Table 1: Comparison between the analytical solution and the Obreschkoff–Shooting method with absolute error applied in Application 3.1 with step size $h = 0.1$

x_i	Analytical$\times 10^{-5}$	OS$\times 10^{-5}$	AE $\times 10^{-6}$
0.0	0.0000	0.0000	0.0000
0.1	−0.0281	−0.0287	0.0059
0.2	−0.0558	−0.0571	0.0134
0.3	−0.0828	−0.0850	0.0222
0.4	−0.1088	−0.1120	0.0322
0.5	−0.1335	−0.1377	0.0428
0.6	−0.1566	−0.1620	0.0540
0.7	−0.1779	−0.1845	0.0654
0.8	−0.1973	−0.2049	0.0768
0.9	−0.2144	−0.2232	0.0880
1.0	−0.2292	−0.2390	0.0987
1.1	−0.2414	−0.2523	0.1087
1.2	−0.2511	−0.2629	0.1179
1.3	−0.2581	−0.2707	0.1261
1.4	−0.2623	−0.2756	0.1331
1.5	−0.2637	−0.2775	0.1388
1.6	−0.2623	−0.2766	0.1429
1.7	−0.2581	−0.2726	0.1455
1.8	−0.2511	−0.2657	0.1464
1.9	−0.2414	−0.2560	0.1454
2.0	−0.2292	−0.2434	0.1425
2.1	−0.2144	−0.2282	0.1377
2.2	−0.1973	−0.2104	0.1308
2.3	−0.1779	−0.1901	0.1219
2.4	−0.1566	−0.1677	0.1108
2.5	−0.1335	−0.1432	0.0977
2.6	−0.1088	−0.1170	0.0823
2.7	−0.0828	−0.0893	0.0649
2.8	−0.0558	−0.0603	0.0453
2.9	−0.0281	−0.0304	0.0237
3.0	0.0000	0.0000	0.0000

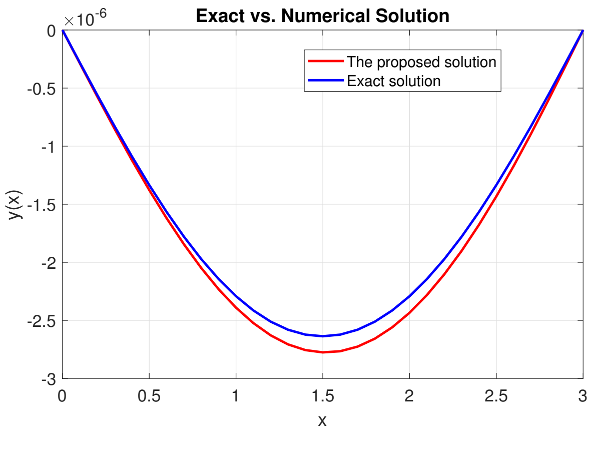

Figure 2: Comparison between the analytical solution and numerical solution obtained by Obreschkoff–Shooting method for Application 3.1 with step size $h = 0.1$

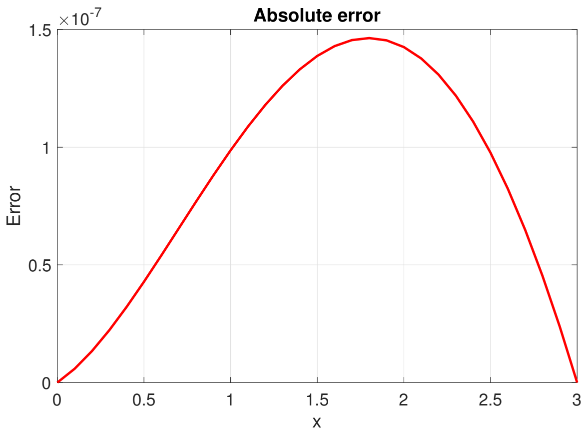

Figure 3: Absolute error involving the numerical and analytical solutions for Application 3.1

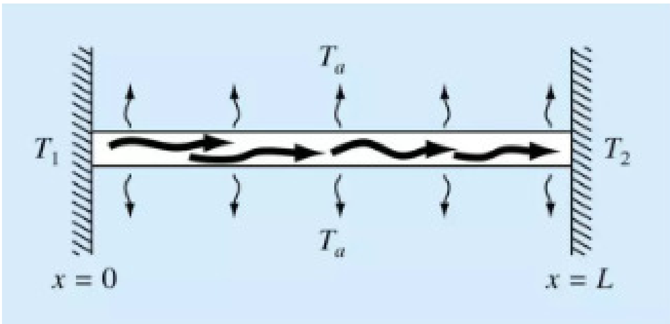

Figure 4: A noninsulated uniform rod positioned two bodies of constant but different tempreture

Actually, equation (17) *can be solved analytically under these conditions. In particular, for a 10-m rod with* $T_a = 20$, $T_1 = 40$, $T_2 = 200$, *and* $h^* = 0.01$, *the solution is*

$$T(x) = 73.4523e^{0.1x} - 53.4523e^{-0.1x} + 20. \tag{19}$$

In a similar manner to all steps discussed in the previous application, we can implement the Obreschkoff–Shooting method to obtain an approximate solution of problem (17)-(18) *numerically. To see the numerical solution approximated by using the considered method for Application 3.2, we compare it with its analytical solution given in* (19) *to be shown in Figure 5. In addition, we plot in Figure 6 the absolute error involving the numerical and analytical solutions for Application 3.2.*

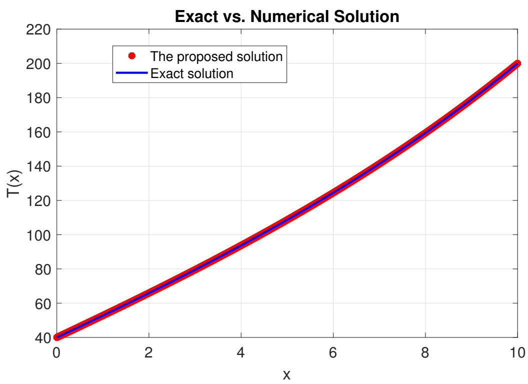

Figure 5: Comparison between the analytical solution and numerical solution obtained by Obreschkoff–Shooting method for Application 3.2 with step size $h = 0.1$

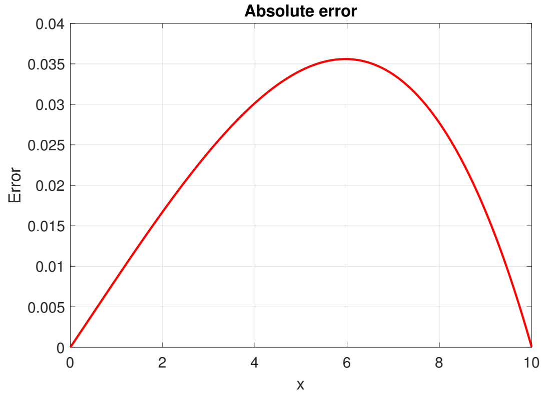

Figure 6: Absolute error involving the numerical and analytical solutions for Application 3.2

4 Conclusion

This study has succeeded in presenting a new numerical method for estimating boundary value problem (BVP) solutions. A modified shooting method has been used to implement this strategy, which has turned the BVP into a linear system of two initial value problems. That is when the so-called Obreschkoff technique has been used to solve this problem. In the end, a linear combination of the solutions to the two systems of equations has represented the numerical solution of the main BVP. To validate the recommended numerical technique, two physical applications have been given.

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